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POSINORMAL TERRACED MATRICES

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Dedicated to Professor Russell A. Stokes

Abstract. This paper is a study of some properties of a collection of bounded linear operators resulting from terraced matrices M acting through multiplication on ℓ^2; the term *terraced matrix* refers to a lower triangular infinite matrix with constant row segments. Sufficient conditions are found for M to be *posinormal*, meaning that $MM^*=M^*PM$ for some positive operator P on ℓ^2; these conditions lead to new sufficient conditions for the hyponormality of M. Sufficient conditions are also found for the adjoint M^* to be posinormal, and it is observed that, unless M is essentially trivial, M^* cannot be hyponormal. A few examples are considered that exhibit special behavior.

1. Introduction

Assume that $a = \{a_n\}$ is a sequence of complex numbers such that the *terraced matrix* M, a lower triangular infinite matrix with constant row segments, acts through multiplication to give a bounded linear operator on ℓ^2.

$$M = M(a) = \begin{pmatrix} a_0 & 0 & 0 & \dots \\ a_1 & a_1 & 0 & \dots \\ a_2 & a_2 & a_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

These matrices have been studied in [4, 6]. The best-known terraced matrix, the Cesàro matrix C, occurs when $a_n = \frac{1}{n+1}$ for all $n \geq 0$; in [1] it is proved that C is a bounded linear operator on ℓ^2 and that C is hyponormal.

For an operator M on ℓ^2 to be *posinormal*, there must exist a positive operator P on ℓ^2 satisfying $MM^* = M^*PM$. These operators were introduced and studied in [7], where it was observed that the set of all posinormal operators on ℓ^2 is an enormous collection that includes every invertible operator and all the hyponormal operators, although the immensity was overstated regarding weighted shifts, as has been noted in [3]; the correct statement is the following.

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Proposition 1.1. *Every unilateral weighted shift with weight sequence* $\{w_n\}$ *satisfying* $\sup_n |\frac{w_n}{w_{n+1}}| < +\infty$ *is posinormal.*

The sup condition ensures the boundedness of the interrupter P.

Recall that M is said to be *hyponormal* on ℓ^2 if

$$\langle [M^*, M]f, f\rangle \equiv \langle (M^*M - MM^*)f, f\rangle \geq 0$$

for all f in ℓ^2. In this study, we will find sufficient conditions for a terraced matrix M to be posinormal, and this will lead us to new sufficient conditions for M to be hyponormal.

Example 1.1. Before proceeding, we observe that not all terraced matrices are posinormal. For consider the matrix M associated with a sequence having $a_0 = 0$ and $a_1 > 0$, and take $f = e_0 - e_1$ where $\{e_n\}$ is the standard orthonormal basis for ℓ^2. Then $Mf = 0$ while $M^*f = -a_1(e_0 + e_1)$, so $f \in \mathrm{Ker}M$ but $f \notin \mathrm{Ker}M^*$. Thus M cannot be posinormal (see [7, Corollary 2.3]).

2. Main results

The first theorem will present sufficient conditions for the posinormality of a terraced matrix.

Theorem 2.1. *If* $a = \{a_n\}$ *is a positive decreasing sequence that converges to* 0 *and* $\{(n+1)(1-\frac{a_{n+1}}{a_n})\}$ *is a bounded sequence, then* $M = M(a)$ *is posinormal.*

Proof. We will display an operator B on ℓ^2 that satisfies $M^* = BM$; consequently, $M = M^*B^*$ also, and it will follow from [7, Theorem 2.1] that M is posinormal.

We define $B = [b_{mn}]$ by

$$b_{mn} = \begin{cases} 1 - \frac{a_{n+1}}{a_n} & \text{if} \quad m \leq n; \\ -\frac{a_{n+1}}{a_n} & \text{if} \quad m = n+1; \\ 0 & \text{if} \quad m > n+1. \end{cases}$$

We must show that B is a bounded operator on ℓ^2. Let $R = M(s)$ where $s = \{1 - \frac{a_{n+1}}{a_n} : n = 0, 1, 2, \ldots\}$, so R is a terraced matrix with all of its entries nonnegative. Since the diagonal matrix D with diagonal$\{(n+1)(1 - \frac{a_{n+1}}{a_n}) : n = 0, 1, 2, \ldots\}$ is bounded, $R = DC$ is bounded. We observe that $(B^* - R)$ is the adjoint of a unilateral weighted shift; since $\{a_n\}$ is positive and decreasing, $(B^* - R)$ is bounded. Therefore $B^* = R + (B^* - R)$ is a bounded operator, and hence B is bounded also. A direct computation shows that $M^* = BM$, as needed. □

For fixed $p > 1$, the p-Cesàro matrix is the terraced matrix associated with the sequence defined by $a_n = \frac{1}{(n+1)^p}$ for all n. These matrices were studied in [5] and in [7], which contains a proof of the next result.

Corollary 2.1. *If* M *is a* p*-Cesàro matrix for some fixed* $p > 1$, *then* M *is posinormal.*

Proof. The hypothesis of the theorem is satisfied since $1-\frac{(n+1)^p}{(n+2)^p} \leq \frac{p}{n+2}$ for all n and for all $p > 1$ (see [2, Theorem 42, 2.15.3, page 40]). $\square$

Corollary 2.2. *If $a = \{a_n\}$ is a positive decreasing sequence that converges to 0 and $\{na_n\}$ is an increasing sequence that converges to $L < +\infty$, then $M = M(a)$ is posinormal.*

Example 2.1. We consider the case where $a = \{a_n\}$ is determined as follows: Choose a_0 in (0,1) and then define the rest of the sequence recursively by $a_{n+1} = a_n(1 - a_n)$. It can be shown that the sequence $\{na_n\}$ is increasing to 1, so $M = M(a)$ is posinormal by Corollary 2.2. Also it can be verified that BB^* is the diagonal operator with diagonal $\{a_0, 1-a_0, 1-a_1, 1-a_2, \dots\}$ and hence B^* is a contraction. It follows that B is also a contraction and therefore $M = M(a)$ is hyponormal for this choice of a. We note that a different approach was used in [8] to show that this operator is hyponormal.

What we saw in Example 2.1 encourages us to continue looking at B in hopes of discovering more examples of terraced matrices that are hyponormal. It turns out that, with a strengthening of the conditions in the hypothesis of Corollary 2.2, we are led to new sufficient conditions for the hyponormality of M, as presented in the next theorem. The proof relies on our knowledge of B.

Theorem 2.2. *Assume $a = \{a_n\}$ is a sequence that satisfies the following:*

(1) *$\{a_n\}$ is a strictly decreasing sequence that converges to 0;*
(2) *$\{(n+1)a_n\}$ is a strictly increasing sequence that converges to $L < +\infty$; and*
(3) *$\frac{1}{a_{n+1}} \geq \frac{1}{2}(\frac{1}{a_n} + \frac{1}{a_{n+2}})$ for all n.*

Then $M = M(a)$ is hyponormal.

Proof. We know that, because of conditions (1) and (2), M is posinormal with $M^* = BM$; the entries of B are described in the proof of Theorem 2.1. For M to be hyponormal, we must have $\langle (M^*M - MM^*)f, f\rangle = \langle (M^*M - (M^*B^*)(BM))f, f\rangle = \langle (I - B^*B)Mf, Mf\rangle \geq 0$ for all f in ℓ^2. Consequently, we can conclude that M will be hyponormal when $Q \equiv I - B^*B \geq 0$; we note that the range of M contains all the e_n's from the standard orthonormal basis for ℓ^2.

The entries of $Q = [q_{mn}]$ are given by

$$q_{mn} = \begin{cases} \frac{(a_n-a_{n+1})((n+2)a_{n+1}-na_n)}{a_n^2} & \text{if} \quad m = n; \\ \frac{(a_m-a_{m+1})((n+2)a_{n+1}-(n+1)a_n)}{a_m a_n} & \text{if} \quad m > n; \\ \frac{(a_n-a_{n+1})((m+2)a_{m+1}-(m+1)a_m)}{a_m a_n} & \text{if} \quad m < n. \end{cases}$$

In order to show that Q is positive, it suffices to show that Q_N, the N^{th} finite section of Q (involving rows $m = 0, 1, 2, \dots, N$ and columns $n = 0, 1, 2, \dots, N$), has positive determinant for each positive integer N. We proceed in the following way.

For columns $n = 1, 2, \ldots, N$, we multiply the n^{th} column from Q_N by $t(n) = \frac{a_n((n+1)a_n - na_{n-1})}{a_{n-1}((n+2)a_{n+1}-(n+1)a_n)}$ and then subtract from the $(n-1)^{st}$ column. Call the new matrix Q'_N and note that det $Q'_N =$ det Q_N. We then work with the rows of Q'_N. For $m = 1, 2, \ldots, N$, we multiply the m^{th} row from Q'_N by $t(m)$ and then subtract from the $(m-1)^{st}$ row.

The resulting matrix is tridiagonal with the following form:

$$Y_N = \begin{pmatrix} d_0 & s_0 & 0 & \ldots & 0 & 0 \\ s_0 & d_1 & s_1 & \ldots & 0 & 0 \\ 0 & s_1 & d_2 & \ldots & . & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & . & \ldots & d_{N-1} & s_{N-1} \\ 0 & 0 & 0 & \ldots & s_{N-1} & d_N \end{pmatrix}$$

where $s_n = -\frac{(a_{n+1}-a_{n+2})((n+2)a_{n+1}-(n+1)a_n)}{a_n((n+3)a_{n+2}-(n+2)a_{n+1})}$ $(0 \leq n \leq N-1)$,

$$d_n = \frac{(a_n - a_{n+1})((n+2)a_{n+1} - na_n)}{a_n^2} + (n+3)s_n^2 \quad (0 \leq n \leq N-1),$$

and

$$d_N = \frac{(a_N - a_{N+1})((N+2)a_{N+1} - Na_N)}{a_N^2}.$$

In transforming Y_N into a triangular matrix with the same determinant, we find that the new matrix has diagonal entries δ_n which are given by a recursion formula: $\delta_0 = d_0, \delta_n = d_n - \frac{s_{n-1}^2}{\delta_{n-1}}$ $(1 \leq n \leq N)$. An induction argument using conditions (1) through (3) shows that $\delta_n \geq \frac{a_{n+1}s_n^2}{a_{n+1}-a_{n+2}} > 0$ for $0 \leq n \leq N-1$; since d_N departs from the pattern set by the earlier d_n's, δ_N must be handled separately: $\delta_N = d_N - \frac{s_{N-1}^2}{\delta_{N-1}} \geq \frac{(a_N - a_{N+1})[(N+2)a_{N+1}-(N+1)a_N]}{a_N^2} > 0$. So det $Q_N = \prod_{j=0}^{N} \delta_n > 0$, and the proof is complete. □

We note that a specialized version of the procedure of the preceding proof was used in [7] to show the hyponormality of M for the case $a_n = \frac{1}{n+k}$ for fixed $k > 0$.

Example 2.2. Consider the case where $a = \{a_n\}$ is given by $a_n = \frac{n+3}{(n+2)^2}$. This positive sequence does not satisfy the inequality

$$a_n(1 - a_n) \leq a_{n+1} \leq \frac{a_n}{1 + a_n} \quad \text{for all } n,$$

the sufficient condition for the hyponormality of M presented in [8]. However, this example does satisfy the three conditions in the hypothesis of Theorem 2.2, so we now know that M is hyponormal for this choice of a.

Example 2.3. We note that the sequences given by $0 < a_0 < 1$ and then recursively by $a_{n+1} = a_n(1 - a_n)$ for all n, or by $a_{n+1} = \frac{a_n}{1+a_n}$ for all n, satisfy conditions (1) through (3) of Theorem 2.2. These two sequences were involved in the inequality from [8] that was mentioned in the previous example. The first sequence was also discussed in Example 2.1. Both sequences result in associated hyponormal terraced matrices M.

Example 2.4. For the Cesàro matrix C, it is demonstrated in [9] that there does not exist a hyponormal terraced matrix M satisfying $U^*MU = C$, where U is the unilateral shift. What if we are willing to relax the requirement on M and settle for posinormality? If we consider any sequence $s = \{s_n\}$ that satisfies $s_0 > 1$ and $s_n = \frac{1}{n}$ for $n \geq 1$, then $\{ns_n\}$ is a nondecreasing sequence that converges to limit $L = 1$, so Corollary 2.2 guarantees that $M = M(s)$ is posinormal. It is straightforward to verify that the equation $C = U^*MU$ is satisfied.

3. Results for the adjoint

The principal result in this section gives sufficient conditions for the posinormality of the adjoint of a terraced matrix.

Theorem 3.1. *If $a = \{a_n\}$ is a positive decreasing sequence that converges to 0 and $\{\frac{1}{a_{n+1}} - \frac{1}{a_n}\}$ is a bounded sequence, then $M^* = M(a)^*$ is posinormal.*

Proof. We define $T = [t_{mn}]$ by

$$t_{mn} = \begin{cases} \frac{a_m}{a_n} & \text{if } \ n = 0; \\ a_m(\frac{1}{a_n} - \frac{1}{a_{n-1}}) & \text{if } \ 0 < n \leq m; \\ -1 & \text{if } \ n = m+1; \\ 0 & \text{if } \ n > m+1. \end{cases}$$

By hypothesis, the diagonal matrix D with diagonal

$$\{\frac{1}{a_0}, \frac{1}{a_1} - \frac{1}{a_0}, \frac{1}{a_2} - \frac{1}{a_1}, \frac{1}{a_3} - \frac{1}{a_2}, \ldots\}$$

is bounded, so MD is bounded. Therefore $T = MD - U^*$ is a bounded operator. A routine computation shows that $M = TM^*$ and hence $M^* = MT^*$. By [7, Theorem 2.1], M^* is posinormal. $\square$

Corollary 3.1. *If $a = \{a_n\}$ is a positive decreasing sequence that converges to 0 and $\{(n+1)a_n\}$ is an increasing sequence that converges to $L < +\infty$, then $M^* = M(a)^*$ is posinormal.*

We note that, in contrast with what we saw earlier for the matrix B, T cannot be a contraction since $\|Te_0\|^2 = \sum_{m=0}^{\infty}(\frac{a_m}{a_0})^2 > 1$. So there will not be any cases where T can help us prove hyponormality for M^*. The next two theorems further address this issue, more generally, for sequences $a = \{a_n\}$ of complex numbers.

Theorem 3.2. *The adjoint M^* of a terraced matrix is hyponormal if and only if the associated sequence $\{a_n\}$ is essentially trivial (that is, $a_j = 0$ for all $j > 0$).*

Proof. The definition of hyponormality applied to M^* requires that $\langle (MM^* - M^*M)e_0, e_0\rangle = -\sum_{j=1}^{\infty} |a_j|^2 \geq 0$, and this occurs if and only if $a_j = 0$ for all $j > 0$. □

What we saw in Theorem 3.2 leads us to ask whether M^* can be a dominant operator. Recall that a *dominant* operator A is one for which $\mathrm{Ran}(A - \lambda) \subset \mathrm{Ran}(A - \lambda)^*$ for all λ in the spectrum of A (see [10]); hyponormal operators are necessarily dominant. In [7] it was shown that A is dominant if and only if $A - \lambda$ is posinormal for all complex λ.

Theorem 3.3. *Suppose the terraced matrix M is determined by a sequence $\{a_n\}$ with $a_j \neq 0$ for some $j > 0$. Then M^* cannot be dominant.*

Proof. We note that $e_0 \in \mathrm{Ker}(M - a_0)^*$ but $e_0 \notin \mathrm{Ker}(M - a_0)$. Thus $(M - a_0)^*$ is not posinormal (see [7, Corollary 2.3]). It follows that M^* is not dominant. □

Thus we see that the adjoint of a terraced matrix is dominant if and only if the associated sequence $\{a_n\}$ is essentially trivial.

4. Conclusion

We close with some examples and a theorem that will tie together some of the results from this study. Recall that M is said to be *cohyponormal* (*codominant*) if M^* is hyponormal (dominant). Similarly, M is *coposinormal* if M^* is posinormal.

Theorem 4.1. *Assume $a = \{a_n\}$ is a sequence satisfying the following conditions:*

(1) *$\{a_n\}$is a strictly decreasing sequence that converges to* 0;
(2) *$\{(n+1)a_n\}$ is a strictly increasing sequence that converges to $L < +\infty$; and*
(3) $\frac{1}{a_{n+1}} \geq \frac{1}{2}(\frac{1}{a_n} + \frac{1}{a_{n+2}})$ *for all n.*

Then, for this choice of a and for $M = M(a)$,

(a) *M is posinormal, coposinormal, dominant, and hyponormal;*
(b) *M is not cohyponormal and not codominant; and*
(c) *M has norm $\|M\| = 2L$ and spectrum $\sigma(A) = \{\lambda : |\lambda - L| \leq L\}$.*

Proof. (a) This claim is justified by Corollary 2.2, Corollary 3.1, and Theorem 2.2.

(b) This statement follows from Theorems 3.2 and 3.3.

(c) The assertion about the norm and the spectrum is an immediate consequence of the main results from [6]. This assertion depends only on conditions (1) and (2). □

Example 4.1. Consider the case where a is given by $a_n = \frac{1}{\sqrt{(n+1)(n+2)}}$ for all n. This example satisfies conditions (1) through (3) of Theorem 4.1 with $L = 1$. When W is the unilateral weighted shift with weights $\{(n+1)a_n\}$, we note that it is straightforward to compute that $(M - W)(M - W)^* = I$, so

$\|M - W\| = 1$. A direct calculation verifies that $\|M^*a\| = \frac{\pi}{\sqrt{6}}$. We note that this positive sequence also satisfies $a_n(1 - a_n) \leq a_{n+1} \leq \frac{a_n}{1+a_n}$ for all n, the criterion for the hyponormality of M presented in [8].

Example 4.2. Other examples that satisfy conditions (1) and (2) of Theorem 4.1 with $L = 1$ include $a_n = \ln(1 + \frac{1}{n+1})$, $a_n = \sin(\frac{1}{n+1})$, and $a_n = \arctan(\frac{1}{n+1})$. This study has not settled the question of hyponormality, or of dominance, for the posinormal terraced matrices associated with these sequences.

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