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GENERALIZED GOTTLIEB SUBGROUPS AND SERRE FIBRATIONS

Jae-Ryong Kim

Abstract. Let $\pi : E \to B$ be a Serre fibration with fibre F. We prove that if the inclusion map $i : F \to E$ has a left homotopy inverse r and $\pi : E \to B$ admits a cross section $\rho : B \to E$, then $G_n(E, F) \cong \pi_n(B) \oplus G_n(F)$. This is a generalization of the case of trivial fibration which has been proved by Lee and Woo in [8]. Using this result, we will prove that $\pi_n(X^A) \cong \pi_n(X) \oplus G_n(F)$ for the function space X^A from a space A to a weak H_*-space X where the evaluation map $\omega : X^A \to X$ is regarded as a fibration.

1. Introduction

D. H. Gottlieb [1, 2] introduced and studied the Gottlieb subgroups $G_n(X)$ of $\pi_n(X)$, which is defined to be the set of all elements $\alpha \in \pi_n(X)$ for which there is a representation map f of α and an affiliated map $F : X \times S^n \to X$ of type $(1_X, f)$; that is $F|_X = 1_X$, $F|_{S^n} = f$.

Let X^A be the space of all mappings from A to X with the compact open topology in the category of spaces which are homotopy equivalent to CW complexes. It is well known that if X is a (pointed) CW complex and A is a finite (pointed) CW complex, then X^A has the homotopy type of a CW complex. If $a_o \in A$ is a base point, the evaluation map $\omega : X^A \to X$ given by $\omega(f) = f(a_o)$ is continuous.

Gottlieb [2, Proposition 1-1] proved that if X is a CW complex, then

$$\omega_*(\pi_n(X^X, 1_X)) = G_n(X, x_o),$$

where x_o is the base point of X. Varadarajan [12] generalized $G_1(X)$ to $G(A, X)$ for any space A and called the maps $f : A \to X$ represented by elements of $G(A, X)$ *cyclic*. In other words, $f : A \to X$ is said to be cyclic if there exists a map $F : X \times A \to X$ such that $Fj \simeq \nabla(1 \vee f)$, i.e., the following diagram is homotopy commutative:

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$$\begin{array}{ccc} X \times A & \xrightarrow{F} & X \\ j \uparrow & \nearrow \bigtriangledown(1 \vee f) & \\ X \vee A. & & \end{array}$$

Since $j : X \vee A \to X \times A$ is a cofibration, this is equivalent to saying that we can find a map $G : X \times A \to X$ such that $Gj = \bigtriangledown(1 \vee f)$. Such a map G is said to be an *associated map* of f. The set of all homotopy classes of cyclic maps from A to X is denoted by $G(A, X)$ and called the *Gottlieb subset* of $[A, X]$.

2. The generalized Gottlieb subgroups

Kim and Woo [6] gave a generalization of the Gottlieb subgroups $G_n(X)$.

Definition. The generalized Gottlieb subgroups $G_n^f(X, A, x_o)$ are defined by

$$G_n^f(X, A, x_0) = \mathrm{Im}(\omega_* : \pi_n(X^A, f) \to \pi_n(X, x_0)),$$

where $\omega : X^A \to X$ is the evaluation map from the function space X^A to X and $f : A \to X$ is a given map. $G_n^f(X, A, x_o)$ of $\pi_n(X, x_o)$ consists of all elements $\alpha \in \pi_n(X, x_o)$ for which there is an affiliated map $F : A \times S^n \to X$ with $F|_A = f$ and $F|_{S^n} = \alpha$.

The elementary properties of the generalized Gottlieb subgroups can be found in [5], [6], and [8].

Theorem 2.1. *If the fibration $p : X^A \to X$ admits a cross section $\alpha : X \to X^A$, then $G_n^f(X, A) = \pi_n(X)$ for $n \geq 1$.*

As a corollary, we can compute the homotopy groups of some function spaces.

Corollary 2.2. *Let X be an H-space. Then*

$$\pi_n(X^{S^q}) \cong \pi_n(X) \oplus \pi_{n+q}(X), n \geq 1.$$

We will use Theorem 2.1 to find a quite similar equation about the function space X^A instead of X^{S^q} in Corollary 2.2.

3. Exact sequences of the generalized Gottlieb subgroups

In the sequel, we will use the notation $*$ to denote the base point of any spaces. All spaces are assumed connected CW complexes. In [9] and [10], Lee and Woo have introduced the subgroups $G_n^{Rel}(X, A)$ of the relative homotopy groups $\pi_n(X, A)$ which are defined by the image of $\omega_\# : \pi_n(X^A, A^A, i) \to \pi_n(X, A, *)$. Equivalently, $G_n^{Rel}(X, A)=\{\alpha \in \pi_n(X, A) \mid \exists$ map $H : (X \times I^n, A \times \partial I^n) \to (X, A)$ such that $[H|_{* \times I^n}] = \alpha$ and $H|_{X \times u}=1_X$ for $u \in J^{n-1}$ for $n \geq 2\}$. Here J^{n-1} is the union of all $n-1$ faces of I^n except for the initial face. Notice that $G_1^{Rel}(X, A)$ need not to be a group.

The inclusion map $i : A \to X$ and the evaluation map ω induce the following commutative diagram

$$\begin{array}{ccccccc}
\cdots \xrightarrow{j_*} & \pi_{n+1}(X^A, A^A) & \xrightarrow{\partial} & \pi_n(A^A) & \xrightarrow{i_*} & \pi_n(X^A) & \longrightarrow \cdots \\
& \downarrow \omega_* & & \downarrow \omega_* & & \downarrow \omega_* & \\
\cdots \xrightarrow{j_*} & G_{n+1}^{Rel}(X, A) & \xrightarrow{\partial} & G_n(A) & \xrightarrow{i_*} & G_n(X, A) & \longrightarrow \cdots \\
& \downarrow \cap & & \downarrow \cap & & \downarrow \cap & \\
\cdots \xrightarrow{j_*} & \pi_{n+1}(X, A) & \xrightarrow{\partial} & \pi_n(A) & \xrightarrow{i_*} & \pi_n(X) & \longrightarrow \cdots
\end{array}$$

where the top and the bottom rows are exact.

The middle row will be called the *G-sequence* of the CW-pair (X, A).

Lee and Woo [10] have given some conditions under which the G-sequence becomes exact. For example, if the inclusion $i : A \to X$ has a left homotopy inverse or is homotopic to a constant map, then the G-sequence is exact. Lee and Woo [8, Theorem] also proved that

Theorem 3.1. *Let $F \xrightarrow{i} F \times B \xrightarrow{p} B$ be a trivial fibration. Then*

$$G_n(F \times B, F) \cong \pi_n(B) \oplus G_n(F).$$

This result was generalized by Hirato, Kuribayashi, and Oda [4] from the viewpoint of rational homotopy theory. In [4, Theorem 1.6], the authors established the following.

Theorem 3.2. *Let $F \xrightarrow{i} X \xrightarrow{p} Y$ be a separable fibration of connected rational spaces with $\dim \oplus_{q \geq 0} H^q(F : \mathbb{Q}) < \infty$ or $\dim \oplus_{i \geq 2} \pi_i(X) \otimes \mathbb{Q} < \infty$. Suppose that F is simply-connected and $\pi_i(Y)$ acts on $H^i(F : \mathbb{Q})$ nilpotently for any i. Then the sequence*

$$0 \to G_n(F) \overset{i_\#}{\to} G_n^i(X, F) \overset{p_\#}{\to} \pi_n(Y) \to 0$$

is exact for $n > 1$.

Theorem 3.2 motivated us to consider the generalization of Theorem 3.1 to the Serre fibrations.

4. Serre fibration

Definition. A map $\pi : E \to B$ is called a Serre fibration if it has the homotopy lifting property with respect to I^n for each $n \geq 0$.

Every locally trivial bundle $\pi : E \to B$ is a Serre fibration.

Let $\pi : E \to B$ be a Serre fibration. Choose $* \in E$ and $* = \pi(*) \in B$. Let $F = \pi^{-1}(*)$ be the fiber. Thus π induces a map $\pi : (E, F) \to (B, *)$.

Theorem 4.1. *Let $\pi : E \to B$ be a Serre fibration. Assume that F, E, B are CW-complexes. Then the induced morphism $\pi_* |_{G_n^{Rel}(E,F)}$: $G_n^{Rel}(E, F) \to \pi_n(B)$ is a 1-1 correspondence (isomorphism for $n \geq 2$).*

Proof. Since the map π induces a 1-1 correspondence

$$\pi_* : \pi_n(E,F) \to \pi_n(B),$$

it is sufficient to show that $\pi_* |_{G_n^{Rel}(E,F)}: G_n^{Rel}(E,F) \to \pi_n(B)$ is onto. Let $[f] \in \pi_n(B)$. Then $f : (I^n, \partial I^n) \to (B,*)$ is a continuous map of pairs. Let $g : J^{n-1} \to E$ be the trivial map $g(x) = * \in F$ for all x. Then the diagram

$$\begin{array}{ccc} J^{n-1} & \xrightarrow{g} & E \\ \downarrow & & \downarrow \pi \\ I^n & \xrightarrow{f} & B \end{array}$$

commutes. Hence by the condition of Serre fibration [3, Proposition 11.7], we can have a lifting $\psi : I^n \to E$ with $\pi_*([\psi]) = [f]$ such that $\psi(\partial I^n) \subset F$, $\psi(J^{n-1}) = *$. Let $\varphi : (I^n, I^{n-1} \times 0) \equiv (I^n, J^{n-1})$ be the homeomorphism defined in [3, Lemma 11.6].

Define a map

$$\bar{\psi} : F \times (0,0,\ldots,0) \times I \sqcup * \times I^{n-1} \times I \to E$$

by

$$\bar{\psi}(e,(0,0,\ldots,0),t) = e,\ \bar{\psi}(*,u,t) = \psi\varphi(u,t),$$

where $(0,0,\ldots,0) \in I^{n-1}, ((0,0,\ldots,0),0) \in J^{n-1}$. Since $\varphi(\partial I^n) = \partial I^n$ and $\psi\varphi(u,0) \in \psi(J^{n-1}) = \{*\}$ for $(u,0) \in I^{n-1} \times 0$, $\bar{\psi}(* \times \partial I^n) \subset F$ and $\bar{\psi}(*,u,0) = *$.

Consequently we have that $\bar{\psi}$ is well-defined and continuous by the pasting lemma. If we consider a map $\bar{\psi}_0 : F \times I^{n-1} \times 0 \equiv F \times I^{n-1} \to E$ defined by $\bar{\psi}_0(e,u,0) = e$, then we have the following commutative diagram

$$\begin{array}{ccc} F \times I^{n-1} & \xrightarrow{\bar{\psi}_0} & E \\ \uparrow & & \uparrow \pi_E \\ F \times (0,\ldots,0) \sqcup * \times I^{n-1} & \xrightarrow{\bar{\psi}^*} & E^I \end{array}$$

where $\bar{\psi}^*$ is adjoint of $\bar{\psi}$ and $\pi_E(w) = w(0)$. Since $(F \times I^{n-1}, F \times (0,\ldots,0) \sqcup * \times I^{n-1})$ is a CW-pair, it has the absolute homotopy extension property. Hence we have an extension $\psi^E : F \times I^{n-1} \times I \to E$ whose adjoint $\psi^{E^*} : F \times I^{n-1} \to E^I$ commutes the diagram:

$$\begin{array}{cccc} F \times I^{n-1} & \xrightarrow{\bar{\psi}_0} & & E \\ \uparrow & & \searrow \psi^{E^*} & \uparrow \pi_E \\ F \times (0,\ldots,0) \sqcup * \times I^{n-1} & \xrightarrow{\bar{\psi}^*} & & E^I. \end{array}$$

Using this extension ψ^E, we can define a new map

$$\bar{\psi}^E : F \times I^{n-1} \times I \sqcup E \times I^{n-1} \times 0 \to E$$

by

$$\bar{\psi}^E|_{F \times I^{n-1} \times I} = \psi^E,\ \bar{\psi}^E|_{E \times I^{n-1} \times 0}(e,u,0) = e.$$

Then $\bar{\psi}^E|_{F\times I^{n-1}\times 0} = \bar{\psi}_0$. Hence $\bar{\psi}^E$ is also well-defined and continuous. Since $(E\times I^{n-1}, F\times I^{n-1})$ is a CW-pair, it has also the absolute homotopy extension property. Hence we get again an extension $\Psi : E \times I^n \to E$. And $\Psi(1_E \times \varphi^{-1})|_{*\times I^n} = \psi$, $\Psi(1_E \times \varphi^{-1})(F \times \partial I^n) \subset F$, and $\Psi(1_E \times \varphi^{-1})|_{E\times u} = 1_E$ for $u \in J^{n-1}$ for $n \geq 2$. This implies that $[\psi] \in G_n(E, F)$. This completes the proof. □

Consider the following commutative diagram:

$$\begin{array}{ccc} G^{Rel}_{n+1}(E,F) & \xrightarrow{\partial} & G_n(F) \\ \downarrow \pi_* & & \downarrow i \\ \pi_{n+1}(B) & \xrightarrow{d} & \pi_n(F) \end{array}$$

where ∂ is the boundary homomorphism in the homotopy sequences of the pair (E, F) and d is the connecting homomorphism in the homotopy sequence of the fibration $F \to E \to B$.

Since $\pi_* : G^{Rel}_{n+1}(E,F) \to \pi_{n+1}(B)$ is a 1-1 correspondence (isomorphism if $n \geq 1$), we have

$$d(\pi_{n+1}(B)) = i\partial\pi_*^{-1}(\pi_{n+1}(B)) = i\partial(G^{Rel}_{n+1}(E,F)) \subseteq i(G_n(F)) = G_n(F).$$

Thus we have the following sequence which is a chain complex:

$$.. \to G_n(F) \xrightarrow{i_*} G_n(E,F) \xrightarrow{\pi_*} \pi_n(B) \xrightarrow{d} G_{n-1}(F) \to \cdots$$
$$\cdots \to G_0(F) \to G_0(E,F) \to \pi_0(B).$$

This sequence will be called *G-sequence of the (Serre) fibration.*

Corollary 4.2. *Let $\pi : E \to B$ be a Serre fibration with an even-dimensional sphere $S^{2n}(n \geq 1)$ as the fibre. Assume that E, B are CW-complexes. Then the induced homomorphism $\pi_* : \pi_{2n+1}(E) \to \pi_{2n+1}(B)$ is onto.*

Proof. In the exact sequence

$$\pi_{2n+1}(E) \xrightarrow{\pi_*} \pi_{2n+1}(B) \xrightarrow{d} \pi_{2n}(S^{2n})$$

we have $d(\pi_{2n+1}(B)) \subset G_{2n}(S^{2n})$. Since $G_{2n}(S^{2n}) = 0$, the homomorphism d is trivial and this implies our corollary. □

Let $\pi : E \to B$ be a Serre fibration. Assume that F, E, B are CW-complexes and the inclusion map $i : F \to E$ has a left homotopy inverse. Then the G-sequence of the pair (E, F) is exact [10]. That is, the sequence

$$\to G_n(F) \xrightarrow{i_*} G_n(E,F) \xrightarrow{j_*} G^{Rel}_n(E,F) \xrightarrow{\partial} G_{n-1}(F) \to$$

is exact.

Now we consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\to G_n(F) & \xrightarrow{i_*} & G_n(E,F) & \xrightarrow{j_*} & G_n^{Rel}(E,F) & \xrightarrow{\partial} & G_{n-1}(F) & \to \\
\| & & \| & & \downarrow \pi_* & & \| & \\
\to G_n(F) & \xrightarrow{i_*} & G_n(E,F) & \xrightarrow{\bar{\pi}_*} & \pi_n(B) & \xrightarrow{d} & G_{n-1}(F) & \to
\end{array}$$

where $\bar{\pi}_* = \pi_* \circ j_*$ and $d = \partial \circ \pi_*^{-1}$.

Since the G-sequence of the pair (E, F) is exact and π_* is an isomorphism, the G-sequence of the Serre fibration is also exact. Therefore we get the following Theorem.

Theorem 4.3. *Let $\pi : E \to B$ be a Serre fibration. Assume that F, E, B are CW-complexes and the inclusion map $i : F \to E$ has a left homotopy inverse r. Then the G-sequence of the Serre fibration is exact. Moreover we can derive a monomorphism $\phi : G_n(E, F) \to G_n(F) \oplus \pi_n(B)$.*

Proof. Let $\alpha \in G_n(E, F)$, then there exists a homotopy $H : F \times I^n \to E$ such that

$$[H\mid_{*\times I^n}] = \alpha \quad \text{and } H\mid_{F\times u} = i \quad \text{for } u \in \partial I^n.$$

If we define $\bar{H} = r \circ H : F \times I^n \to F$, then we have

$$[\bar{H}\mid_{*\times I^n}] = r_*(\alpha) \quad \text{and } \bar{H}\mid_{F\times u} = 1_F \quad \text{for } u \in \partial I^n.$$

Therefore $r_*(\alpha) \in G_n(F)$ and we can derive a homomorphism $\phi : G_n(E, F) \to G_n(F) \oplus \pi_n(B)$ defined by $\phi(\alpha) = (r_*(\alpha),\ \bar{\pi}_*(\alpha))$. Now we show that ϕ is a monomorphism. Suppose $\phi(\alpha) = (r_*(\alpha),\ \bar{\pi}_*(\alpha)) = 0$. Then $\alpha \in \mathrm{Ker}\bar{\pi}_* = \mathrm{Im} i_*$. By the exactness, there is a $\delta \in G_n(F)$ such that $i_*(\delta) = \alpha$. Thus $\delta = r_* i_*(\delta) = r_*(\alpha) = 0$. Hence $\alpha = i_*(\delta) = 0$. □

The following Theorem 4.4 is one of the generalizations of the Theorem 3.1.

Theorem 4.4. *Let $\pi : E \to B$ be a Serre fibration. Assume that F, E, B are CW-complexes, the inclusion map $i : F \to E$ has a left homotopy inverse r and π admits a cross section $\rho : B \to E$. Then $G_n(E, F) \cong G_n(F) \oplus \pi_n(B)$.*

Proof. Define $\mathcal{I} : G_n(F) \oplus \pi_n(B) \to G_n(E, F)$ by $\mathcal{I}(\delta, \beta) = i_*(\delta) + \rho_*(\beta)$. Then $\mathcal{I}$ is well defined. In fact, since $\bar{\pi}_*$ is an epimorphism, there is an $\alpha \in G_n(E, F)$ such that $\bar{\pi}_*(\alpha) = \beta$. Then $\rho_*(\beta) - \alpha \in \mathrm{Ker}\bar{\pi}_* = \mathrm{Im} i_*$. Hence $\rho_*(\beta) = \alpha + i_*(\gamma)$ for some $\gamma \in G_n(F)$. This implies $i_*(\delta) + \rho_*(\beta) \in G_n(E, F)$. Clearly $\mathcal{I}$ is a homomorphism since i_* and ρ_* are homomorphisms. Suppose $\mathcal{I}(\delta, \beta) = 0$. Then $0 = \bar{\pi}_*(\mathcal{I}(\delta, \beta)) = \bar{\pi}_*(i_*(\delta)) + \bar{\pi}_*(\rho_*(\beta)) = \bar{\pi}_* \rho_*(\beta) = \beta$. Hence $\beta = 0$ and $i_*(\delta) = 0$. But this means $\delta = 0$ because we have exactness at $G_n(F)$. We now show that $\mathcal{I}$ is onto. Let $\alpha \in G_n(E, F)$. Then $\bar{\pi}_*(\alpha - \rho_* \bar{\pi}_*(\alpha)) = 0$. By exactness, there is $\delta \in G_n(F)$ with $i_*(\delta) = \alpha - \rho_* \bar{\pi}_*(\alpha)$. Thus $\alpha = i_*(\delta) + \rho_*(\bar{\pi}_*(\alpha)) = \mathcal{I}(\delta, \bar{\pi}_*(\alpha))$. This completes the proof. □

Definition. We will call a topological space X an *H_*-space* ([7, 13]) if the following conditions are satisfied:

(i) A continuous multiplication $x \cdot y$ is defined for each pair of elements in X.
(ii) There is a fixed element $*$ in X satisfying $x \cdot * = x$ for all $x \in X$. We shall call such element $*$ the right identity.
(iii) To each $x \in X$, there is an right inverse $x^{-1} \in X$ defined continuously by x such that $x \cdot x^{-1} = *$.
(iv) For each pair of elements x, x' in X, we have $x^{-1} \cdot (x \cdot x') = x'$.

According to the definition of H_*-space, the right identity $*$ and left inverse x^{-1} of x are unique. Moreover the right inverse x^{-1} of x is the left inverse of x. An H_*-space need not to be an H-space.

Example 1. Let X^A be the space of all mappings from A to X with the compact open topology in the category of spaces which are homotopy equivalent to CW complexes. If $* \in A$ is a base point, the evaluation map $\omega : X^A \to X$ given by $\omega(f) = f(*)$ is continuous. Consequently $\omega : X^A \to X$ is a fibration with fibre $F = \omega^{-1}(*) = \{g \in X^A | g(*) = *\}$ over the right identity $* \in X$.

Theorem 4.5. *Let X be an H_*-space. Then the function space X^A is homeomorphic to $X \times F$.*

Proof. Let $g \in X^A$. Then the map $x \cdot g$ defined by $(x \cdot g)(a) = x \cdot g(a)$ is continuous. Hence $x \cdot g \in X^A$. Clearly $g = x \cdot (x^{-1} \cdot g) = x^{-1} \cdot (x \cdot g)$ for any $x \in X$. We shall define two maps $\phi : X^A \to X \times F$ and $\psi : X \times F \to X^A$ as follows:

$$\phi(g) = (g(*), \quad g(*)^{-1} \cdot g), \quad g \in X^A,$$
$$\psi(x, \; f) = x \cdot f, \quad x \in X, f \in F.$$

Now the continuity of these maps follows from the compact open topology on the function space X^A [7]. Moreover we can easily find that ϕ is the inverse map of ψ. □

This theorem resembles the theorem of Koh [7], where he dealt function space from the n-sphere S^n to an H_*-space X.

Moreover Koh [7] has given a condition which provides that the function space X^A is homeomorphic to the product space $X \times F$. Let $X = S^r$ and $A = S^p$. Then the arc components of F are elements of the pth homotopy group of X. Denote the arc component ${X^A}_\alpha$ of X^A which contains $\alpha = F_\alpha \in \pi_p(X)$. Then ${X^A}_\alpha$ is also a fibration over X. He proved that

Theorem 4.6. *X^A_α is homeomorphic to $X \times F_\alpha$ if $r = 1, 3$ or 7. Conversely, if ${X^A}_{i_r}$ and $X \times F_{i_r}$ have the same homotopy type, then $r = 1, 3$ or 7, where i_r is represented by the identity map $S^r \to S^r$.*

Definition. We call a space X a *weak* H_*-space if it satisfies the conditions (i), (ii), (iii) in the definition of H_*-space and the following condition (iv)$'$ instead of (iv):

(iv)$'$ The right inverse $*^{-1}$ of the right identity $*$ is the right identity itself.

If X is a (weak) H_*-space, then we can easily find out that the function space X^A and its subspace F are (weak) H_*-spaces. Above theorem tells us that function space X^A from A to a weak H_*-space may not be homeomorphic to the product space $X \times F$.

Example 2. Let $X = \{e, x, y, z\}$ be a system with multiplication defined by

$$\begin{array}{llll} e \cdot e = e, & e \cdot x = y, & e \cdot y = z, & e \cdot z = x, \\ x \cdot e = x, & x \cdot x = y, & x \cdot y = z, & x \cdot z = e, \\ y \cdot e = y, & y \cdot x = z, & y \cdot y = e, & y \cdot z = x, \\ z \cdot e = z, & z \cdot x = e, & z \cdot y = x, & z \cdot z = y. \end{array}$$

This system X satisfies the conditions of a weak H_*-space. But the fact that $e^{-1} \cdot (e \cdot y) = e^{-1} \cdot z = e \cdot z = x \neq y$ implies X is not an H_*-space.

Theorem 4.7. *Let X be a weak H_*-space. Assume that X, A are CW-complexes. Then $\pi_n(X^A) \cong G_n(X^A, F) \cong G_n(F) \oplus \pi_n(X)$.*

Proof. Let X be a weak H_*-space with right identity $*$ as base point. If we define a map $r : X^A \to F$ by $r(g) : A \to X$, $r(g)(a) = g(a) \cdot g(*)^{-1}$, then r is well defined and continuous. If we define $\rho : X \to X^A$ by $\rho(x) : A \to X$, $\rho(x)(a) = x$ for any $a \in A$, then ρ becomes a cross section and it holds $r \circ i(f) = f$. Hence the fibration $\omega : X^A \to X$ satisfies the assumptions of Theorem 4.4. Thus we have $G_n(X^A, F) \cong G_n(F) \oplus \pi_n(X)$. Consider the evaluation map $\bar{\omega} : (X^A)^F \to X^A$ given by $\bar{\omega}(g) = g(*)$ where the function space X^A is regarded as a weak H_*-space. The trivial map $* : A \to * \in X$ is the right identity of this weak H_*-space X^A. Then $\bar{\omega} : (X^A)^F \to X^A$ is also a fibration with fibre $\bar{F} = \bar{\omega}^{-1}(*)$ over the right identity. By virtue of the weak H_*-space X^A, we have the following continuous maps $\bar{r} : (X^A)^F \to \bar{F}$ and $\bar{\rho} : X^A \to (X^A)^F$ such that $\bar{r}\bar{i} = 1_{\bar{F}}$ and $\bar{\omega}\bar{\rho} = 1_{X^A}$. Hence the fibration $\bar{\omega} : (X^A)^F \to X^A$ satisfies the assumption of Theorem 2.1. Therefore we have $\pi_n(X^A) \cong G_n(X^A, F)$. This completes the proof. □

This theorem tells us that if the fibre F is not a G-space, then the function space X^A and the product space $X \times F$ are not homotopy equivalent.

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Department of Mathematics
Kookmin University
Seoul 136-702, Korea
E-mail address: kimjr@kookmin.ac.kr