Bull. Korean Math. Soc. **46** (2009), No. 1, pp. 61–66

A NOTE ON THE GENERALIZED MYERS THEOREM

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Abstract. We provide a generalized Myers theorem under integral curvature bound and use this result to obtain a closure theorem in general relativity.

1. Introduction

One of the most important results in global Riemannian or Lorentzian geometry is Myers theorem, which says that if a complete Riemannian manifold M satisfies $\mathrm{Ric}(v,v) \geq (n-1)a > 0$ for all unit vectors v, then M is compact and $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{a}}$. We can find various kinds of generalizations and variations of this theorem in [2, 3, 4, 5, 6, 8, 9, 10, 11].

In particular, C. Sprouse [11] obtained that for a complete Riemannian manifold with (nonpositive) lower Ricci curvature bounds, one has bounded diameter provided that the Ricci curvature below some positive constant is small in a suitable integral sense. The precise statement of the result in [11] is the following

Theorem 1.1 ([11]). *Let (M,g) be a complete Riemannian manifold of dimension n satisfying $\mathrm{Ric}(v,v) \geq -a(n-1)$ for all unit vectors v and some $a > 0$. Then for any $R, \delta > 0$, there exists $\epsilon = \epsilon(n,a,R,\delta)$ such that if*

$$\sup_x \frac{1}{\mathrm{vol}(B(x,R))} \int_{B(x,R)} \max\{(n-1) - \mathrm{Ric}_-(x), 0\} d\mathrm{vol} < \epsilon(n,k,R,\delta),$$

then M is compact with $\mathrm{diam}(M) \leq \pi + \delta$.

Here, $\mathrm{Ric}_-(x)$ is the lowest eigenvalue of the Ricci tensor, $\mathrm{Ric}(x)$.

Theorem 1.1 indeed generalized some of the previous results which are related to Myers theorem.

In this paper, we use the line integral of the 'bad' part of Ricci curvature to obtain the same conclusion of the above theorem without any assumption on

Received March 12, 2008.

2000 *Mathematics Subject Classification.* 53C20.

Key words and phrases. Myers theorem, Ricci curvature.

This work was supported by 2008 research grant from Korea National University of Education and was also supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2007-331-C00027).

the pointwise lower bound of Ricci curvature. We also need not take average of the integral over metric balls.

So our main theorem is the following

Theorem 1.2. *Let (M, g) be a complete Riemannian manifold of dimension n. Then for any $\delta > 0, a > 0$, there exists an $\epsilon = \epsilon(n, a, \delta)$ satisfying the following:*

If there is a point p such that along each geodesic γ emanating from p, the Ricci curvature satisfies

$$\int_0^\infty \max\{(n-1)a - \mathrm{Ric}(\gamma'(t), \gamma'(t)), 0\}dt < \epsilon(n, a, \delta),$$

then M is compact with $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{a}} + \delta$. *Here, $\epsilon(n, a, \delta)$ can be expressed explicitly in terms of n, a and δ.*

The proof of Theorem 1.2 depends basically on the Riccati equation for the mean curvature of the metric balls centered at p. Recall that in the Lorentzian case, we have a similar Riccati-type formula which is called Raychaudhuri equation in general relativity. So, by using the similar arguments, we may also prove the following

Theorem 1.3. *Let (M, g) be a globally hyperbolic spacetime. Then for any $\delta > 0, a > 0$, there exists an $\epsilon = \epsilon(n, a, \delta)$ satisfying the following:*

If there is a point p such that along each future directed timelike geodesic γ emanating from p with $l(\gamma) = \sup\{t \geq 0 : d(p, \gamma(t)) = t\}$, the Ricci curvature satisfies

$$\int_0^{l(\gamma)} \max\{(n-1)a - \mathrm{Ric}(\gamma'(t), \gamma'(t)), 0\}dt < \epsilon(n, a, \delta),$$

then we have $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{a}} + \delta$.

Remark. For the precise definition of 'global hyperbolicity' and 'diameter' of a spacetime, we refer to chapter 11 in [1]. Here, we just note that $l(\gamma)$ can be interpreted as the "largest" parameter value t so that γ is the unique maximal geodesic between $\gamma(0)$ and $\gamma(t)$. We also note that Theorem 1.3 generalizes the classical result where one assume the pointwise lower bound of Ricci curvature (see Theorem 2.3 in [9] or p. 405 in [1]).

Using Theorem 1.2, we also obtain a closure theorem analogous to that of [7] as follows.

Let M be a static spacetime with a spatial hypersurface V which is a Riemannian manifold with the induced metric from M. Then the unit tangent vectors to the future directed timelike geodesics orthogonal to V define a smooth unit timelike vector field T in a neighborhood of V. As indicated in [7], all integral curves of T represent the worldlines of the inertial observers who start from V simultaneously, since V is their common "rest space".

Let X be a vector in T_pV and extend it along the normal geodesic c through p by making it invariant under the flow generated by T (see p. 814 in [7] for details). Then, the vector field $v(X) = \nabla_T X$ and $a(X) = \nabla_T \nabla_T X$ along c is the "3-velocity" and "3-acceleration" of X , respectively. This means that $\langle v(X), X\rangle \geq 0$ (resp. ≤ 0) implies the recess (resp. approaching) of nearby inertial observers in the direction of X and that $\langle a(X), X\rangle \leq 0$ indicates a deceleration of the recess or the approaching in the direction of X. We also introduce $\Theta = \mathrm{div} T$ which measures the average rate of expansion of the normal geodesics.

We now state our closure theorem as follow.

Theorem 1.4. *Let V be a spatial hypersurface in M and assume that V is complete in the induced metric. Then for any $\delta > 0, a > 0$, there exists an $\epsilon = \epsilon(n, a, \delta)$ satisfying the following:*

If there is a point p in V such that along each geodesic γ in V emanating from p, the condition

$$\int_0^\infty \max\{(n-1)a - \mathrm{Ric}(X, X) + \langle v(X), X\rangle\Theta + \langle a(X), X\rangle, 0\}dt < \epsilon(n, a, \delta),$$

is satisfied, where $X = \gamma'(t)$, then V is compact and $\mathrm{diam}(V) \leq \frac{\pi}{\sqrt{a}} + \delta$.

2. Proof of Theorem 1.2

For a point p in M and along a geodesic $\gamma : [0, \infty) \to M$ with $\gamma(0) = p$, we first divide $[0, \infty)$ into two parts E_1 , E_2 as follows.

For any small positive ϵ ($< a^2$) (to be determined later), let

$$E_1 = \{t \in [0, \infty) : \mathrm{Ric}(\gamma'(t), \gamma'(t)) \geq (n-1)(a - \sqrt{\epsilon})\},$$
$$E_2 = \{t \in [0, \infty) : \mathrm{Ric}(\gamma'(t), \gamma'(t)) < (n-1)(a - \sqrt{\epsilon})\}.$$

Then we have

$$\begin{aligned}
\epsilon &> \int_0^\infty \max\{(n-1)a - \mathrm{Ric}(\gamma'(t), \gamma'(t)), 0\}dt \\
&> \int_{E_2} \{(n-1)a - \mathrm{Ric}(\gamma'(t), \gamma'(t))\}dt \\
&> \int_{E_2} \{(n-1)a - (n-1)(a - \sqrt{\epsilon})\}dt \\
&= \mu[E_2](n-1)\sqrt{\epsilon},
\end{aligned}$$

where μ is the usual measure on $\mathbb{R}$. Thus we obtain

$$\mu[E_2] < \frac{\sqrt{\epsilon}}{n-1}.$$

Now recall that the mean curvature function h defined by $h(x) = (\Delta r)(x)$, where $r(x) = d(p, x)$ satisfies the following inequality:

$$h' + \frac{h^2}{n-1} \leq -\mathrm{Ric}(\gamma'(t), \gamma'(t)),$$

where $h' = \frac{d}{dt}h(\gamma(t))$. So on E_1, we have

$$\frac{(\frac{h}{n-1})'}{(a-\sqrt{\epsilon})+(\frac{h}{n-1})^2} \leq -1.$$

On the other hand, on E_2 we can estimate the following:

$$\begin{aligned}
\frac{(\frac{h}{n-1})'}{(a-\sqrt{\epsilon})+(\frac{h}{n-1})^2} &\leq \frac{-\mathrm{Ric}(\gamma'(t),\gamma'(t))/(n-1)-(\frac{h}{n-1})^2}{(a-\sqrt{\epsilon})+(\frac{h}{n-1})^2} \\
&= -1+\frac{(a-\sqrt{\epsilon})-\mathrm{Ric}(\gamma'(t),\gamma'(t))/(n-1)}{(a-\sqrt{\epsilon})+(\frac{h}{n-1})^2} \\
&\leq \frac{(a-\sqrt{\epsilon})-\mathrm{Ric}(\gamma'(t),\gamma'(t))/(n-1)}{(a-\sqrt{\epsilon})}.
\end{aligned}$$

Thus, we have for $0 < r < \infty$,

$$\begin{aligned}
\int_0^r \frac{(\frac{h}{n-1})'}{(a-\sqrt{\epsilon})+(\frac{h}{n-1})^2}dt &= \int_{[0,r]\cap E_1} \frac{(\frac{h}{n-1})'}{(a-\sqrt{\epsilon})+(\frac{h}{n-1})^2}dt \\
&\quad + \int_{[0,r]\cap E_2} \frac{(\frac{h}{n-1})'}{(a-\sqrt{\epsilon})+(\frac{h}{n-1})^2}dt \\
&\leq \int_{[0,r]\cap E_1} -1dt \\
&\quad + \int_{[0,r]\cap E_2} \frac{(a-\sqrt{\epsilon})-\mathrm{Ric}(\gamma'(t),\gamma'(t))/(n-1)}{(a-\sqrt{\epsilon})}dt \\
&\leq -\mu\{[0,r]\cap E_1\} + \frac{\epsilon/(n-1)}{a-\sqrt{\epsilon}} \\
&= -r+\mu\{[0,r]\cap E_2\} + \frac{\epsilon/(n-1)}{a-\sqrt{\epsilon}} \\
&\leq -r+\frac{\sqrt{\epsilon}}{n-1}+\frac{\epsilon/(n-1)}{a-\sqrt{\epsilon}}.
\end{aligned}$$

If we let $\tau(\epsilon) = \frac{\sqrt{\epsilon}}{n-1} + \frac{\epsilon/(n-1)}{a-\sqrt{\epsilon}}$, then the above inequality can be rewritten as follow:

$$\int_0^r \frac{(\frac{h}{n-1})'}{(a-\sqrt{\epsilon})+(\frac{h}{n-1})^2}dt \leq -r+\tau(\epsilon).$$

The integral of the left hand side can be computed explicitly and we therefore have

$$\frac{1}{a(\epsilon)}\arctan\left(\frac{h(\gamma(r))}{(n-1)a(\epsilon)}\right) \leq -r+\tau(\epsilon),$$

where $a(\epsilon) = \sqrt{a-\sqrt{\epsilon}}$.

Now it is easy to check that taking "tan" on both sides for $\tau(\epsilon) < r < \frac{\pi}{a(\epsilon)} + \tau(\epsilon)$, we have

$$h(\gamma(r)) \leq (n-1)a(\epsilon)\cot\{a(\epsilon)(r - \tau(\epsilon))\}.$$

Since $\cot\{a(\epsilon)(r - \tau(\epsilon))\}$ goes to $-\infty$ as $r \to \frac{\pi}{a(\epsilon)} + \tau(\epsilon)$, we know that $h(\gamma(r))$ goes to $-\infty$ before $r = \frac{\pi}{a(\epsilon)} + \tau(\epsilon)$. This means that γ should have a conjugate point (with respect to $\gamma(0)$) before $\frac{\pi}{a(\epsilon)} + \tau(\epsilon)$ and so cannot be minimal beyond $\frac{\pi}{a(\epsilon)} + \tau(\epsilon)$.

Furthermore, we note that $\frac{\pi}{a(\epsilon)} + \tau(\epsilon) \downarrow \frac{\pi}{\sqrt{a}}$ as $\epsilon \to 0$. Thus for any given $\delta > 0$, we can choose an ϵ explicitly so that $\frac{\pi}{a(\epsilon)} + \tau(\epsilon) = \frac{\pi}{\sqrt{a}} + \delta$ and this completes the proof.

3. Proof of Theorem 1.3

Let $\epsilon = \epsilon(n, a, \delta)$ be the explicit constant appearing in the proof of Theorem 1.2 and suppose that $\operatorname{diam}(M) > \frac{\pi}{\sqrt{a}} + \delta$. We may then find p, q in M with $d(p,q) > \frac{\pi}{\sqrt{a}} + \delta$ by definition of $\operatorname{diam}(M)$. Since M is globally hyperbolic, there exists a maximal timelike geodesic segment γ joining p and q. We also know that $l(\gamma) > \frac{\pi}{\sqrt{a}} + \delta$ by definition of $l(\gamma)$. Now recall that we have the following Raychaudhuri equation along γ as follow (cf. p. 48 in [3]):

$$\theta' + \frac{\theta^2}{n-1} + \operatorname{Ric}(\gamma'(t), \gamma'(t)) + \operatorname{tr}(\sigma^2) = 0,$$

where θ is the *expansion tensor* and σ is the *shear tensor* along γ.

From this equation, we have a similar inequality as in Riemannian case:

$$\theta' + \frac{\theta^2}{n-1} \leq -\operatorname{Ric}(\gamma'(t), \gamma'(t)).$$

Thus, using the same arguments as in the previous section, we may obtain that θ goes to $-\infty$ before $\frac{\pi}{\sqrt{a}} + \delta$. This, in turns, means that γ should have a conjugate point (with respect to $\gamma(0)$) before $\frac{\pi}{\sqrt{a}} + \delta$ and so cannot be maximal beyond $\frac{\pi}{\sqrt{a}} + \delta$. But this contradicts the fact $l(\gamma) > \frac{\pi}{\sqrt{a}} + \delta$ and we complete our proof.

4. Proof of Theorem 1.4

By a standard computation (cf. [8]), we may find that for any unit vectors $X \in T_pV$, we have

$$\begin{aligned}\operatorname{Ric}_V(X,X) =& \operatorname{Ric}(X,X) - \langle a(X), X\rangle - \langle v(X), X\rangle\Theta \\ &+ \langle v(X), X\rangle^2 + \langle v(X), e_2\rangle^2 + \langle v(X), e_3\rangle^2,\end{aligned}$$

where Ric_V is the Ricci tensor of V in the induced metric and $\{X, e_2, e_3\}$ is an orthonormal frame in T_pV.

From this, we have the following inequality.

$$(n-1)a - \mathrm{Ric}_V(X,X) \leq (n-1)a - \mathrm{Ric}(X,X) + \langle a(X), X\rangle + \langle v(X), X\rangle\Theta.$$

Combining this inequality with Theorem 1.2 gives the desired result.

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