J. Appl. Math. & Informatics Vol. **27***(2009), No. 1 - 2, pp. 351 - 363*
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FIXED POINT THEOREMS FOR SIX WEAKLY COMPATIBLE MAPPINGS IN D^*-METRIC SPACES

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ABSTRACT. In this paper, we give some new definitions of D^*-metric spaces and we prove a common fixed point theorem for six mappings under the condition of weakly compatible mappings in complete D^*-metric spaces. We get some improved versions of several fixed point theorems in complete D^*-metric spaces.

AMS Mathematics Subject Classification: 54E40; 54E35; 54H25
Key words and Phrases : D-metric contractive mapping; complete D^*-metric space; common fixed point theorem.

1. Introduction and preliminaries

In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point.His result is called Banach's Fixed Point Theorem or the Banach Contraction Principle. This theorem provides a technique for solving a variety of problems of applied nature in mathematical science and engineering.Many authors have extended, generalized and improved Banach's Fixed Point Theorem in different ways. In [17], Jungck introduced the notion of *compatible* mappings which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems(see, e.g.,([3, 4, 5, 6, 8, 10, 11, 19, 20, 21, 25]). Dhage[7] introduced the concept of generalized metric or D-metric spaces and claimed that D-metric convergence defines a Hausdorff topology and that D-metric is sequentially continuous in all the three variables.Many authors have taken these claims for granted and used them in proving fixed point theorems in D-metric spaces.Rhoades [17] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently, motivated by the concept of compatibility for metric space, Singh and Sharma

Received September 23, 2007. Revised June 30, 2008. Accepted July 12, 2008. *Corresponding author.

[23] introduced the concept of D-compatibility of maps in D-metric space and proved some fixed point theorems using a contractive condition. Unfortunately, almost all theorems in D-metric spaces are not valid (see [14, 15, 16]).In this paper, we introduce D^*-metric which is a probable modification of the definition of D-metric introduced by Dhage [7] and prove some basic properties in D^*-metric spaces.

In what follows (X, D^*) will denote a D^*-metric space, $\mathbb{N}$ the set of all natural numbers, and $\mathbb{R}^+$ the set of all positive real numbers.

Definition 1.1. Let X be a nonempty set. A generalized metric (or D^*-metric) on X is a function: $D^* : X^3 \longrightarrow \mathbb{R}^+$ that satisfies the following conditions for each $x, y, z, a \in X$.

(1) $D^*(x, y, z) \geq 0$,

(2) $D^*(x, y, z) = 0$ if and only if $x = y = z$,

(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$,(symmetry) where p is a permutation function,

(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called a generalized metric (or D^*-metric) space.

Immediate examples of such a function are the following :

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,

(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

(c) If $X = \mathbb{R}^n$ then we define

$$D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$$

for every $p \in \mathbb{R}^+$.

(d) If $X = \mathbb{R}^+$ then we define

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise} , \end{cases}$$

Remark 1.2. In a D^*-metric space, we prove that $D^*(x, x, y) = D^*(x, y, y)$. For

(i) $D^*(x, x, y) \leq D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y)$, and similarly

(ii) $D^*(y, y, x) \leq D^*(y, y, y) + D^*(y, x, x) = D^*(y, x, x)$.

Hence by (i),(ii) we get $D^*(x, x, y) = D^*(x, y, y)$.

Let (X, D^*) be a D^*-metric space. For $r > 0$ define

$$B_{D^*}(x, r) = \{y \in X : D^*(x, y, y) < r\}$$

Example 1.3. Let $X = \mathbb{R}$. Denote $D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in \mathbb{R}$. Thus

$$\begin{aligned} B_{D^*}(1, 2) &= \{y \in \mathbb{R} : D^*(1, y, y) < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| + |y - 1| < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| < 1\} = (0, 2) \end{aligned}$$

Definition 1.4. Let (X, D^*) be a D^*-metric space and $A \subset X$.

(1) If for every $x \in A$ there exist $r > 0$ such that $B_{D^*}(x, r) \subset A$, then subset A is called open subset of X.

(2) Subset A of X is said to be D^*-bounded if there exists $r > 0$ such that $D^*(x, y, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $D^*(x_n, x_n, x) = D^*(x, x, x_n) \to 0$ as $n \to \infty$.That is for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \Longrightarrow D^*(x, x, x_n) < \epsilon \ (*)$$

This is equivalent with, for each $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0 \Longrightarrow D^*(x, x_n, x_m) < \epsilon \ (**)$$

Indeed, if have $(*)$,then

$$D^*(x_n, x_m, x) = D^*(x_n, x, x_m) \leq D^*(x_n, x, x) + D^*(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon$$

Conversely, set $m = n$ in $(**)$ we have $D^*(x_n, x_n, x) < \epsilon$.

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exits $n_0 \in \mathbb{N}$ such that $D^*(x_n, x_n, x_m) < \epsilon$ for each $n, m \geq n_0$. The D^*-metric space (X, D^*) is said to be complete if every Cauchy sequence is convergent.

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $r > 0$ such that $B_{D^*}(x, r) \subset A$. Then τ is a topology on X (induced by the D^*-metric D^*).

Lemma 1.5. *Let (X, D^*) be a D^*-metric space. If $r > 0$, then ball $B_{D^*}(x, r)$ with center $x \in X$ and radius r is open ball.*

Proof. Let $z \in B_{D^*}(x, r)$, hence $D^*(x, z, z) < r$. If set $D^*(x, z, z) = \delta$ and $r' = r - \delta$ then we prove that $B_{D^*}(z, r') \subseteq B_{D^*}(x, r)$. Let $y \in B_{D^*}(z, r')$, by triangular inequality we have $D^*(x, y, y) = D^*(y, y, x) \leq D^*(y, y, z) + D^*(z, x, x) < r' + \delta = r$. Hence $B_{D^*}(z, r') \subseteq B_{D^*}(x, r)$. That is ball $B_{D^*}(x, r)$ is open ball. □

Definition 1.6. Let (X, D^*) be a D^*- metric space. D^* is said to be continuous function on $X^3 \times (0, \infty)$ if

$$\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z).$$

Whenever a sequence $\{(x_n, y_n, z_n)\}$ in $X^3 \times (0, \infty)$ converges to a point $(x, y, z) \in X^3 \times (0, \infty)$ i.e.

$$\lim_{n\to\infty} x_n = x, \ \lim_{n\to\infty} y_n = y, \ \lim_{n\to\infty} z_n = z$$

Lemma 1.7. *Let (X, D^*) be a D^*- metric space. Then D^* is continuous function on $X^3 \times (0, \infty)$.*

Proof. Since sequence $\{(x_n, y_n, z_n)\}$ in $X^3 \times (0, \infty)$ converges to a point $(x, y, z) \in X^3 \times (0, \infty)$ i.e.

$$\lim_{n\to\infty} x_n = x, \ \lim_{n\to\infty} y_n = y, \ \lim_{n\to\infty} z_n = z$$

for each $\epsilon > 0$ there exist $n_1 \in \mathbb{N}$ such that for every $n \geq n_1 \Longrightarrow D^*(x, x, x_n) < \frac{\epsilon}{3}$ $n_2 \in \mathbb{N}$ such that for every $n \geq n_2 \Longrightarrow D^*(y, y, y_n) < \frac{\epsilon}{3}$, and similarly there exist $n_3 \in \mathbb{N}$ such that for every $n \geq n_3 \Longrightarrow D^*(z, z, z_n) < \frac{\epsilon}{3}$. If set $n_0 = \max\{n_1, n_2, n_3\}$, then for every $n \geq n_0$ by triangular inequality we have

$$\begin{aligned} D^*(x_n, y_n, z_n) &\leq D^*(x_n, y_n, z) + D^*(z, z_n, z_n) \\ &\leq D^*(x_n, z, y) + D^*(y, y_n, y_n) + D^*(z, z_n, z_n) \\ &\leq D^*(z, y, x) + D^*(x, x_n, x_n) + D^*(y, y_n, y_n) + D^*(z, z_n, z_n) \\ &< D^*(x, y, z) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = D^*(x, y, z) + \epsilon \end{aligned}$$

Hence we have $D^*(x_n, y_n, z_n) - D^*(x, y, z) < \epsilon$

$$\begin{aligned} D^*(x, y, z) &\leq D^*(x, y, z_n) + D^*(z_n, z, z) \\ &\leq D^*(x, z_n, y_n) + D^*(y_n, y, y) + D^*(z_n, z, z) \\ &\leq D^*(z_n, y_n, x_n) + D^*(x_n, x, x) + D^*(y_n, y, y) + D^*(z_n, z, z) \\ &< D^*(x_n, y_n, z_n) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = D^*(x_n, y_n, z_n) + \epsilon \end{aligned}$$

That is, $D^*(x, y, z) - D^*(x_n, y_n, z_n) < \epsilon$. Therefore we have $|D^*(x_n, y_n, z_n) - D^*(x, y, z)| < \epsilon$, that is $\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)$. □

Lemma 1.8. *Let (X, D^*) be a D^*-metric space. If sequence $\{x_n\}$ in X converges to x,then x is unique.*

Proof. Let $x_n \longrightarrow y$ and $y \neq x$. Since $\{x_n\}$ converges to x and y, for each $\epsilon > 0$ there exist $n_1 \in \mathbb{N}$ such that for every $n \geq n_1 \Longrightarrow D^*(x, x, x_n) < \frac{\epsilon}{2}$ and $n_2 \in \mathbb{N}$ such that for every $n \geq n_2 \Longrightarrow D^*(y, y, x_n) < \frac{\epsilon}{2}$. If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by triangular inequality we have

$$D^*(x, x, y) \leq D^*(x, x, x_n) + D^*(x_n, y, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

Hence $D^*(x, x, y) = 0$ is a contradiction. So, $x = y$. □

Lemma 1.9. *Let (X, D^*) be a D^*-metric space. If sequence $\{x_n\}$ in X is converges to x ,then sequence $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $x_n \longrightarrow x$ for each $\epsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1 \Longrightarrow D^*(x_n, x_n, x) < \frac{\epsilon}{2}$ and $n_2 \in \mathbb{N}$ such that for every $m \geq n_2 \Longrightarrow D^*(x, x_m, x_m) < \frac{\epsilon}{2}$. If set $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ by triangular inequality we have

$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x) + D^*(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence sequence $\{x_n\}$ is a Cauchy sequence. □

In 1998, Jungck and Rhoades [10] introduced the following concept of weak compatibility.

Definition 1.10. Let A and S be mappings from a D^*-metric space (X, D^*) into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.11. The pair (A, S) satisfies the property (E.A) [1], if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} D^*(Ax_n, u, u) = \lim_{n\to\infty} D^*(Sx_n, u, u) = 0$$

for some $u \in X$.

Example 1.12. Let $X = \mathbb{R}$ and

$$D^*(x, y, z) = |x - y| + |x - z| + |y - z|,$$

for every $x, y, z \in X$. Define A and S by $Ax = 2x + 1$, $Sx = x + 2$. Define the sequence $\{x_n\}$ by $x_n = 1 + \frac{1}{n}$, $n = 1, 2, ...$ We have

$$\lim_{n\to\infty} D^*(Ax_n, 3, 3) = \lim_{n\to\infty} D^*(Sx_n, 3, 3) = 0$$

Then, the pair (A, S) satisfies the property (E.A). However, A and S are not weakly compatible.

The following example shows that there are some pairs of mappings which do not satisfy the property (E.A).

Example 1.13. Let $X = \mathbb{R}$ and

$$D^*(x, y, z) = |x - y| + |x - z| + |y - z|,$$

for every $x, y, z \in X$. Define A and B by $Ax = x + 1$ and $Sx = x + 2$. Assume that there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} D^*(Ax_n, u, u) = \lim_{n\to\infty} D^*(Sx_n, u, u) = 0$$

for some $u \in X$. Therefore

$$\lim_{n\to\infty} D^*(x_n + 1, u, u) = \lim_{n\to\infty} D^*(x_n + 2, u, u) = 0.$$

We conclude that $x_n \to u - 1$ and $x_n \to u - 2$ which is a contradiction. Hence, the pair (A, S) do not satisfy property (E.A).

Recently, Y. Liu et al [12] defined a common property (E.A) as follows.

Definition 1.14. The pairs (A, S) and (B, T) of a D^*-metric space (X, D^*) satisfy a common property (E.A) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that for some $u \in X$

$$\begin{aligned}\lim_{n\to\infty} D^*(Ax_n, u, u) &= \lim_{n\to\infty} D^*(Sx_n, u, u) = \lim_{n\to\infty} D^*(By_n, u, u) \\ &= \lim_{n\to\infty} D^*(Ty_n, u, u) = 0. \end{aligned} \tag{1.1}$$

If $B = A$ and $T = S$ in (1.1), we obtain the definition of property (E.A).

Example 1.15. Let $X = [1, \infty)$ and

$$D^*(x, y, z) = |x - y| + |x - z| + |y - z|,$$

for every $x, y, z \in X$. Define A, B, S, T by

$$Ax = 2 + \frac{x}{3}, Bx = 2 + \frac{x}{2}, Sx = 1 + \frac{2}{3}x, Tx = 1 + x.$$

Define sequences $\{x_n\}$ and $\{y_n\}$ by $x_n = 3 + \frac{1}{n}$, $y_n = 2 + \frac{1}{n}$, $n = 1, 2, ...$

$$\lim_{n\to\infty} D^*(Ax_n, 3, 3) = \lim_{n\to\infty} D^*(By_n, 3, 3) = \lim_{n\to\infty} D^*(Sx_n, 3, 3)$$
$$= \lim_{n\to\infty} D^*(Ty_n, 3, 3) = 0.$$

Therefore, the pairs (A, S) and (B, T) satisfy a common property (E.A)

2. Main results

Let Φ be the set of all increasing and continuous functions $\phi : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$, such that $\phi(s) < s$ for every $s \in (0, \infty)$, $\phi(0) = 0$.

Example 2.1. Let $\phi : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ defined by $\phi(s) = ks$ for every $0 < k < 1$.

Theorem 2.2. *Let S and T be self-mappings of a complete D^*- metric space (X, D^*) satisfying the following conditions:*

$$\int_0^{D^*(Tx,TSy,Sz)} \varphi(s)ds \leq \phi(\int_0^{L(x,y,z)} \varphi(s)ds), \tag{2.1}$$

where

$$L(x, y, z) = \max\{D^*(x, Sy, z), D^*(x, Sy, Tx), D^*(Tx, x, x), D^*(Tx, Sz, Sz)\},$$

for all $x, y \in X$, $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a continuous map and $\phi \in \Phi$. Then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point . Then there exist $x_1, x_2 \in X$ such that

$$Tx_0 = x_1 \text{ and } Sx_1 = x_2.$$

Inductively, construct sequence $\{x_n\}$ in X such that

$$Tx_{2n} = x_{2n+1} \text{ and } Sx_{2n+1} = x_{2n+2},$$

for $n = 0, 1, 2, \cdots$.

Now, we prove that $\{x_n\}$ is a Cauchy sequence. Let $d_m = D^*(x_m, x_m, x_{m+1})$. Replacing $x_{2n}, x_{2n-1}, x_{2n+1}$ by x, y, z respectively in (2.1), then we have

$$\begin{aligned}\int_0^{D^*(x_{2n+1},x_{2n+1},x_{2n+2})} \varphi(s)ds &= \int_0^{D^*(Tx_{2n},TSx_{2n-1},Sx_{2n+1})} \varphi(s)ds \\ &\leq \phi(\int_0^{L(x_{2n},x_{2n-1},x_{2n+1})} \varphi(s)ds)......(2.2),\end{aligned}$$

where

$$\begin{aligned}L(x_{2n}, x_{2n-1}, x_{2n+1}) &= \max\begin{pmatrix}D^*(x_{2n}, Sx_{2n-1}, x_{2n+1}), D^*(x_{2n}, Sx_{2n-1}, Tx_{2n}), \\ D^*(Tx_{2n}, x_{2n}, x_{2n}), D^*(Tx_{2n}, Sx_{2n+1}, Sx_{2n+1})\end{pmatrix} \\ &= \max\begin{pmatrix}D^*(x_{2n}, x_{2n}, x_{2n+1}), D^*(x_{2n}, x_{2n}, x_{2n+1}), \\ D^*(x_{2n+1}, x_{2n}, x_{2n}), D^*(x_{2n+1}, x_{2n+2}, x_{2n+2})\end{pmatrix}\end{aligned}$$

Hence we get $L(x_{2n}, x_{2n-1}, x_{2n+1}) = \max\{d_{2n}, d_{2n}, d_{2n}, d_{2n+1}\}$. We now prove that $d_{2n+1} \leq d_{2n}$ for every $n \in \mathbb{N}$. If $d_{2n+1} > d_{2n}$ for some $n \in \mathbb{N}$, by inequality (2.2), we have

$$\int_0^{d_{2n+1}} \varphi(s)ds \leq \phi(\int_0^{d_{2n+1}} \varphi(s)ds) < \int_0^{d_{2n+1}} \varphi(s)ds,$$

which is a contradiction. Hence $d_{2n+1} \leq d_{2n}$.

Now, replacing x, y, z by $x_{2n}, x_{2n-1}, x_{2n-1}$ respectively in (2.1), we obtain

$$\begin{aligned}\int_0^{D^*(x_{2n+1},x_{2n+1},x_{2n})} \varphi(s)ds &= \int_0^{D^*(Tx_{2n},TSx_{2n-1},Sx_{2n-1})} \varphi(s)ds \\ &\leq \phi(\int_0^{L(x_{2n},x_{2n-1},x_{2n-1})} \varphi(s)ds),.\end{aligned}$$

where

$$\begin{aligned}L(x_{2n}, x_{2n-1}, x_{2n-1}) &= \max\begin{pmatrix} D^*(x_{2n}, Sx_{2n-1}, x_{2n-1}), D^*(x_{2n}, Sx_{2n-1}, Tx_{2n}), \\ D^*(Tx_{2n}, x_{2n}, x_{2n}), D^*(Tx_{2n}, Sx_{2n-1}, Sx_{2n-1}) \end{pmatrix} \\ &= \max\begin{pmatrix} D^*(x_{2n}, x_{2n}, x_{2n-1}), D^*(x_{2n}, x_{2n}, x_{2n+1}), \\ D^*(x_{2n+1}, x_{2n}, x_{2n}), D^*(x_{2n+1}, x_{2n}, x_{2n}) \end{pmatrix}\end{aligned}$$

Hence we get

$$L(x_{2n}, x_{2n-1}, x_{2n+1}) = \max\{d_{2n-1}, d_{2n}, d_{2n}, d_{2n}\}.$$

We prove that $d_{2n} \leq d_{2n-1}$, for every $n \in \mathbb{N}$. If $d_{2n} > d_{2n-1}$ for some $n \in \mathbb{N}$, by inequality (2.2), we have

$$\int_0^{d_{2n}} \varphi(s)ds \leq \phi(\int_0^{d_{2n}} \varphi(s)ds) < \int_0^{d_{2n}} \varphi(s)ds,$$

is a contradiction. Hence $d_{2n} \leq d_{2n-1}$.

Hence for every $n \in \mathbb{N}$ we have $d_n \leq d_{n-1}$. Thus sequence $\{d_n\}$ is lower bounded and decreasing sequence, hence it is lead to 0. It follows

$$\lim_{n\to\infty} \int_0^{D^*(x_n,x_n,x_{n+1})} \varphi(s)ds = 0.$$

Therefore

$$\lim_{n\to\infty} D^*(x_n, x_n, x_{n+1}) = 0. \tag{2}$$

Now, we prove that $\{x_{2n}\}$ is Cauchy sequence. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence in X. Then there is an $\epsilon > 0$ such that for each integer k, there exist integers $2m(k)$ and $2n(k)$ with $m(k) > n(k) \geq k$ such that

$$D^*(x_{2n(k)}, x_{2m(k)}, x_{2m(k)}) \geq \epsilon \text{ and } D^*(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}) < \epsilon. \tag{3}$$

From(3) , we have

$$\begin{aligned}\epsilon &\leq D^*(x_{2n(k)}, x_{2m(k)}, x_{2m(k)}) \\ &\leq D^*(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}) + D^*(x_{2m(k)-1}, x_{2m(k)}, x_{2m(k)}) \\ &\leq \epsilon + d_{2m(k)-1}\end{aligned}$$

Letting $k \longrightarrow \infty$ and using (2), we get

$$lim_k D^*(x_{2n(k)}, x_{2m(k)}, x_{2m(k)}) = \epsilon \tag{4}$$

Similarly,using (2) and (4),we can show that

$$lim_k D^*(x_{2n(k)+1}, x_{2m(k)}, x_{2m(k)}) = lim_k D^*(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}) = \epsilon. \tag{5}$$

Replacing x, y, z by $x_{2m(k)}, x_{2n(k)+1}, x_{2m(k)}$ in (2.1), we have

$$\int_0^{D^*(x_{2m(k)}, x_{2n(k)+1}, x_{2n(k)+1})} \varphi(s)ds \leq \phi(\int_0^{L(x_{2m(k)}, x_{2n(k)+1}, x_{2m(k)})} \varphi(s)ds),$$

where $L(x_{2m(k)}, x_{2n(k)+1}, x_{2m(k)})$

$$\begin{aligned} &= \max\begin{pmatrix} D^*(x_{2m(k)}, Sx_{2n(k)+1}, x_{2m(k)}), D^*(x_{2m(k)}, Sx_{2n(k)+1}, Tx_{2m(k)}), \\ D^*(Tx_{2m(k)}, x_{2m(k)}, x_{2m(k)}), D^*(Tx_{2m(k)}, Sx_{2m(k)}, Sx_{2m(k)}) \end{pmatrix} \\ &= \max\begin{pmatrix} D^*(x_{2m(k)}, x_{2n(k)+2}, x_{2m(k)}), D^*(x_{2m(k)}, x_{2n(k)+1}, x_{2m(k)+1}), \\ D^*(x_{2m(k)+1}, x_{2m(k)}, x_{2m(k)}), D^*(x_{2m(k)+1}, x_{2m(k)+1}, x_{2m(k)+1}) \end{pmatrix} \end{aligned}$$

Making $k \to \infty$ and using (2),(4) and (5), we obtain

$$\int_0^{\epsilon} \varphi(s)ds \leq \phi(\int_0^{\epsilon} \varphi(s)ds) < \int_0^{\epsilon} \varphi(s)ds$$

which is a contradiction. This establishes the fact that $\{x_{2n}\}$ is a Cauchy sequence.

$$\begin{aligned} D^*(x_{2n+1}, x_{2m+1}, x_{2m+1}) \leq\; & D^*(x_{2n+1}, x_{2n}, x_{2n}) + D^*(x_{2n}, x_{2m}, x_{2m}) \\ & + D^*(x_{2m}, x_{2m+1}, x_{2m+1}) \end{aligned}$$

Making $n, m \to \infty$ we get $\lim_{n,m\to\infty} D^*(x_{2n+1}, x_{2m+1}, x_{2m+1}) = 0$. Similarly , we get

$$\lim_{n,m\to\infty} D^*(x_{2n+1}, x_{2m}, x_{2m}) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence, and due to the completeness of X, $\{x_n\}$ converges to some x in X. That is, $\lim_{n\to\infty} x_n = x$. Hence

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = x$$

Now we show that $Sx = x$. From the inequality (2.1), we get

$$\begin{aligned} \int_0^{D^*(Tx_{2n}, TSx_{2n+1}, Sx)} \varphi(s)ds &= \int_0^{D^*(x_{2n+1}, x_{2n+2}, Sx)} \varphi(s)ds \\ &\leq \phi(\int_0^{L(x_{2n}, x_{2n+1}, x)} \varphi(s)ds), \end{aligned}$$

where

$$\begin{aligned} L(x_{2n}, x_{2n+1}, x) &= \max\begin{pmatrix} D^*(x_{2n}, Sx_{2n+1}, x), D^*(x_{2n}, Sx_{2n+1}, Tx_{2n}), \\ D^*(Tx_{2n}, x_{2n}, x_{2n}), D^*(Tx_{2n}, Sx, Sx) \end{pmatrix} \\ &= \max\begin{pmatrix} D^*(x_{2n}, x_{2n+2}, x), D^*(x_{2n}, x_{2n+2}, x_{2n+1}), \\ D^*(x_{2n+1}, x_{2n}, x_{2n}), D^*(x_{2n+1}, Sx, Sx) \end{pmatrix} \end{aligned}$$

On making $n \longrightarrow \infty$, we get

$$\int_0^{D^*(x,x,Sx)} \varphi(s)ds \le \phi(\int_0^{D^*(x,x,Sx)} \varphi(s)ds) < \int_0^{D^*(x,x,Sx)} \varphi(s)ds,$$

which is a contradiction.Therefore, it follows that $Sx = x$. Next we prove that $Tx = x$. For this, replacing x, y, z by x_{2n}, x, x in inequality (2.1), we have

$$\begin{aligned} \int_0^{D^*(Tx_{2n},TSx,Sx)} \varphi(s)ds &= \int_0^{D^*(Tx_{2n},Tx,x)} \varphi(s)ds \\ &\le \phi(\int_0^{L(x_{2n},x,x)} \varphi(s)ds), \end{aligned}$$

where

$$\begin{aligned} L(x_{2n}, x, x) &= \max\begin{pmatrix} D^*(x_{2n}, Sx, x), D^*(x_{2n}, Sx, Tx_{2n}), \\ D^*(Tx_{2n}, x_{2n}, x_{2n}), D^*(Tx_{2n}, Sx, Sx) \end{pmatrix} \\ &= \max\begin{pmatrix} D^*(x_{2n}, x, x), D^*(x_{2n}, x, x_{2n+1}), \\ D^*(x_{2n+1}, x_{2n}, x_{2n}), D^*(x_{2n+1}, x, x) \end{pmatrix} \end{aligned}$$

As $n \longrightarrow \infty$, we have

$$\int_0^{D^*(x,Tx,x)} \varphi(s)ds \le \phi(\int_0^{D^*(x,x,Tx)} \varphi(s)ds) < \int_0^{D^*(x,x,Tx)} \varphi(s)ds,$$

which is a contradiction. So it follows that $Tx = x$. Hence ,$Tx = Sx = x$, that is, x is a common fixed of T, S . The uniqueness of x follows from the inequality (2.1). □

Theorem 2.3. *Let (X, D^*) be a D^*-metric space, and A, B, C, R, S and T be self-mappings of X satisfying the following conditions:*

$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq R(X) \text{ and } C(X) \subseteq S(X)$$

$$\int_0^{D^*(Ax,By,Cz)} \varphi(s)ds \le \phi(\int_0^{L(x,y,z)} \varphi(s)ds), \tag{2.3}$$

where $L(x,y,z) = \max\{D^*(Sx,Ty,Rz), D^*(Ax,Ty,Rz), D^*(Sx,By,Rz), D^*(Sx,Ty,Cz)\}$, for all $x,y,z \in X$, $\varphi : \mathbb{R} \to \mathbb{R}_+$ is a continuous mapping and $\phi \in \Phi$. Suppose that two of the pairs (A,S), (C,R) and (B,T) satisfy the common property (E.A); pairs (A,S) , (C,R) and (B,T) are weakly compatible, and one of $R(X), T(X)$ and $S(X)$ is a closed subset of X. Then A, B, C, R, S and T have a unique common fixed point in X.

Proof. Suppose that (A,S) and (B,T) satisfy a common property (E.A). Then, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for some $u \in X$.

$$\begin{aligned} \lim_{n\to\infty} D^*(Ax_n, u, u) &= \lim_{n\to\infty} D^*(Sx_n, u, u) \\ &= \lim_{n\to\infty} D^*(By_n, u, u) = \lim_{n\to\infty} D^*(Ty_n, u, u) = 0. \end{aligned}$$

As $B(X) \subseteq R(X)$, there exists a sequence $\{z_n\}$ in X such that $By_n = Rz_n$. Thus $\lim_{n\to\infty} Rz_n = u$. Now we prove that $\lim_{n\to\infty} Cz_n = u$. Replacing x_n, y_n, z_n by x, y, z respectively in (2.3) ,we obtain

$$\int_0^{D^*(Ax_n,By_n,Cz_n)} \varphi(s)ds \le \phi(\int_0^{L(x_n,y_n,z_n)} \varphi(s)ds),$$

where

$$L(x_n, y_n, z_n) = \max\left\{ \begin{array}{l} D^*(Sx_n, Ty_n, Rz_n), D^*(Ax_n, Ty_n, Rz_n), \\ D^*(Sx_n, By_n, Rz_n), D^*(Sx_n, Ty_n, Cz_n) \end{array} \right\}.$$

Hence $\lim_{n\to\infty} L(x_n, y_n, z_n) = \max\left\{0, 0, 0, D^*(u, u, \lim_{n\to\infty} Cz_n)\right\}$. On making $n \to \infty$ in above inequality , we get

$$\begin{aligned} \int_0^{D^*(u,u,\lim_{n\to\infty} Cz_n)} \varphi(s)ds &\le \phi(\int_0^{D^*(u,u,\lim_{n\to\infty} Cz_n)} \varphi(s)ds) \\ &< \int_0^{D^*(u,u,\lim_{n\to\infty} Cz_n)} \varphi(s)ds, \end{aligned}$$

which is a contradiction. Hence $\lim_{n\to\infty} Cz_n = u$. Assume that $S(X)$ is a closed subset of X. Then, there exists $v \in X$ such that $Sv = u$.

If $u \neq Av$, then using (2.3) we obtain

$$\int_0^{D^*(Av,By_n,Cz_n)} \varphi(s)ds \le \phi(\int_0^{L(v,y_n,z_n)} \varphi(s)ds),$$

where $L(v, y_n, z_n) = \max\left\{D^*(Sv, Ty_n, Rz_n), D^*(Av, Ty_n, Rz_n), D^*(Sv, By_n, Rz_n), D^*(Sv, Ty_n, Cz_n)\right\}$. As $n \to \infty$, it follows that

$$\int_0^{D^*(Av,u,u)} \varphi(s)ds \le \phi(\int_0^{D^*(Av,u,u)} \varphi(s)ds) < \int_0^{D^*(Av,u,u)} \varphi(s)ds,$$

which is a contradiction. Therefore, $Av = Sv = u$. Since $A(X) \subseteq T(X)$, there exists $w \in X$ such that $Av = Tw = u$. If $u \neq Bw$, using (2.3) we obtain

$$\int_0^{D^*(Av,Bw,Cz_n)} \varphi(s)ds \le \phi(\int_0^{L(v,w,z_n)} \varphi(s)ds),$$

where $L(v, w, z_n) = \max\{D^*(Sv, Tw, Rz_n), D^*(Av, Tw, Rz_n), D^*(Sv, Bw, Rz_n), D^*(Sv, Tw, Cz_n)\}$. As $n \to \infty$, it follows that

$$\int_0^{D^*(u,Bw,u)} \varphi(s)ds \le \phi(\int_0^{D^*(u,Bw,u)} \varphi(s)ds) < \int_0^{D^*(u,Bw,u)} \varphi(s)ds,$$

which is a contradiction. Therefore, $Bw = u$. Since $B(X) \subseteq R(X)$, there exists $e \in X$ such that $Re = Bw = u$. If $e \neq Re$, using (2.3) we obtain

$$\int_0^{D^*(Av,Bw,Ce)} \varphi(s)ds \leq \phi(\int_0^{L(v,w,e)} \varphi(s)ds),$$

where $L(v,w,e) = \max\{D^*(Sv,Tw,Re), D^*(Av,Tw,Re), D^*(Sv,Bw,Re), D^*(Sv,Tw,Ce)\}$. Thus by the last inequality, we get

$$\int_0^{D^*(u,u,Ce)} \varphi(s)ds \leq \phi(\int_0^{D^*(u,u,Ce)} \varphi(s)ds) < \int_0^{D^*(u,u,Ce)} \varphi(s)ds,$$

which is a contradiction. Hence $Ce = u$.That is,

$$Av = Sv = Bw = Tw = Re = Ce = u.$$

By weak compatibility of the pairs (A,S) , (B,T) and (R,C) , we get $Au = Su$, $Bu = Tu$ and $Ru = Cu$. If $u \neq Au$, then using (2.3),we have

$$\int_0^{D^*(Au,Bw,Ce)} \varphi(s)ds \leq \phi(\int_0^{L(u,w,e)} \varphi(s)ds),$$

where

$$\begin{aligned} L(u,w,e) &= \max\{D^*(Su,Tw,Re), D^*(Au,Tw,Re), D^*(Su,Bw,Re), \\ &\quad D^*(Su,Tw,Ce)\} \\ &= D^*(Au,u,u), \end{aligned}$$

it follows that

$$\int_0^{D^*(Au,u,u)} \varphi(s)ds \leq \phi(\int_0^{D^*(Au,u,u)} \varphi(s)ds) < \int_0^{D^*(Au,u,u)} \varphi(s)ds,$$

which is a contradiction. Hence $Au = Su = u$. Similarly, we can prove that $Bu = Tu = u$ and $Ru = Cu = u$. Thus u is a common fixed point of A, B, C, R, S and T. The uniqueness of u follows from inequality (2.2). □

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