A FIFTH ORDER NUMERICAL METHOD FOR SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE EQUATIONS WITH NEGATIVE SHIFT

P. PRAMOD CHAKRAVARTHY*, K. PHANEENDRA AND Y.N. REDDY

ABSTRACT. In this paper, a fifth order numerical method is presented for solving singularly perturbed differential-difference equations with negative shift. In recent papers the term negative shift has been using for delay. Similar boundary value problems are associated with expected first exit time problem of the membrane, potential in models for neuron and in variational problems in control theory. In the numerical treatment for such type of boundary value problems, first we use Taylor approximation to tackle terms containing small shifts which converts it to a boundary value problem for singularly perturbed differential equation. The two point boundary value problem is transformed into general first order ordinary differential equation system. A discrete approximation of a fifth order compact difference scheme is presented for the first order system and is solved using the boundary conditions. Several numerical examples are solved and compared with exact solution. It is observed that present method approximates the exact solution very well.

AMS Mathematics Subject Classification : 65L10
Key Words : Singular perturbation problems, differential-difference equations, negative shift, boundary layer, fifth order compact difference scheme.

1. Introduction

The boundary value problems for singularly perturbed differential-difference equations arise in various practical problems in biomechanics and physics such as in variation problems in control theory and depolarization in Stein's model. The depolarization in Stein's model [12] is continuous time, continuous state space, Markov process whose sample paths have discontinuities of first kind. Lange and Miura [6-10] gave an asymptotic approach in study of class of boundary-value problems for linear second-order differential-difference equations in which the highest order derivative is multiplied by small parameter. In [2] M.K.Kadalbajoo

and K.K.Sharma presented a numerical method to solve boundary value problems for singularly perturbed differential-difference equations of mixed type, i.e., containing both terms having a negative shift and terms having a positive shift. In [3] they presented a numerical method to solve boundary value problems for singularly perturbed differential-difference equations with negative shift. In [4], they described a numerical approach based on finite difference method to solve a mathematical model arising from a model of neuronal variability. The mathematical modeling of the determination of the expected time for generation of action potentials in nerve cells by random synaptic inputs in dendrites includes a general boundary value problem for singularly perturbed differential-difference equation with small shifts. In [5] Kailash C.Patidar and Kapil K.Sharma presented non-standard finite difference methods for second order, linear, singularly perturbed differential-difference equations. The non-standard finite difference methods are ε-uniformly convergent.

The objective of this paper is to determine the fifth order numerical method to the boundary value problems for singularly perturbed differential-difference equations with negative shift. In this method, we approximate the shifted term by Taylor series and apply a difference scheme, provided shifts are of $O(\varepsilon)$. The two point boundary value problem is transformed into general first order ordinary differential equation system. A discrete approximation of a fifth order compact scheme is presented for the first order system and is solved using the boundary conditions. Several numerical examples are solved and compared with exact solution. It is observed that the present method approximates the exact solution very well.

2. Fifth order numerical method

We consider the boundary value problem for a singularly perturbed differential-difference equation, which contains only negative shift in the differentiated term

$$\varepsilon y''(x) + a(x)y'(x - \delta) + b(x)y(x) = f(x) \tag{1}$$

on (0,1), under the boundary conditions

$$y(x) = \phi(x) \ on - \delta \leq x \leq 0, \quad y(1) = \gamma, \tag{2}$$

where ϵ is a small parameter, $0 < \epsilon << 1$ and δ is also a small shifting parameter, $0 < \delta << 1$, $a(x), b(x), f(x), \delta(\epsilon)$ and $\phi(x)$ are smooth functions and γ is a constant. Now there are two cases according to the sign of $a(x)$. If $a(x) \geq M > 0$ throughout the interval [0, 1], where M is a positive constant, then boundary layer will be in the neighborhood of 0, i.e., on the left side of the interval [0, 1]. If $a(x) \leq M < 0$ throughout the interval [0, 1], then boundary layer will be in the neighborhood of 1, i.e., on the right side of the interval [0, 1].

Since the solution $y(x)$ of boundary value problem (1) and (2) is sufficiently differentiable, so we expand the retarded term $y'(x-\delta)$ by Talor series, we obtain

$$y'(x - \delta) \approx y'(x) - \delta y''(x) \tag{3}$$

$$y(0) = \phi_0, \ y(1) = \gamma$$

using (3) in (1),we obtain

$$(\epsilon - \delta a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \tag{4}$$

Equation (4) is a second order singular perturbation problem.

We solve (4) subject to the boundary conditions $y(0) = \phi_0, \ y(1) = \gamma$ by using fifth order numerical method described by Pramod Chakravarthy et. al [11].

To describe the fifth order method, we consider the second order differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = r(x) \tag{5}$$

with boundary conditions $y(0) = \alpha$ and $y(1) = \beta$

The first order linear system corresponding to the above BVP is

$$Y' = A(x)Y + R(x), \ x\epsilon[0,1] \tag{6}$$

with the boundary conditions $B_1 Y(0) + B_2 Y(1) = D$, where A, B_1 and B_2 are 2x2 matrices. Y, R, D are two dimensional vectors.

Now we divide the interval $[0, 1]$ into N equal parts with constant mesh length H. Let $0 = x_0, x_1, x_2, \ldots \ldots x_N = 1$ be the mesh points. Again we divide each subinterval $[x_i, \ x_{i+1}]$ into four equal smaller sub intervals. Let $t_1, t_2, \ldots \ldots, t_5$ are the grids in the subinterval $[x_i, \ x_{i+1}]$ and corresponding values of the variables and its derivatives are Y_1, Y_2, Y_3, Y_4, Y_5 and $Y_1', Y_2', Y_3', Y_4', Y_5'$.

By considering Taylor's expansions of Y_1, Y_2, Y_3, Y_4, Y_5 at the fractional grid t_3 have (Ref. Dianyun Peng [1]), we have

$$\frac{h^{n+1}}{(n+1)} Y_3^{(n+1)} = \sum_{j=1}^{5} a_j^n Y_j + a_6^n Y_3' + O(h^6 Y_3^{(6)}), n = 1, 2, 3, 4 \tag{7}$$

where $h = \frac{x_{i+1}-x_i}{4}$ and the coefficients a_j^n are given by:

$$a_1^1 = \frac{-1}{24}, a_2^1 = \frac{2}{3}, a_3^1 = \frac{-5}{4}, a_4^1 = \frac{2}{3}, a_5^1 = \frac{-1}{24}, a_6^1 = 0$$

$$a_1^2 = \frac{1}{48}, a_2^2 = \frac{-2}{3}, a_3^2 = 0, a_4^2 = \frac{2}{3}, a_5^2 = \frac{-1}{48}, a_6^2 = \frac{-5}{4}$$

$$a_1^3 = \frac{1}{24}, a_2^3 = \frac{-1}{6}, a_3^3 = \frac{1}{4}, a_4^3 = \frac{-1}{6}, a_5^3 = \frac{1}{24}, a_6^3 = 0$$

$$a_1^4 = \frac{-1}{48}, a_2^4 = \frac{1}{6}, a_3^4 = 0, a_4^4 = \frac{-1}{6}, a_5^4 = \frac{1}{48}, a_6^4 = \frac{1}{4} \tag{8}$$

By taking the Taylor's series expansions of $Y_1', Y_2', Y_3', Y_4', Y_5'$ at the grid point t_3 and substituting (7), we get

$$Y_k' = \frac{1}{h} \sum_{j=1}^{5} b_j^k Y_j + b_6^k Y_3' + O(h^5 Y_3^{(6)}) \text{ for } k = 1, 2, 4, 5 \tag{9}$$

where

$$b_j^1 = -4a_j^1 + 12a_j^2 - 32a_j^3 + 80a_j^4 + Sgn(j-6)$$

$$b_j^2 = -2a_j^1 + 3a_j^2 - 4a_j^3 + 5a_j^4 + Sgn(j-6)$$

$$b_j^4 = 2a_j^1 + 3a_j^2 + 4a_j^3 + 5a_j^4 + Sgn(j-6)$$

$$b_j^5 = 4a_j^1 + 12a_j^2 + 32a_j^3 + 80a_j^4 + Sgn(j-6)$$

$$Sgn(x) = \left\{ \begin{array}{l} 1, x \geq 0 \\ 0, x < 0 \end{array} \right\}.$$

The variable Y and its derivative Y' at grids $t_1, t_2, \ldots t_5$ subject to equations

$$Y_j' = A_j Y_j + R_j, j = 1, 2, 3, 4, 5 \tag{10}$$

where A_j and R_j are values of A and R at grids t_j. Substituting (10) in (9), we get six linear algebraic equations with respect to five unknown variables Y_1, Y_2, Y_3, Y_4, Y_5.

By eliminating Y_2, Y_3, Y_4 from the above equations a relation between Y_1 and Y_5 can be obtained as follows:

$$\frac{1}{h} S_i Y_i + \frac{1}{h} T_i Y_{i+1} = F_i \text{ for } i = 1, 2, \ldots N - 1 \tag{11}$$

where S_i and T_i are 2x2 matrices and F_i is a two dimensional vector.

By assuming

$$c_1 = b_2^1 b_4^5 - b_2^5 b_4^1$$

$$W_1 = ((b_5^5 b_4^1 - b_5^1 b_4^5)I - hb_4^1 A_5)/c_1,$$

$$W_2 = ((b_3^5 b_4^1 - b_3^1 b_4^5)I + h(b_6^5 b_4^1 - b_6^1 b_4^5)A_3)/c_1,$$

$$W_3 = ((b_1^5 b_4^1 - b_4^5 b_1^1)I + hb_4^5 A_1)/c_1,$$

$$G_1 = (b_4^5 R_1 - b_4^1 R_5 + (b_6^5 b_4^1 - b_6^1 b_4^5)R_3)/c_1,$$

$$W_4 = ((b_1^5 b_2^1 - b_1^1 b_2^5)I + hb_2^5 A_1)/c_2,$$

$$W_5 = ((b_2^1 b_3^5 - b_2^5 b_3^1)I + h(b_6^5 b_2^1 - b_6^1 b_2^5)A_3)/c_2,$$

$$W_6 = ((b_2^1 b_5^5 - b_2^5 b_5^1)I - hb_2^1 A_5)/c_2,$$

$$G_2 = (b_2^5 R_1 - b_2^1 R_5 + (b_6^5 b_2^1 - b_6^1 b_2^5)R_3)/c_2,$$

$$W_7 = b_1^2 I + (b_2^2 - hA_2)W_3 + b_4^2 W_4,$$

$$W_8 = b_3^2 I + b_4^2 W_5 + hb_6^2 A_3 + (b_2^2 I - hA_2)W_2,$$

$$W_9 = b_5^2 I + b_4^2 W_6 + (b_2^2 I - hA_2)W_1,$$

$$G_3 = R_2 - b_6^2 R_3 - (b_2^2 I - hA_2)G_1 - b_4^2 G_2,$$

$$W_{10} = b_2^4 W_3 + (b_4^4 - hA_4)W_4 + b_1^4 I,$$

$$W_{11} = b_3^4 I + b_2^4 W_2 + hb_6^4 A_3 + (b_4^4 - hA_4)W_5,$$

$$W_{12} = b_5^4 I + b_2^4 W_1 + (b_4^4 - hA_4)W_6,$$
$$G_4 = R_4 - b_6^4 R_3 - b_2^4 G_1 - (b_4^4 - hA_4)G_2$$

We get

$$S_i = W_{11}W_7 - W_8W_{10}, T_i = W_{11}W_9 - W_8W_{12}, F_i = G_3W_{11} - W_8G_4$$

The formula (11) is the fifth order compact difference scheme of equation (6) in the ith subinterval. Solving the above system together with the boundary conditions $y(0) = \alpha$ and $y(1) = \beta$ we get the solution.

3. Numerical Examples

Layer on the left side: If $a(x) \geq M > 0$ throughout the interval $[0,1]$, where M is some positive constant, Then the boundary layer will be in the neighborhood of 0, i.e., on the left side of the interval $[0, 1]$. To demonstrate the efficiency of the method, we consider some numerical experiments.

Example 3.1. Consider the $a(x) = 1$, $b(x) = -1$, $f(x) = 0$, $\phi(x) = 1$ and $\gamma = 1$. The singular perturbed delay differential equation is

$$\varepsilon y''(x) + y'(x - \delta) - y(x) = 0; \ x \epsilon[0, 1]$$

with $y(0) = 1$ and $y(1) = 1$.

The exact solution is given by $y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{(e^{m_2} - e^{m_1})}$ Where

$m_1 = (-1 - \sqrt{1 + 4(\varepsilon - \delta)})/2(\varepsilon - \delta)$ and $m_2 = (-1 + \sqrt{1 + (4\varepsilon - \delta)})/2(\varepsilon - \delta)$

The numerical results are given in tables 1(a), 1(b), 1(c), 1(d) and 1(e) for different choices ε and δ.

Example 3.2. Now consider an example of the BVP with variable coefficients. Consider equation (1) with $a(x) = e^{-0.5x}$, $b(x) = -1$, $f(x) = 0$, $\phi(x) = 1$ and $\gamma = 1$.

The variable coefficient singular perturbed delay differential equation is

$$\varepsilon y''(x) + e^{-0.5x} y'(x - \delta) - y(x) = 0; \ x\epsilon[0, 1] \text{ with } y(0) = 1 \text{ and } y(1) = 1.$$

for which exact solution is not known.

The numerical results are given in tables 2(a), 2(b) for different values of ε and δ.

Remark. We have considered numerical results for several test examples to show the effect of small shifts on boundary layer solution of the problem. From the numerical experiments presented here, we observe as δ increases, the thickness of the boundary layer decreases and maximum error decreases as the grid size h decreases, which shows the convergence to the computed solution.

Layer on the right side: If $a(x) \leq M < 0$ throughout the interval $[0,1]$, then the boundary layer will be in the neighborhood of 1, i.e., on the right side of the

interval $[0, 1]$. To demonstrate the efficiency of the method, we consider some numerical experiments.

Example 3.3. Consider equation (1) with $a(x) = -1$, $b(x) = -1$, $f(x) = 0$, $\phi(x) = 1$ and $\gamma = -1$. The singular perturbed delay differential equation is

$$\varepsilon y''(x) - y'(x - \delta) - y(x) = 0; \ x\epsilon[0, 1]$$

$$\text{with } y(0) = 1 \text{ and } y(1) = -1.$$

The exact solution is given by $y(x) = \frac{[(e^{m_2}+1)e^{m_1 x}-(1+e^{m_1})e^{m_2 x}]}{(e^{m_2}-e^{m_1})}$ where

$m_1 = (1 - \sqrt{1 + 4(\varepsilon + \delta)})/2(\varepsilon + \delta)$ and $m_2 = (1 + \sqrt{1 + (4\varepsilon + \delta)})/2(\varepsilon + \delta)$

The numerical results are given in tables 3(a), 3(b), 3(c) and 3(d) for different choices of ε and δ.

Example 3.4. Now we consider an example of the BVP with variable coefficients. Consider equation (1) with $a(x) = -e^x$, $b(x) = -x$, $f(x) = 0$, $\phi(x) = 1$ and $\gamma = 1$.

The variable coefficient singularly perturbed delay differential equation is
$\varepsilon y''(x) - e^x y'(x - \delta) - xy(x) = 0$; $x\epsilon[0, 1]$ with $y(0) = 0$, and $y(1) = 1$, for which exact solution is not known.

The numerical results are given in table 4(a), 4(b) for different choices of ε and δ.

Remark. We have considered several numerical experiments to show the effect of small shifts on boundary layer solution. From the numerical experiments presented here, we observe that as δ increases, the thickness of the right boundary layer increases. As the grid size h decreases, the maximum error decreases, which shows the convergence to the computed solution.

Table 1(a) Numerical Results of Example 3.1, $\varepsilon = 0.1, \delta = 0.01$ and N=100

X	Approximate solution	Exact solution
0	1.000000000000000	1.000000000000000
0.01	0.93533760733629	0.93533760733569
0.02	0.87845885453881	0.87845885453774
0.05	0.74622776641336	0.74622776641149
0.10	0.61656559127633	0.61656559127428
0.20	0.53207801010147	0.53207801010024
0.30	0.54028252516007	0.54028252515951
0.40	0.57955600542291	0.57955600542269
0.50	0.63171565754760	0.63171565754752
0.60	0.69164811960337	0.69164811960334
0.70	0.75819626296802	0.75819626296801
0.80	0.83142699534590	0.83142699534590
0.90	0.91181469770927	0.91181469770927
1.00	1.00000000000000	1.00000000000000

Least square error = 8.179682163245023e-012
Maximum error = 2.088107464714994e-012

4. Conclusion

We have described a fifth order numerical method for solving boundary value problems for singularly perturbed differential-difference equation with small shifts. Here we have discussed both the cases, when boundary layer is on the left side and when boundary layer is on the right side of the underlying interval. From the numerical experiments considered in the paper, we observe that the small shift affects both the boundary layer solutions in similar fashion but reversely, i.e., as δ increases the thickness of the left boundary layer decreases while that of the right boundary layer increases. This method does not depend on the asymptotic expansion as well as on the matching of coefficients. Thus we have devised an alternative technique of solving boundary value problems for singularly perturbed delay differential equations, which is easily implemented on computer and is also practical.

Table 1(b) Numerical Results of Example 3.1, $\epsilon = 0.1, \delta = 0.08$ and N=100

x	Approximate Solution	Exact solution
0	1.000000000000000	1.000000000000000
0.01	0.75408821248241	0.75408820194106
0.02	0.60790350674732	0.60790349408482
0.05	0.44272012697889	0.44272012012022
0.10	0.41748799161620	0.41748799054409
0.20	0.45632108431666	0.45632108430355
0.30	0.50331795240175	0.50331795240163
0.40	0.55518312397366	0.55518312397366
0.50	0.61239299322917	0.61239299322917
0.60	0.67549816005444	0.67549816005444
0.70	0.74510611533090	0.74510611533090
0.80	0.82188695089678	0.82188695089678
0.90	0.90657980944690	0.90657980944690
1.00	1.00000000000000	1.00000000000000

Least square error = 2.405803586280801e-008
Maximum error = 1.266250471143593e-008

Table 1(c) Numerical Results of Example 3.1, $\varepsilon = 0.01, \delta = 0.001$ and N=100

x	Approximate Solution	Exact solution
0	1.00000000000000	1.00000000000000
0.01	0.57981739052165	0.57981589273995
0.02	0.44538818375023	0.44538720735391
0.05	0.39231676539590	0.39231668086627
0.10	0.40982631559573	0.40982631497375
0.20	0.45251844452165	0.45251844452164
0.30	0.49966824483544	0.49966824483544
0.40	0.55173078132630	0.55173078132630
0.50	0.60921793251678	0.60921793251678
0.60	0.67269491183332	0.67269491183332
0.70	0.74278582466705	0.74278582466705
0.80	0.82017980457538	0.82017980457538
0.90	0.90563778884021	0.90563778884021
1.00	1.00000000000000	1.00000000000000

Least square error = 1.864418701257692e-006
Maximum error = 1.497781700332546e-006

Table 1(d) Numerical Results of Example 3.1, $\varepsilon = 0.01, \delta = 0.008$ and N=100

X	Approximate Solution	Exact solution
0	1.00000000000000	1.00000000000000
0.01	0.37947809549342	0.37652221318501
0.02	0.37612587063888	0.37607259487377
0.05	0.38747359985959	0.38747359974888
0.10	0.40729922618261	0.40729922618262
0.20	0.45004559992807	0.45004559992808
0.30	0.49727823917792	0.49727823917793
0.40	0.54946798102107	0.54946798102108
0.50	0.60713507726878	0.60713507726879
0.60	0.67085438056860	0.67085438056860
0.70	0.74126107480500	0.74126107480501
0.80	0.81905700691014	0.81905700691014
0.90	0.90501768320301	0.90501768320301
1.00	1.00000000000000	1.00000000000000

Least square error = 2.956362472803127e-003
Maximum error = 2.955882308403768e-003

Table 1(e) Numerical Results of Example 3.1, $\varepsilon = 0.01, \delta = 0.008$ and N=1000

X	Approximate Solution	Exact solution
0	1.000000000000000	1.000000000000000
0.001	0.75155464424333	0.75155463473756
0.002	0.60116064093887	0.60116062941928
0.005	0.42202642323194	0.42202641682515
0.010	0.37652221423168	0.37652221423168
0.100	0.40729922618275	0.40729922618262
0.200	0.45004559992821	0.45004559992808
0.300	0.49727823917805	0.49727823917793
0.400	0.54946798102119	0.54946798102108
0.500	0.60713507726889	0.60713507726879
0.600	0.67085438056869	0.67085438056860
0.700	0.74126107480509	0.74126107480501
0.800	0.81905700691020	0.81905700691014
0.900	0.90501768320304	0.90501768320301
1.000	1.00000000000000	1.00000000000000

Least square error = 2.207710596547204e-008
Maximum error = 1.151959416745285e-008

Table 2(a) Numerical Results of Example 3.2, $\varepsilon = 0.1$, and N=100

x	Approximate Solution; $\delta = 0$	Approximate Solution; $\delta = 0.03$	Approximate Solution; $\delta = 0.08$
0	1.00000000000000	1.00000000000000	1.00000000000000
0.01	0.93704982052527	0.91161146840726	0.73708838608371
0.02	0.88095406306475	0.83612445319956	0.57861648271836
0.05	0.74690959132383	0.66969757495947	0.38827922023551
0.10	0.60641129699105	0.52137367952998	0.34308121177832
0.20	0.49466658403196	0.43747849380685	0.37068595105221
0.30	0.48262803914204	0.44854243455544	0.41266453862454
0.40	0.51037065418540	0.48844568405741	0.46175103340312
0.50	0.55789825916983	0.54206594598835	0.51904660508508
0.60	0.61893964968797	0.60656078885841	0.58616261868148
0.70	0.69227494576921	0.68247497103046	0.66508326507048
0.80	0.77878226043080	0.77158045051582	0.75824503391702
0.90	0.88043494198740	0.87636979723337	0.86864839173032
1.00	1.00000000000000	1.00000000000000	1.00000000000000

Table 2(b) Numerical Results of Example 3.2, $\varepsilon = 0.01$, and N=1000

x	Approximate solution; $\delta = 0$	Approximate solution; $\delta = 0.008$
0	1.00000000000000	1.00000000000000
0.001	0.93160114087772	0.71659489436671
0.002	0.86979688313026	0.54477707837492
0.005	0.71789433033423	0.33880072595677
0.010	0.54799743765769	0.28470464470229
0.050	0.30100727254446	0.29098707906127
0.100	0.31154907887158	0.30643411316279
0.200	0.34635378201876	0.34115441548927
0.300	0.38713375183003	0.38185994187319
0.400	0.43511323814654	0.42984114084066
0.500	0.49187582752134	0.48671931566182
0.600	0.55941806174273	0.55454434933653
0.700	0.64027320415474	0.63592499725349
0.800	0.73767678409435	0.73420381171598
0.900	0.85578956287997	0.85369367409840
1.000	1.00000000000000	1.00000000000000

Table 3(a) Numerical Results of Example 3.3, $\varepsilon = 0.01, \delta = 0.007$ and N=100

x	Approximate Solution	Exact solution
0	1.00000000000000	1.00000000000000
0.10	0.90632668834811	0.90632668834811
0.20	0.82142806601206	0.82142806601206
0.30	0.74448217878491	0.74448217878490
0.40	0.67474406763231	0.67474406763231
0.50	0.61153855629959	0.61153855629958
0.60	0.55425371447229	0.55425371447229
0.70	0.50233491142056	0.50233491142061
0.80	0.45527078268309	0.45527078269823
0.90	0.40915991523934	0.40915991823444
0.97	0.15674311908056	0.15674317819657
0.98	-0.03402568555112	-0.03402561387860
0.99	-0.37783620393470	-0.37783613876263
1.00	-1.00000000000000	-1.00000000000000

Least square error = 2.624647110010439e-009
Maximum error = 1.042466474254766e-009

Table 3(b) Numerical Results of Example 3.3, $\varepsilon = 0.01, \delta = 0.025$ and N=100

X	Approximate Solution	Exact solution
0	1.00000000000000	1.00000000000000
0.10	0.90780519660681	0.90780519660681
0.20	0.82411027491793	0.82411027491793
0.30	0.74813158877369	0.74813158877369
0.40	0.67915771762727	0.67915771762727
0.50	0.61654239883634	0.61654239883636
0.80	0.45750360389700	0.45750360394282
0.90	0.34677344960943	0.34677345004892
0.95	0.08382770350674	0.08382770446910
0.96	-0.02830692371227	-0.02830692267780
0.97	-0.17761947652789	-0.17761947548542
0.98	-0.37689906062553	-0.37689905969172
0.99	-0.64332958128641	-0.64332958065906
1.00	-1.00000000000000	-1.00000000000000

Least square error = 2.624647110010439e-009

Maximum error = 1.042466474254766e-009

Table 3(c) Numerical Results of Example 3.3, $\varepsilon = 0.001, \delta = 0.0007$ and N=1000

X	Approximate Solution	Exact solution
0	1.0000000000000	1.0000000000000
0.100	0.90499073260046	0.90499073260046
0.200	0.81900822609272	0.81900822609272
0.300	0.74119485453746	0.74119485453745
0.500	0.60704468300378	0.60704468300376
0.700	0.49717458898595	0.49717458898593
0.800	0.44993839551673	0.44993839551671
0.900	0.40719007818376	0.40719007818374
0.990	0.36842267600774	0.36842267894261
0.995	0.29844515309722	0.29844518102688
0.996	0.24036655498587	0.24036659526283
0.997	0.13597067236771	0.13597072682033
0.998	-0.05191733207893	-0.05191726664118
0.999	-0.39030905134213	-0.39030899236287
1.000	-1.00000000000000	-1.00000000000000

Least square error = 1.171028077548662e-007

Maximum error = 6.543775547818154e-008

Table 3(d) Numerical Results of Example 3.3, $\varepsilon = 0.001, \delta = 0.0025$ and N=1000

X	Approximate Solution	Exact solution
0	1.00000000000000	1.0000000000000
0.100	0.90515196815104	0.90515196815105
0.200	0.81930008544771	0.81930008544772
0.300	0.74159108484932	0.74159108484932
0.500	0.60758563918431	0.60758563918432
0.700	0.49779496610051	0.49779496610052
0.800	0.45058009330156	0.45058009330157
0.900	0.40784345826111	0.40784345826111
0.990	0.29500272894226	0.29500272932812
0.995	0.04451443209256	0.044514432090164
0.996	-0.06426244481058	-0.06426244394840
0.997	-0.20903441451370	-0.20903441365235
0.998	-0.40175337222480	-0.40175337145993
0.999	-0.65833970770268	-0.65833970719326
1.000	-1.00000000000000	-1.00000000000000

Least square error $= 2.209589813446237e-009$

Maximum error $= 8.621774266748261e-010$

Table 4(a) Numerical Results of Example 3.4, $\varepsilon = 0.1$, and N=100

x	Approximate solution; $\delta = 0$	Approximate solution; $\delta = 0.008$
0	1.00000000000000	1.00000000000000
0.01	0.99920314305062	0.99881185785093
0.10	0.98888269558388	0.98549173811356
0.30	0.95005036233944	0.94335642465092
0.50	0.89782975581026	0.89361471853488
0.70	0.84043249907747	0.85353050475096
0.90	0.80212705329284	0.88363945461353
0.91	0.80466506435304	0.89026779259037
0.92	0.80875047942108	0.89775369919748
0.93	0.81484740738283	0.90617743780868
0.94	0.82356629236103	0.91562707688116
0.95	0.83571212026914	0.92619927507559
0.96	0.85234916512936	0.93800014727553
0.97	0.87488817396376	0.95114622003515
0.98	0.90520407196102	0.96576548590128
0.99	0.94579528936361	0.98199856707403
1.00	1.00000000000000	1.00000000000000

Table 4(b) Numerical Results of Example 3.4, $\varepsilon = 0.01$, and N=1000

x	Approximate solution; $\delta = 0$	Approximate solution; $\delta = 0.008$
0	1.00000000000000	1.00000000000000
0.001	0.99998980890339	0.99998233567297
0.010	0.99985473614977	0.99978071478265
0.100	0.99450235296924	0.99383396557864
0.300	0.96196060389908	0.96041896366450
0.500	0.91166082582984	0.90976066283490
0.700	0.85367025803677	0.85176071685649
0.900	0.79462166136873	0.79295425247490
0.990	0.78457431560877	0.86737397649755
0.991	0.78914581433962	0.87602334229006
0.992	0.79522047032765	0.88547237933664
0.993	0.80326596422511	0.89579296316360
0.995	0.82791720553639	0.91936929365837
0.996	0.84638762196155	0.93280362821805
0.997	0.87069816360978	0.94746802588880
0.998	0.90267632481818	0.96347323088163
0.999	0.94472419901100	0.98093999780019
1.000	1.00000000000000	1.00000000000000

REFERENCES

1. Dianyun Peng , *High order Numerical method for two point boundary value problems*, Vol. 120(1995), 253-259.

2. M.K.Kadalbajoo and K.K.Sharma, *Numerical Analysis of boundary value Problems for singularly perturbed diffrential-difference equations with small shifts of mixed type*, Journal of Optimization theory and Applications, Vol. 115(2002), No.1, 145-163.

3. M.K.Kadalbajoo and K.K.Sharma, *Numerical Analysis of singularly perturbed delay diffrential equations with layer behaviour*, Applied Mathematics and Computation, Vol. 157(2004), 11-28.

4. M.K.Kadalbajoo and K.K.Sharma, *Numerical Treatment of Mathematical Model of a neuronal variability*, Journal of Mathematical Analysis and Applications, Vol. 307(2005), 606-627.

5. Kailash C. Patidar and Kapil K. Sharma, *ε-Uniformly convergent non-standard finite difference methods for Singularly Perturbed differential difference equations with small delay*, Applied Mathematics and Computation, Vol. 175(2006), issue 1, 864-890.

6. C.G.Lange and R.M.Miura, *Singular Perturbation Analysis of boundary-value problems for differential difference equations. V. Small shifts with layer behaviour*, SIAM Journal on Applied Mathematics, Vol. 54(1994), 249-272.

7. C.G.Lange and R.M.Miura, *Singular Perturbation Analysis of boundary-value problems for differential difference equations*, SIAM Journal on Applied Mathematics, Vol. 42(1982), 502-531.

8. C.G.Lange and R.M.Miura, *Singular Perturbation Analysis of boundary-value problems for differential difference equations II. Rapid oscillations and resonances*, SIAM Journal on Applied Mathematics, Vol. 45(1985), 687-707.

9. C.G.Lange and R.M.Miura, *Singular Perturbation Analysis of boundary-value problems for differential difference equations III. Turning point problems*, SIAM Journal on Applied Mathematics, Vol. 45(1985), 708-734.

10. C.G.Lange and R.M.Miura, *Singular Perturbation Analysis of boundary-value problems for differential difference equations VI. Small shifts with rapid oscillations*, SIAM Journal on Applied Mathematics, Vol. 54(1994), 273-283.

11. P. Pramod Chakravarthy, K. Phaneendra and Y.N. Reddy, *A Fifth Order Method for Singular Perturbation Problems*, Journal of Applied Mathematics & Informatics, to appear.

12. R.B.Stein, *A theoretical analysis of neuronal variability*, Biophysical Journal, Vol. 5(1965), 173-194.

Pramod Chakravarthy received his Ph.D. at National Institute of Technology, Warangal (India) in 2004. His research interests are Numerical Analysis, Singular Perturbation Problems. He is working in the Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur, Maharastra-440011, India.

Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur, Maharastra-440011, India.
e-mail:pramodpodila@yahoo.co.in

K. Phaneendra is persuing Ph.D. in Numerical Analysis in National Insitute of Technology, Warangal (India). He is working in the Department of Mathematics, Kakatiya Institute of Technology & Science, Warangal, Andhrapradesh-506015, India.
e-mail:kollojuphaneendra@yahoo.co.in

Y.N. Reddy is Professor & Head of the Deaprtment of Mathematics at National Institute of Technology, Warangal, Andhrapradesh-506004, India. His area of research is Numerical Analysis, Singular Perturbation Problems and Singular Boundary Value Problems.
e-mail:ynreddy@nitw.ac.in