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DIMENSION AND INTERVAL DIMENSION OF CERTAIN BIPARTITE ORDERED SETS

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ABSTRACT. We consider the dimension and interval dimension of bipartite ordered sets and provide a sufficient condition for when the two parameters are equal.

1. Introduction

Let X be a set. An *order* R on X is a reflexive, antisymmetric and transitive binary relation on X. Then $P = (X, R)$ is called an *ordered set.* For $a, b \in X$, we usually write $a \leq b$ for $(a, b) \in R$ and also $a < b$ when $a \leq b$ and $a \neq b$. In this paper, we assume that every set is finite. An order R on a set is called an *extension* of another order S on the same set if $S \subseteq R$. An order R is *linear* if $(a, b) \in R$ or $(b, a) \in R$ for any $a, b \in X$. Szpilarjn [3] showed that any order has a linear extension and that the intersection of all linear extensions of an order is the order itself. Dushnik and Miller [1] later defined the *dimension* of an ordered set $P = (X, R)$, denoted by $\dim(P)$, to be the minimal cardinality of a family of linear extensions of R whose intersection is R itself.

An incomparable pair (a, b) in an ordered set P is called a *critical pair* if $x < a$ implies $x < b$ and $x > b$ implies $x > a$. It is well known that the dimension of ordered set P is the least positive integer t for which there exists a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of the order of P reversing all critical pairs in P. An ordered set $P = (X \cup Y, I_P)$ is *bipartite* if X and Y are disjoint nonempty sets and I_P is an order on $X \cup Y$ such that $\emptyset \neq \{(x, y) \in I_P \mid x \neq y\} \subseteq X \times Y$. In [4], Trotter defined the *interval dimension* of a bipartite ordered set P, denoted by $\dim_I(P)$, as the least positive integer t for which there exists a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of the order of P reversing all critical pairs only in $X \times Y$. Hence the interval dimension of P is quite easier to compute than the dimension of P in some cases. However, Trotter [4] has shown that $\dim(P) - 1 \leq \dim_I(P) \leq \dim(P)$ for any bipartite ordered set P. In this paper, we find a sufficient condition for

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a bipartite ordered set P to have

$$\dim(P) = \dim_I(P).$$

2. Preliminaries

Let G and M be nonempty sets. We define a *context* as a triple (G, M, I) with $I \subseteq G \times M$ (see [5]). A subset F of $G \times M$ is called a *Ferrers relation* if g_1Fm_1 and g_2Fm_2 implies g_1Fm_2 or g_2Fm_1 for $g_1, g_2 \in G$ and $m_1, m_2 \in M$. For a Ferrers relation F in $G \times M$, we define

$$C(F) = \{g \in G \mid (g,m) \in F\} \text{ and } R(F) = \{m \in M \mid (g,m) \in F\}.$$

For $g \in G$ and $m \in M$, let

$$gF = \{m \in M \mid (g,m) \in F\} \text{ and } Fm = \{g \in G \mid (g,m) \in F\}.$$

Now observe that

(i) $gF \subseteq g'F$ or $gF \supseteq g'F$ for $g, g' \in G$,
(ii) $Fm \subseteq Fm'$ or $Fm \supseteq Fm'$ for $m, m' \in M$.

The *Ferrers dimension* of a context (G, M, I), denoted by $\text{fdim}(G, M, I)$, is defined to be the smallest number of Ferrers relations $F_1, F_2, \ldots, F_n$ in $G \times M$ with $I = \bigcap F_i$. Observe that the complement of a Ferrers relation in $G \times M$ is again a Ferrers relation in $G \times M$. Therefore, one can alternatively define $\text{fdim}(G, M, I)$ as the minimum number of Ferrers relations $F_1, F_2, \ldots, F_n$ in $G \times M$ such that $(G \times M) - I = \bigcup F_i$. For better visualization we shall use this alternative definition throughout this paper.

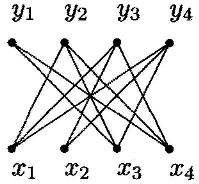
Let $P = (X, R)$ be an ordered set. Then (X, X, R) is a context and it is well known (cf. [2]) that

$$\dim(P) = \text{fdim}(X, X, R).$$

Similarly, if $P = (X \cup Y, I_P)$ is a bipartite ordered set, then (X, Y, I_P) is also a context and it can be easily seen that

$$\dim_I(P) = \text{fdim}(X, Y, I_P).$$

For instance, considering the bipartite ordered set $P = (X \cup Y, I_P)$ in Figure 1, we can find three Ferrers relations in $X \times Y$ to show that $\dim_I(P) = 3$ as in Table 1.



P

Figure 1

$X \setminus Y$	y_1	y_2	y_3	y_4
x_1	F_1	O	O	O
x_2	F_1	F_1	O	O
x_3	O	O	F_2	O
x_4	O	O	O	F_3

Here O means an order relation

Table 1

To state our main result in the next section, we need some definitions. Let $P = (X \cup Y, I_P)$ be a bipartite ordered set. Throughout this paper we consider $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ with *fixed* indexing. For a Ferrers relation F in $X \times Y$, a nonempty subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ of X is called a *C_1-set* of F if

$$x_{i_1}F \supseteq x_{i_2}F \supseteq \cdots \supseteq x_{i_m}F \text{ with } i_1 > i_2 > \cdots > i_m,$$

a nonempty subset $\{y_{j_1}, y_{j_2}, \ldots, y_{j_n}\}$ of Y is called an *R_1-set* of F if

$$Fy_{j_1} \supseteq Fy_{j_2} \supseteq \cdots \supseteq Fy_{j_n} \text{ with } j_1 < j_2 < \cdots < j_n,$$

a nonempty subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ of X is called a *C_2-set* of F if

$$x_{i_1}F \supseteq x_{i_2}F \supseteq \cdots \supseteq x_{i_m}F \text{ with } i_1 < i_2 < \cdots < i_m,$$

and a nonempty subset $\{y_{j_1}, y_{j_2}, \ldots, y_{j_n}\}$ of Y is called an *R_2-set* of F if

$$Fy_{j_1} \supseteq Fy_{j_2} \supseteq \cdots \supseteq Fy_{j_n} \text{ with } j_1 > j_2 > \cdots > j_n.$$

A family $\mathcal{F}$ of Ferrers relations in $X \times Y$ called a (*optimal*) *realizer* of P if $|\mathcal{F}| = \text{fdim}(X, Y, I_P)$ and $\cup_{F \in \mathcal{F}} F = X \times Y - I_P$. Let $\mathcal{E}$ be a nonempty subfamily of a realizer $\mathcal{F}$ of P. We say that $\mathcal{E}$ is a *lower left cover* of $\mathcal{F}$ if for each $F \in \mathcal{E}$ there exist a C_1-set X_F and an R_1-set Y_F of F such that for some k and l

$$\bigcup_{F \in \mathcal{E}} X_F = \{x_k, x_{k+1}, \ldots, x_r\} \text{ and } \bigcup_{F \in \mathcal{E}} Y_F = \{y_1, y_2, \ldots, y_l\}.$$

Similarly, we say $\mathcal{E}$ is an *upper right cover* of $\mathcal{F}$ if for each $F \in \mathcal{E}$ there exist a C_2-set X_F and an R_2-set Y_F of F such that for some k and l

$$\bigcup_{F \in \mathcal{E}} X_F = \{x_1, x_2, \ldots, x_k\} \text{ and } \bigcup_{F \in \mathcal{E}} Y_F = \{y_l, y_{l+1}, \ldots, y_s\}.$$

Finally, for a Ferrers relation F in $X \times Y$, let

$$\underline{F} = \bigcup_{(x_k, y_l) \in F} \{(x_u, y_v) \mid k \le u \text{ and } v \le l\}$$

and

$$\overline{F} = \bigcup_{(x_k, y_l) \in F} \{(x_u, y_v) \mid u \le k \text{ and } l \le v\}.$$

3. Main theorem

In this section we prove the following main theorem.

Theorem. *Let $P = (X \cup Y, I_P)$ be a bipartite ordered set with $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ and let $\mathcal{F}$ be a realizer of P. Suppose that there are a lower left cover $\mathcal{F}_1$ and an upper right cover $\mathcal{F}_2$ of $\mathcal{F}$ with $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ such that $\bigcap_{F \in \mathcal{F}_1} \underline{F} \cap \bigcap_{F \in \mathcal{F}_2} \overline{F} = \emptyset$. Then*

$$\dim(P) = \dim_I(P).$$

Proof. Since $\dim_I(P) \leq \dim(P)$, it is enough to show that $\dim(P) \leq \dim_I(P)$.

For each $F \in \mathcal{F}_1$, let

$$H_F = F \cup F_X \cup F_Y \cup F_0,$$

where

(i) F_X is a maximal Ferrers relation in $X \times X$ with the property that $F_X \subseteq \{(x_i, x_j) \in X \times X \mid i > j\}$ and that $xF \subset x'F \Longrightarrow xF_X \subseteq x'F_X$ for $x, x' \in X$,

(ii) F_Y is a maximal Ferrers relation in $Y \times Y$ with the property that $F_Y \subseteq \{(y_u, y_v) \in Y \times Y \mid u > v\}$ and that $Fy \subset Fy' \Longrightarrow F_Y y \subseteq F_Y y'$ for $y, y' \in Y$, and

(iii) $F_0 = \{(y, x) \mid (x, y) \in X \times Y - \underline{F}\}$.

For each $F \in \mathcal{F}_2$, let

$$H'_F = F \cup F'_X \cup F'_Y \cup F'_0,$$

where

(i) F'_X is a maximal Ferrers relation in $X \times X$ with the property that $F'_X \subseteq \{(x_i, x_j) \in X \times X \mid i < j\}$ and that $xF \subset x'F \Longrightarrow xF'_X \subseteq x'F'_X$ for $x, x' \in X$,

(ii) F'_Y is a maximal Ferrers relation in $Y \times Y$ with the property that $F'_Y \subseteq \{(y_u, y_v) \in Y \times Y \mid u < v\}$ and that $Fy \subset Fy' \Longrightarrow F'_Y y \subseteq F'_Y y'$ for $y, y' \in Y$, and

(iii) $F'_0 = \{(y, x) \mid (x, y) \in X \times Y - \overline{F}\}$.

We shall prove that H_F and H'_F are Ferrers relations in $(X \cup Y) \times (X \cup Y)$. Since the case for H'_F can be treated similarly, we only show that each H_F is a Ferrers relation in $(X \cup Y) \times (X \cup Y)$.

Let $X_F = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$ with $i_1 > i_2 > \cdots > i_m$ and $Y_F = \{y_{j_1}, y_{j_2}, \ldots, y_{j_n}\}$ with $j_1 < j_2 < \cdots < j_n$ which are a C_1-set and an R_1-set, respectively, of F. Now we have the following useful observations:

(1) $x_t F_X = \{x_1, x_2, \ldots, x_{t-1}\}$ if and only if $t \in \{2, 3, \ldots, i_m - 1, i_m, \ldots, i_2, i_1\}$.

(2) If $x_i \in C(F)$, then $x_i F_X = x_{i_t} F_X$ for some $x_{i_t} \in X_F$ with $i_t \leq i$.

(3) If $xF \supset x_{i_p} F$ for some $x_{i_p} \in X_F$, then $xF_X \supseteq x_{i_{p-1}} F_X$.

(4) If $t > i_m$, then $x_t F_X \supseteq x_{i_m} F_X$.

(5) $(F_X \cup F_0)x_i \supseteq (F_X \cup F_0)x_j$ for all i, j with $1 \leq i < j \leq r$.

(1)$'$ $F_Y y_w = \{y_{w+1}, y_{w+2}, \ldots, y_s\}$ if and only if $w \in \{j_1, j_2, \ldots, j_n, j_n + 1, j_n + 2, \ldots, s - 1\}$.

(2)$'$ If $y_j \in R(F)$, then $F_Y y_j = F_Y y_{j_w}$ for some $y_{j_w} \in Y_F$ with $j_w \geq j$.

(3)$'$ If $Fy \supset Fy_{j_q}$ for some $y_{j_q} \in Y_F$, then $F_Y y \supseteq F_Y y_{j_{q-1}}$.

(4)$'$ If $w < j_n$, then $F_Y y_w \supseteq F_Y y_{j_n}$.

(5)$'$ $y_i(F_Y \cup F_0) \subseteq y_j(F_Y \cup F_0)$ for all i, j with $1 \leq i < j \leq s$.

Clearly, $F \cup F_X$ and $F \cup F_Y$ are Ferrers relations in $(X \cup Y) \times (X \cup Y)$. From (5) and $(5)'$, we see that $F_X \cup F_0$ and $F_Y \cup F_0$ are also Ferrers relations in $(X \cup Y) \times (X \cup Y)$. Then it remains to show the following two cases:

(I) If $(x, x') \in F_X$ and $(y, y') \in F_Y$, then $(y, x') \in F_0$ or $(x, y') \in F$.

(II) If $(y, x) \in F_0$ and $(x', y') \in F$, then $(x', x) \in F_X$ or $(y, y') \in F_Y$.

To see (I), suppose that $(x_i, x_j) \in F_X$ and $(y_u, y_v) \in F_Y$ but $(y_u, x_j) \notin F_0$. Then we have $i > j$, $u > v$ and $(x_j, y_u) \in \underline{F}$. Further, we know that there are $x_k \in X_F$ and $y_w \in Y_F$ with $j < k \leq i$ and $v \leq w < u$ such that $x_i F_X = x_k F_X$ and $F_Y y_v = F_Y y_w$. Since $(x_j, y_u) \in \underline{F}$, it follows that there is a pair $(x_p, y_q) \in F$ with $p \leq j$ and $q \geq u$. Now $p < k$ and so $x_i F_X = x_k F_X \supset x_p F_X$ whence $(x_i, y_q) \in F$. Similarly, since $v \leq w < u \leq q$, we have $F_Y y_v = F_Y y_w \supset F_Y y_q$. Hence $(x_i, y_v) \in F$, as desired.

To see (II), let $(y_v, x_u) \in F_0$ and $(x_{u'}, y_{v'}) \in F$. Since $u' \geq i_m$ or $v' \leq j_n$, it follows from (4) that if $u < i_m$ or $v > j_n$, then $(x_{u'}, x_u) \in F_X$ or $(y_v, y_{v'}) \in F_Y$, we are done. Now we may assume that there are $x_{i_p} \in X_F$ and $y_{j_q} \in Y_F$ such that

$$i_p \leq u < i_p - 1 \text{ and } j_q < v \leq j_{q+1}.$$

Then we have the following three cases to consider.

Case 1. $u' \leq u$ and $v' < v$.

Since $u' \leq u$, it follows that $(x_{u'}, x_u) \notin F_X$. Suppose that $(y_v, y_{v'}) \notin F_Y$. Then $y_{v'} \notin Y_F$ and hence there is an element $y_{j_w} \in Y_F$ with $j_{q+1} \leq j_w$ such that

$$F_Y y_{v'} = F_Y y_{j_w} \subseteq F_Y y_{j_{q+1}} \subset F_Y y_{j_q}. \tag{6}$$

Since $y_{v'}, y_{j_{q+1}} \in R(F)$, it follows that $F_{y_{v'}} \subseteq F y_{j_{q+1}}$ or $F_{y_{v'}} \supset F y_{j_{q+1}}$. If $F y_{v'} \subseteq F y_{j_{q+1}}$, then $(x_{u'}, y_{j_{q+1}}) \in F$. Since $u' \leq u$ and $v \leq j_{q+1}$, we have $(x_u, y_v) \in \underline{F}$, which is a contradiction. If $F y_{v'} \supset F y_{j_{q+1}}$, then $F_Y y_{v'} \supseteq F_Y y_{j_q}$ by $(3)'$, which contradicts to (6).

Case 2. $u < u'$ and $v \leq v'$.

This case can be done by a similar method to the preceding case.

Case 3. $u < u'$ and $v' < v$.

Suppose that $(x_{u'}, x_u) \notin F_X$ and $(y_v, y_{v'}) \notin F_Y$. Then $x_{u'} \notin X_F$ and $y_{v'} \notin Y_F$ and so there is $x_w \in X_F$ with $w \leq i_p$ such that

$$x_{u'} F_X = x_w F_X \subseteq x_{i_p} F_X \tag{7}$$

and there is $y_{w'} \in Y_F$ with $j_{q+1} \leq w'$ such that

$$F_Y y_{v'} = F_Y y_{w'} \subseteq F_Y y_{j_{q+1}}. \tag{8}$$

Since $x_{u'}, x_{i_p} \in C(F)$ and $y_{v'}, y_{j_{q+1}} \in R(F)$, it follows that

$$x_{u'} F \subseteq x_{i_p} F \text{ or } x_{u'} F \supset x_{i_p} F$$

and

$$Fy_{v'} \subseteq Fy_{j_{q+1}} \text{ or } Fy_{v'} \supset Fy_{j_{q+1}}.$$

If $x_{u'}F \supset x_{i_p}F$ or $Fy_{v'} \supset Fy_{j_{q+1}}$, then $x_{u'}F_X \supseteq x_{i_{p-1}}F_X \supset x_{i_p}F_X$ or $F_Y y_{v'} \supseteq F_Y y_{j_q} \supset F_Y y_{j_{q+1}}$ by (3) and $(3)'$, which contradicts to (7) or (8). Thus, $x_{u'}F \subseteq x_{i_p}F$ and $Fy_{v'} \subseteq Fy_{j_{q+1}}$, whence $(x_{i_p}, y_{v'}) \in F$ and so $(x_{i_p}, y_{j_{q+1}}) \in F$. Since $i_p \leq u$ and $v \leq j_{q+1}$, we have $(x_u, y_v) \in \underline{F}$, which is also a contradiction.

Consequently, H_F is a Ferrers relation in $(X \cup Y) \times (X \cup Y)$.

By the hypothesis, if $(x, y) \in X \times Y$, then $(x, y) \notin \underline{F}$ for some $F \in \mathcal{F}_1$ or $(x, y) \notin \overline{F}$ for some $F \in \mathcal{F}_2$, which implies that

$$Y \times X \subseteq \bigcup_{F \in \mathcal{F}_1} F_0 \cup \bigcup_{F \in \mathcal{F}_2} F_0'.$$

Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are a lower left cover and an upper right cover, respectively, of $\mathcal{F}$, there exist a C_1-set X_F and an R_1-set Y_F of each $F \in \mathcal{F}_1$ for some k and l such that

$$\bigcup_{F \in \mathcal{F}_1} X_F = \{x_k, x_{k+1}, \ldots, x_r\} \text{ and } \bigcup_{F \in \mathcal{F}_1} Y_F = \{y_1, y_2, \ldots, y_l\},$$

and there exist a C_2-set X_F' and an R_2-set Y_F' of each $F \in \mathcal{F}_2$ for some k' and l' such that

$$\bigcup_{F \in \mathcal{F}_2} X_F' = \{x_1, x_2, \ldots, x_{k'}\} \text{ and } \bigcup_{F \in \mathcal{F}_2} Y_F' = \{y_{l'}, y_{l'+1}, \ldots, y_s\}.$$

Hence we can see from (1) and $(1)'$ that

$$\begin{aligned} &[X \times X - \{(x_i, x_i) \mid 1 \leq i \leq r\}] \cup [Y \times Y - \{(y_j, y_j) \mid 1 \leq j \leq s\}] \\ \subseteq &\bigcup_{F \in \mathcal{F}_1} (F_X \cup F_Y) \cup \bigcup_{F \in \mathcal{F}_2} (F_X' \cup F_Y'), \end{aligned}$$

and so

$$\bigcup_{F \in \mathcal{F}_1} H_F \cup \bigcup_{F \in \mathcal{F}_2} H_F' \cup \bigcup \{F \in \mathcal{F} - (\mathcal{F}_1 \cup \mathcal{F}_2)\} = (X \cup Y) \times (X \cup Y) - I_P.$$

Thus $\dim(P) \leq \dim_I(P)$, as desired. □

4. Corollaries and examples

In this final section we have two simple corollaries and some examples as applications.

Corollary 1. *Let $P = (X \cup Y, I_P)$ be a bipartite ordered set with $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ and let $\mathcal{F}$ be a realizer of P. If there are a lower left cover $\mathcal{F}_1$ and an upper right cover $\mathcal{F}_2$ of $\mathcal{F}$ such that $\underline{F_1} \cap \overline{F_2} = \emptyset$ for some $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$, then*

$$\dim(P) = \dim_I(P).$$

Examples. A bipartite ordered set $P = (X \cup Y, I_P)$ is *t-interval irreducible* for some $t \geq 2$, if $\dim_I(P) = t$ but $\dim_I(P - \{u\}) = t - 1$ for every $u \in X \cup Y$. In [4], it is shown that P_1, P_2, P_3 and P_4 are 3-interval irreducible ordered sets (see Figure 2).

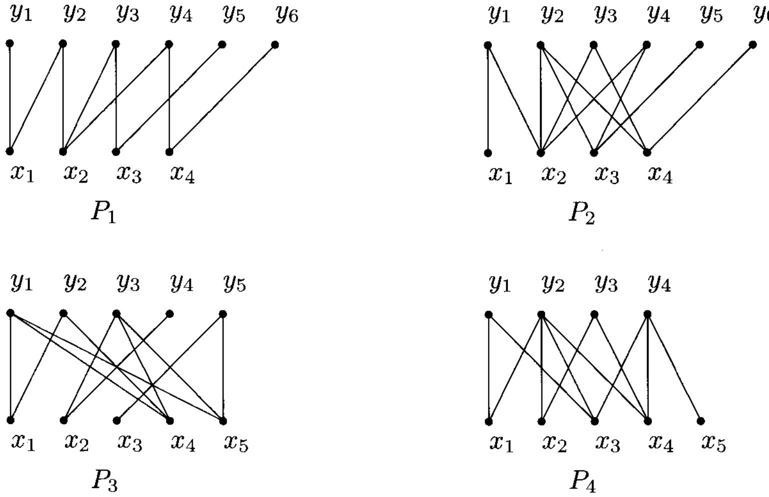


Figure 2

Consider the 3-interval irreducible ordered sets P_1 with $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. Then there is a realizer $\mathcal{F} = \{F_1, F_2, F_3\}$ of P_1 such that

$$\begin{aligned}
F_1 &= \{(x_3, y_2), (x_4, y_2), (x_4, y_3), (x_4, y_5)\}, \\
F_2 &= \{(x_2, y_1), (x_3, y_1), (x_4, y_1), (x_3, y_4)\}, \\
F_3 &= \{(x_1, y_3), (x_1, y_4), (x_1, y_5), (x_1, y_6), (x_2, y_5), (x_2, y_6), (x_3, y_6)\}.
\end{aligned}$$

Thus we see that $\{F_1, F_2\}$ is a lower left cover and $\{F_3\}$ is an upper right cover of $\mathcal{F}$. Since $\underline{F_1} \cap \overline{F_3} = \emptyset$, it follows from Corollary 1 that $\dim(P_1) = \dim_I(P_1) = 3$. Similarly, we can see that $\dim(P_i) = \dim_I(P_i) = 3$ for $i = 2, 3, 4$.

Corollary 2. *Let $P = (X, Y, I_P)$ be a bipartite ordered set with $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ and let $\mathcal{F}$ be a realizer of P with $|\mathcal{F}| \geq 2$. If there are $F_1, F_2 \in \mathcal{F}$ such that $R(F_1) \cap R(F_2) = \emptyset$ and $C(F_1) \cap C(F_2) = \emptyset$, then*

$$\dim(P) = \dim_I(P).$$

Proof. By rearranging the indices of the elements of X and Y, we can get a lower left cover $\{F_1\}$ such that $C(F_1)$ is a C_1-set and $R(F_1)$ is an R_1-set, and similarly we also can get an upper right cover $\{F_2\}$ such that $C(F_2)$ is a C_2-set and $R(F_2)$ is an R_2-set. Hence, we conclude from the main theorem that $\dim(P) = \dim_I(P)$. □

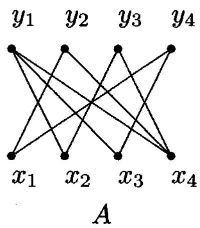


Figure 3

Example. Let A be a bipartite ordered set with $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ (see Figure 3). Then there is a realizer $F = \{F_1, F_2, F_3\}$ of A such that $F_1 = \{(x_1, y_1), (x_1, y_3)\}$, $F_2 = \{(x_2, y_2), (x_3, y_2), (x_3, y_3)\}$, and $F_3 = \{(x_2, y_4), (x_4, y_4)\}$. Thus we see that

$$R(F_1) \cap R(F_3) = \emptyset \text{ and } C(F_1) \cap C(F_3) = \emptyset.$$

It also follows from Corollary 2 that $\dim(A) = \dim_I(A) = 3$.

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