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REMARKS ON KERNEL FOR WAVELET EXPANSIONS IN MULTIDIMENSIONS

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ABSTRACT. In expansion of function by special basis functions, properties of expansion kernel are very important. In the Fourier series, the series are expressed by the convolution with Dirichlet kernel. We investigate some of properties of kernel in wavelet expansions both in one and higher dimensions.

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1. Introduction

For a function f, the Fourier series is given by $\sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx)$, where a_n and b_n are Fourier coefficients. Its partial sum $(S_n f)(x) = \sum_{k=0}^{n}(a_k \cos kx + b_k \sin kx)$ is represented by the convolution with $D_n(x)$

$$(S_n f)(x) = (f * D_n)(x) = \int_0^{2\pi} f(t) D_n(x-t)dt,$$

where $D_n(x) = \dfrac{\sin\left(n+\frac{1}{2}\right)t}{2\pi \sin \frac{1}{2}t}$ is the Dirichlet kernel. So all the convergence properties of the Fourier series are closely related with the properties of the Dirichlet kernel. One of the typical properties is $\int_0^{2\pi} D_n(t)dt = 1$.

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Wavelet theory [1,4] usually begins with a scaling function ϕ whose integer translates are Riesz basis of a subspaces V_0 of $L^2(R)$. It, in turn, becomes part of a family of nested subspaces usually referred as a multiresoultion analysis (MRA). The other spaces are obtained by dilation by factors of two: $f(t) \in V_m$ if and only if $f(2^{-m}t) \in V_0$. An MRA will have the following properties:

(i) $\cdots \subset V_m - 1 \subset V_m \subset \cdots \subset L^2(R)$,

(ii) $\overline{\cup V_m} = L^2(R)$,

(iii) $\cap V_m = 0$.

Implicit in this definition is the existence of a dilation equation which relates the bases of two successive subspaces, i.e., it expands ϕ in terms of $\phi(2\cdot -n)$. This dilation equation is modified to construct a mother wavelet ψ which is then used to form basis (usually orthogonal) of $L^2(R)$ composed of dilations and translations $\left\{\psi(2^m t - n)\right\}$.

2. Properties of wavelet kernel in one dimension

The orthogonal projection $Q_m f$ of $f \in L^2(R)$ onto V_m is given by

$$Q_m f = \sum_{n\in Z}(f, \phi_{mn})\phi_{mn}, \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(R)$.

The quantity $Q_m f$ is partial sum of the wavelet expansion associated with the given scaling function. It is possible to interchange sum and integral in (2.1), we can write $Q_m f$ as an integral operator

$$(Q_m f)(x) = \int_{-\infty}^{\infty} 2^m q(2^m x, 2^m y) f(y) dy, \tag{2.2}$$

where the kernel $q(x,y)$ is defined by

$$q(x,y) = \sum_{n\in Z} \phi(x-n)\phi(y-n) \quad \text{for} \quad x, y \in R. \tag{2.3}$$

We collect some properties of q in the following propositions (compare with [5, (6.2) on p.33]).

Proposition 2.1. *Let ϕ be a continuous scaling function satisfying*

$$|\phi(x)| \leq K(1+|x|)^{-\beta} \quad \textit{for} \quad x \in R \tag{2.4}$$

with constant K and $\beta > 1$. Then

(a) the kernel $q(x,y)$ is continuous and satisfies the estimate

$$|q(x,y)| \leq L(1+|x-y|)^{-\beta} \quad \textit{for all} \quad x, y \in R,$$

where L is a constant;

(*b*) *every function in* V_m *is continuous;*

(*c*) *equation (2.2) holds for every* $f \in L^2(R)$, $m \in Z$ *and* $x \in R$.

Proof. (a) By assumption, the defining series (2.3) for converges locally uniformly. Thus continuity of ϕ implies continuity of q. For the proof of the estimate, we assume without loss of generality that $|x+y| \leq 1$ and $x \geq y$ because $q(x+1, y+1) = q(x,y) = q(y,x)$. If $n \geq 0$ then

$$|x-n| = \left|\frac{x+y}{2} + \frac{x-y}{2} - n\right| \geq \left|\frac{x-y}{2} - n\right| - \frac{1}{2},$$

and

$$|y-n| = \frac{x+y}{2} - \frac{x-y}{2} - n| \geq \frac{x-y}{2} + n - \frac{1}{2} \geq \frac{x-y}{2} - \frac{1}{2}.$$

Thus

$$(1+|x-n|)(1+|y-n|) \geq \frac{1}{4}(1+x-y)(1+|x-y-2n|).$$

The same result holds for $n < 0$ if we replace n by $-n$ on the right hand side. Using (2.3) and (2.4) we have shown that

$$|q(x,y)| \leq K^2 4^\beta (1+x-y)^{-\beta} \\ \times \left(\sum_{n \geq 0} (1+|x-y-2n|)^{-\beta} + \sum_{n<0} (1+|x-y+2n|^{-\beta} \right).$$

Since $\beta > 1$ the sum $\sum_{n \in Z}(1+|t-2n|)^{-\beta}$ is bounded by a constant independent of$t \in R$. Therefore the above inequality implies (a).

(b) This follows from the fact that the expansion $f = \sum_n (f, \phi_{mn})\phi_{mn}$ of every $f \in V_m$ is locally uniformly convergent.

(c) Without of loss of generality we assume that $m = 0$. Since

$$\sum_n \int_{-\infty}^{\infty} |\phi(x-n)\phi(y-n)f(y)|dy \\ \leq \left(\int_{-\infty}^{\infty} |f(y)|^2 dy\right)^{1/2} \sum_n |\phi(x-n)| < \infty$$

we are permitted to interchange sum and integral in (2.1) yielding (2.2) for almost all x . This equation then holds for all x because both sides represent continuous functions of x . This is true for the left hand side by part (b), and it holds for the right hand side by a well known theorem on the continuous parameter dependence of Lebesgue integrals in combination with the estimate given in part (a). □

Proposition 2.2. *If ϕ is a continuous scaling function satisfying (2.4) with $\beta > 1$, then*

$$\int_{-\infty}^{\infty} q(x,y)dy = 1 \quad \textit{for all} \quad x \in R.$$

Proof. Let $d = k/2^n, \quad k \in Z, \quad n \in N$ be a dyadic number number. Then $\phi(2^{-m}x + d)$ belongs to V_0 for all $m \geq n$. In fact, the second property of scaling functions shows that $\phi(2^{-m}x)$ is in V_0, and then $\phi(2^{-m}x+d)$ is in V_0, too. Thus, by Proposition 2.1(c) and definition of Q_0,

$$\phi(2^{-m}x + d) = \int_{-\infty}^{\infty} q(x,y)\phi(2^{-m}y + d)dy.$$

Since $q(x,y)$ is integrable with respect to y by proposition 2.1(a) and ϕ is bounded, we obtain for $m \to \infty$ by the Lebesgue dominated convergence theorem

$$\phi(d) = \int_{-\infty}^{\infty} q(x,y)\phi(d)dy.$$

Since there is a dyadic number d such that $\phi(d) \neq 0$ this yields the desired result. □

3. Properties of wavelet kernel in higher dimensions

Scaling functions in higher dimensions satisfy the same sorts of properties as in one dimension. They generate a multi-resolution analysis $\{V_m\}_{m\in Z}$ of closed subspaces of $L^2(R^d)$ such that

(a) $V_m \subset V_{m+1}, m \in Z$;
(b) $\overline{\bigcup_{j\in Z} V_j} = L^2(R^d)$.

The scaling function $\phi(\mathbf{x}), \mathbf{x} \in \mathbf{R}^d$, has the property that

$$\Big\{\phi(\{\mathbf{x} - \mathbf{n}\})\Big\}_{\mathbf{n}\in\mathbf{z}^d}$$

is an orthonormal basis of V_0. It satisfies a dilation equation

$$\phi(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{c_k}\phi(A\mathbf{x} - \mathbf{k}), \{\mathbf{c_k}\}_{\mathbf{k}\in\mathbf{Z}^d} \in l^2(\mathbf{Z}^d), \tag{3.1}$$

where A is the dilation matrix. In one dimension this is usually taken to be 2 and is often, in particular in the separable case, taken to be $2\mathbf{I}$ in higher dimensions. But some interesting examples use other dilation matrices [2]. In any case, as observed in [3], A must be non-singular, map vectors of integers into integers, and must have eigenvalue all of whose moduli exceed 1.

The multi-resolution spaces $\{V_m\}$ are also related to the dilations based on A:

(c) $f(\mathbf{x}) \in V_m \Longleftrightarrow f(A\mathbf{x}) \in V_{m+1}$.

This ensures that $\left\{|\det A|^{m/2}\phi(A^m\mathbf{x} - \mathbf{n})\right\}_{\mathbf{n}\in\mathbf{Z}^d}$ is an orthonormal basis of V_m. Here $detA$ is the determinant of the matrix A.

The projection onto the subspace V_m of a function $f \in L^2(\mathbf{R}^d)$ is given by the expression

$$f_m(\mathbf{x}) = (P_m f)(\mathbf{x}) = \int_{\mathbf{R}^d} q_m(\mathbf{x}, \mathbf{t}) f(\mathbf{t}) d\mathbf{t}, \tag{3.2}$$

where $q_m(\mathbf{x}, \mathbf{t})$ is the reproducing kernel of V_m. It is given by

$$q_m(\mathbf{x}, \mathbf{t}) = |\det A|^m q_m(A^m\mathbf{x}, A^m\mathbf{t}),$$

where q is given in turn by the scaling function series

$$q(\mathbf{x}, \mathbf{t}) = \sum_{\mathbf{n}\in\mathbf{Z}^d} \phi(\mathbf{x} - \mathbf{n})\overline{\phi(\mathbf{x} - \mathbf{n})}, \tag{3.3}$$

where $\overline{\phi(\mathbf{x} - \mathbf{n})}$ is the complex conjugate of $\phi(\mathbf{x} - \mathbf{n})$. In this paper we only consider real-valued scaling functions, and so we do not need to use the overline. For the moment, we shall assume that ϕ is l-regular in the sense of Meyer[6, p. 21], although many of our calculations are valid for weaker conditions. This guarantees that the series (3.3) converges uniformly on $\mathbf{R}^d$ in terms of $\mathbf{x}$ (or $\mathbf{t}$) when $\mathbf{t}$ (or $\mathbf{x}$) is fixed. We also have the property [6, p. 33] that

$$\left|\partial_{\mathbf{x}}^{\alpha}\partial_{\mathbf{t}}^{\beta} q(\mathbf{x}, \mathbf{t})\right| \leq \mathbf{C}_k(1 + |\mathbf{x} - \mathbf{t}|)^{-k} \quad \text{for every} \quad k \in \mathbf{N}, \tag{3.4}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d), \beta = (\beta_1, \beta_2, \cdots, \beta_d), |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \leq l$, and $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_d) \leq l$. It also follows [6, p. 38] that

$$P(\mathbf{x}) = \int_{\mathbf{R}^d} q_m(\mathbf{x}, \mathbf{t}) P(\mathbf{t}) d\mathbf{t}$$

for every polynomial P of degree$\leq l$.

According to [6, p. 33], l-regular multi-resolution approximation reproduces polynomials up to degree l, that is

$$\int_{\mathbf{R}^d} q(\mathbf{x},\mathbf{t})\mathbf{t}^{\beta} d\mathbf{t} = \mathbf{x}^{\beta} := x_1^{\beta_1} x_2^{\beta_2} \cdots x_d^{\beta_d} \tag{3.5}$$

for every multi-index $\beta = (\beta_1, \beta_2, \cdots, \beta_d) \in \mathbf{N}^d$ such that $|\beta| \leq l$. We shall require only that the scaling function be 0-regular unless stated otherwise.

We recall periodicity of the kernel $q(\mathbf{x},\mathbf{t})$, that is,

$$q(\mathbf{x}+\mathbf{n},\mathbf{t}+\mathbf{n}) = q(\mathbf{x},\mathbf{t}) \quad \text{for all} \quad \mathbf{x},\mathbf{t} \in \mathbf{R}^d, \mathbf{n} \in \mathbf{Z}^d. \tag{3.6}$$

We first take $\gamma = (1,0,0,\cdots,0) \in \mathbf{R}^d$ and consider

$$r(\mathbf{x}) := r_\gamma(\mathbf{x}) = H(x_1) - \int_{t_1>0} q(\mathbf{x},\mathbf{t}) d\mathbf{t}, \tag{3.7}$$

where $H(t)$ is the Heaviside function on $\mathbf{R}$, that is, $H(t) = 1$ if $t > 0$ and $H(t) = 0$ if $t \leq 0$ We collect some properties of $r(\mathbf{x})$.

Proposition 3.1. *The function $r(\mathbf{x})$ satisfies*

$$r(\mathbf{x}+\mathbf{n}') = r(\mathbf{x}) \quad \textit{for all} \quad \mathbf{x} \in \mathbf{R}^d \quad \textit{and} \quad \mathbf{n}' = (0, n_2, \cdots, n_d) \in \mathbf{Z}^d. \tag{3.8}$$

Moreover, for every $k \in \mathbf{N}$ there is a constant C such that

$$|r(\mathbf{x})| \leq C\Big(1 + |x-1|\Big)^{-k} \quad \textit{for all} \quad \mathbf{x} \in \mathbf{R}^d \tag{3.9}$$

Proof. Eq. (3.8) follows from (3.6) and (3.7). Let $x_1 \leq 0$. Then

$$r(\mathbf{x}) = -\int_{t_1>0} q(\mathbf{x},\mathbf{t}) d\mathbf{t}.$$

By (3.4), there is a constant C_1 such that

$$|r(\mathbf{x})| \leq C_1 \int_{t_1>0} (1 + |\mathbf{x}-\mathbf{t})^{-k-d} d\mathbf{t} = C_1 \int_{t_1>|x_1|} (1+|\mathbf{t}|)^{-k-d} d\mathbf{t}. \tag{3.10}$$

If S denotes the surface area of the unit sphere in $\mathbf{R}^{d-1}$ then

$$\begin{aligned}\int_{\mathbf{R}^{d-1}}(1+|\mathbf{t}|)^{-k-d}dt_2\cdots dt_d &= S\int_0^\infty s^{d-2}\left(1+\sqrt{t_1^2+s^2}\right)^{-k-d}ds\\ &\leq S\int_0^\infty \left(1+\sqrt{t_1^2+s^2}\right)^{-k-2}ds\\ &\leq S2^{(k+2)/2}\int_0^\infty (1+|t_1|+s)^{-k-2}ds\\ &= \frac{S2^{(k+2)/2}}{k+1}(1+|t_1|)^{-k-1}.\end{aligned}$$

Using this estimate in (3.10), we find a constant C_2 such that

$$|r(\mathbf{x})| \leq C_2\int_{|x_1|}^\infty (1+t_1)^{-k-1}dt_1 = \frac{C_2}{k}(1+|x_1|)^{-k}.$$

If $x_1 > 0$, we use (3.5) with $\beta = \mathbf{0}$ to write

$$r(\mathbf{x}) = 1 - \int_{t_1>0} q(\mathbf{x},\mathbf{t})d\mathbf{t} = \int_{t_1<0} q(\mathbf{x},\mathbf{t})d\mathbf{t}.$$

Now we estimate $|r(\mathbf{x})$ in a similar way to complete the proof of (3.9) □

We introduce the region

$$F := \{(x_1, x_2, \cdots, x_d) : x_1 \in \mathbf{R}, x_j \in [0,1) \quad for \quad j = 2, 3, \cdots, d\}.$$

Proposition 3.2. *For every $k \in \mathbf{N}$ there is a constant C such that*

$$\int_F |q(\mathbf{x},\mathbf{t})r(\mathbf{x})|d\mathbf{x} \leq C(1+|\mathbf{t}|)^{-k} \quad for\ all \quad \mathbf{t} \in \mathbf{R}^d.$$

Proof. Let $k \in \mathbf{N}$. Using (3.4) and (3.9), there is a constant C_1 such that, for all $\mathbf{x}, \mathbf{t} \in \mathbf{R}^d$,

$$|q(\mathbf{x},\mathbf{t})r(\mathbf{x})| \leq C_1(1+|\mathbf{x}-\mathbf{t}|)^{-k}(1+|x_1|)^{-k-2}.$$

We apply the inequality

$$(1+|\mathbf{a}|)(1+|\mathbf{b}|) \geq 1+|\mathbf{b}-\mathbf{a}| \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbf{R}^d$$

and obtain

$$|q(\mathbf{x},\mathbf{t})r(\mathbf{x})| \leq C_1\Big(1+|\mathbf{x}'-\mathbf{t}|\Big)^{-k}(1+|x_1|)^{-2},$$

where $\mathbf{x}' := (0, x_2, \cdots, x_d)$. It follows that

$$\int_F |q(\mathbf{x},\mathbf{t})r(\mathbf{x})|d\mathbf{x} \leq 2C_1 \quad \sup\Big\{(1+|\mathbf{x}'-\mathbf{t}|)^{-k} : x_2, x_3, \cdots, x_d \in [0,1)\Big\}.$$

Since $|\mathbf{t}| \leq |\mathbf{t}-\mathbf{x}'| + |\mathbf{x}'| \leq |\mathbf{t}-\mathbf{x}'| + \sqrt{d-1}$ we finally obtain

$$\int_F |q(\mathbf{x},\mathbf{t})r(\mathbf{x})|d\mathbf{x} \leq 2C_1(1+\sqrt{d-1})^k(1+|\mathbf{t}|)^{-k}.$$

□

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