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ON THE SUBDIFFERENTIAL OF A NONLINEAR COMPLEMENTARITY PROBLEM FUNCTION WITH NONSMOOTH DATA

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ABSTRACT. In this paper, a system of nonsmooth equations reformulated from a nonlinear complementarity problem with nonsmooth data is studied. The formulas of some subdifferentials for related functions in this system of nonsmooth equations are developed. The present work can be applied to Newton methods for solving this kind of nonlinear complementarity problem.

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1. Introduction

Consider the nonlinear complementarity problem

$$F(x) \geq 0, x \geq 0, x^T F(x) = 0, \tag{1}$$

where $F : \Re^n \to \Re^m$ is a given map. We denote $F(x) = (f_1(x), \dots, f_n(x))^T$, $x = (x_1, \dots, x_n)$. The complementarity problem (1) is to find a solution $x \in \Re^n$, which satisfies (1). The Fischer-Burmeister nonlinear complementarity problem (for short NCP) function, see [2], is defined as the following:

$$\phi(a, b) = \sqrt{a^2 + b^2} - a - b.$$

By the NCP function ϕ, the nonlinear complementarity problem (1) can be reformulated as the following

$$\phi(x_i, f_i(x)) = 0, i = 1, \dots, n. \tag{2}$$

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In other words, $x \in \Re^n$ is a solution of (1) if and only if it is one of (2). Evidently, (2) is a system of nonsmooth equations even if each f_i is smooth.

Recently, Newton methods for a solution of nonsmooth equations are applied to solving the nonlinear complementarity problem. However, most work is related to the case that F is continuously differentiable, to our knowledge, only [3, 4, 7] dealt with the case that F is nonsmooth function. In the present paper, we try to consider the problem when F is locally Lipschitzian function. The formulas of subdifferentials for the functions in the left hand of (1) are investigated. By using these formulas, we can compute the B-differential or the Clarke generalized gradient of each $\phi(x_i, f_i(x))$. This work can be used in Newton methods for solving the nonsmooth equations (2).

Throughout the paper, e_i denotes the unit vector in $\Re^n$ whose i-th component is 1, $B(x,1)$ the unit ball with x as its center, D_H the set where function H is differentiable, x_i the i-th component of x, d_i i-th component of d.

2. Preliminaries

Let $H : \Re^n \to \Re^m$ be locally Lipschitzian. By the definition in [1, 6],

$$\partial_B H(x) = \{\lim_{y \to x} JH(y) \mid x_n \to x, y \in D_H\}$$

is called the B-differential of H; $\partial_{Cl} H(x) = \mathrm{co}\partial_B H(x)$ is called the Clarke generalized Jacobian of F; when H is from $\Re^n$ to $\Re$, $\partial_{Cl} H(x)$ is said to be the Clarke generalized gradient.

As in [6], the locally Lipschitzian function $H : \Re^n \to \Re^m$ is said to be semismooth at x if

$$Vh - H'(x; d) = o(\|d\|), \quad V \in \partial_{Cl} F(x + d).$$

Let us consider the nonsmooth equations:

$$H(z) = 0, \tag{3}$$

where $H : \Re^n \to \Re^n$ is locally Lipschitzian. Newton method for solving the nonsmooth equations (3) is given by

$$x^{k+1} = x^k - V_k^{-1} H(x^k), \tag{4}$$

where V_k is an element of $\partial_B H(x^k)$, $\partial_{Cl} H(x^k)$, $\partial_{Cl} h_1(x^k) \times \cdots \times \partial_{Cl} h_n(x^k)$ or $\partial_B h_1(x^k) \times \cdots \times \partial_B h_n(x^k)$. The locally superlinear convergence of Newton methods were shown when F is semismooth and all elements of corresponding subdifferentials of H at the solution are nonsingular.

Fischer-Burmeister NCP function has properties as follows [2]:

1. $\phi(a, b) = 0 \Longleftrightarrow a \geq 0, b \geq 0, ab = 0$;
2. the square of ϕ is continuously differentiable;
3. ϕ is twice continuously differentiable at every point except the origin, but it is strong semismooth at the origin.

If $a \neq 0$ or $b \neq 0$, then ϕ is continuously differentiable at $(a, b) \in \Re^2$ with

$$\nabla\phi(a,b) = (\frac{a}{\sqrt{a^2+b^2}} - 1, \frac{b}{\sqrt{a^2+b^2}} - 1)^T. \tag{5}$$

The following proposition can be found in [5].

Proposition 1. *Let $f, g : \Re^n \to \Re$ be directionally differentiable and let $G(x) = \phi(f(x), g(x))$. Then, G is directionally differentiable and its directional derivative is of the form:*

$$G'(x;d) = (\frac{f(x)}{\sqrt{f^2(x)+g^2(x)}} - 1)f'(x;d) + (\frac{g(x)}{\sqrt{f^2(x)+g^2(x)}} - 1)g'(x;d)$$
$$if (f(x), g(x)) \neq 0;$$
$$G'(z;d) = \phi(f'(x;d), g'(x;d)) \; if (f(x), g(x)) = 0;$$

Moreover, if both f and g are differentiable at x, then

$$\nabla G(z) = (\frac{f(x)}{\sqrt{f^2(x)+g^2(x)}} - 1)\nabla f(x) + (\frac{g(x)}{\sqrt{f^2(x)+g^2(x)}} - 1)\nabla g(x)$$
$$if (f(x), g(x)) \neq 0.$$

3. The formulas of subdifferentials of related functions

To perform Newton method for solving the system of nonsmooth equations (5), an element of some subdifferential of the function in the left hand side of (5) is required at each interactive step. In this section, we proceed to investigate the structure of the B-differentials or the Clarke generalized for the functions:

$$G_i(x) = \phi(x_i, f_i(x)), i = 1, \ldots, n.$$

By virtue Proposition 1, we have that

$$G_i'(x;d) = (\frac{x_i}{\sqrt{x_i^2+f_i^2(x)}} - 1)d_i + (\frac{f_i(x)}{\sqrt{x_i^2+f_i^2(x)}} - 1)f_i'(x;d)$$
$$if \; (x_i, f_i(x)) \neq 0; \tag{6}$$
$$G_i'(x;d) = \phi(d_i, f_i'(x;d)) \; if \; (x_i, f_i(x)) = 0.$$

If f_i is differentiable, then

$$\nabla G_i(x) = (\frac{x_i}{\sqrt{x_i^2+f_i^2(x)}} - 1)e_i + (\frac{f_i(x)}{\sqrt{x_i^2+f_i^2(x)}} - 1)\nabla f_i(x),$$
$$if \; (x_i, f_i(x)) \neq 0. \tag{7}$$

Lemma 1. *Let $f_i : \Re^n \to \Re$ be locally Lipschitzian and directionally differentiable. If $x_i = 0$ and $f_i(x) > 0$, then G_i is differentiable at x with the gradient $\nabla G_i(x) = -e_i$.*

Proof. Since $x_i = 0$ and $f_i(x) > 0$, by (6), we have $G_i'(x;d) = -e_i$. Therefore, $G_i'(x;\cdot)$ is linear function. This means that G_i is differentiable at x with the gradient $-e_i$. □

Lemma 2. *Let $f_i : \Re^n \to \Re$ be locally Lipschitzian and directionally differentiable. If $x_i = 0$ and $f_i(x) < 0$, then G_i is differentiable at x if and only if f_i is differentiable at x. Moreover, $\nabla G_i(x) = -2\nabla f_i(x)$ if $x_i = 0$ and $f_i(x) < 0$.*

Proof. Since $x_i = 0$ and $f_i(x) < 0$, by (6), we have $G_i'(x; d) = -2f_i'(x; d)$. Thus, the function $G'(x; \cdot)$ is linear if and only if so is $f_i'(x; \cdot)$. Therefore, G_i is differentiable if and only if f_i is differentiable. When G is differentiable, it is easy to see that $\nabla G_i(x) = -2\nabla f_i(x)$. □

Theorem 1. *Let $f_i : \Re^n \to \Re$ be locally Lipschitzian and directionally differentiable. If $(x_i, f_i(x)) \neq 0$, then*

$$\partial_B G_i(x) = (\frac{x_i}{\sqrt{x_i^2 + f_i^2(x)}} - 1)e_i + (\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)\partial_B f_i(x). \tag{8}$$

Proof. The proof is divided into three cases.

(I) Suppose $x_i \neq 0$. For a fixed x, the function $G_i'(x; \cdot)$ is linear if and only if so is $(\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)f_i'(x; \cdot)$ since the first part of the right hand side of (6) is a linear function. Notice that $(\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1) \neq 0$ because of $x_i \neq 0$. Therefore, the function $(\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)f_i'(x; \cdot)$ is linear if and only if so is $f_i'(x; \cdot)$. We know that the function $f_i'(x; \cdot)$ for a fixed x is linear if and only if f_i is differentiable at x. Hence, G_i is differentiable at x if and only if f_i is differentiable at x. In other words, $D_{G_i} = D_{f_i}$. In the differentiable case, the gradient of G_i is given as in (7). By the definition of the B-differential, (7) and $D_{G_i} = D_{f_i}$, and noticing that $y_i \neq 0$ when y near x enough because of $x_i \neq 0$, we have that

$$\begin{aligned}
\partial_B G_i(x) &= \{\lim_{y \to x} \nabla G_i(y) \mid y \to x, y \in D_{G_i}\} \\
&= \lim_{y \to x}\{(\frac{y_i}{\sqrt{y_i^2 + f_i^2(y)}} - 1)e_i + (\frac{f_i(y)}{\sqrt{y_i^2 + f_i^2(y)}} - 1)\nabla f_i(y) \mid y \to x, y \in D_{f_i}\} \\
&= (\frac{x_i}{\sqrt{x_i^2 + f_i^2(x)}} - 1)e_i + (\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)\{\lim_{y \to x} \nabla f_i(y) \mid y \to x, y \in D_{f_i}\} \\
&= (\frac{x_i}{\sqrt{x_i^2 + f_i^2(x)}} - 1)e_i + (\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)\partial_B f_i(x).
\end{aligned}$$

(II) Suppose $x_i = 0$ and $f_i(x) > 0$. By the definition of the B-differential, we have

$$\begin{aligned}
\partial_B G_i(x) \subset &\{\lim_{y \to x} \nabla G_i(y) \mid y \to x, y \in D_{G_i}, y_i = 0\} \\
&\bigcup \{\lim_{y \to x} \nabla G_i(y) \mid y \to x, y \in D_{G_i}, y_i \neq 0\}.
\end{aligned} \tag{9}$$

Noticing that $f_i(y) > 0$ when y near x enough because of the continuity of f_i, according to Lemma 1, it is obtained that

$$\{\lim_{y\to x} \nabla G_i(y) \mid y \to x, y \in D_{G_i}, y_i = 0\} = \{-e_i\}. \tag{10}$$

By the definition of the B-differential, (7) and $D_{G_i} = D_{f_i}$, and noticing that $x = 0$ and $y_i \neq 0$, we have

$$\begin{aligned}
&\{\lim_{y\to x} \nabla G_i(y) \mid y \to x, y \in D_{G_i}, y_i \neq 0\} \\
&= \{\lim_{y\to x} (\frac{y_i}{\sqrt{y_i^2 + f_i^2(y)}} - 1)e_i + (\frac{f_i(y)}{\sqrt{y_i^2 + f_i^2(y)}} - 1)\nabla f_i(y) \mid y \to x, y \in D_{f_i}, y_i \neq 0\} \\
&= (\frac{x_i}{\sqrt{x_i^2 + f_i^2(x)}} - 1)e_i + (\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)\{\lim_{y\to x} \nabla f_i(y) \mid y \to x, y \in D_{f_i}, y_i \neq 0\} \\
&\subset (\frac{x_i}{\sqrt{x_i^2 + f_i^2(x)}} - 1)e_i + (\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)\{\lim_{y\to x} \nabla f_i(y) \mid y \to x, y \in D_{f_i}\} \\
&= (\frac{x_i}{\sqrt{x_i^2 + f_i^2(x)}} - 1)e_i + (\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)\partial_B f_i(x) = \{-e_i\} \ (\textit{since } x = 0).
\end{aligned} \tag{11}$$

Combing (9) with (10) and (11) yields $\partial_B G_i(x) = \{-e_i\}$. Notice that

$$\{-e_i\} = (\frac{x_i}{\sqrt{x_i^2 + f_i^2(x)}} - 1)e_i + (\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)\partial_B f_i(x)$$

if $x_i = 0$. Hence, (8) holds if $x_i = 0, f_i(x) > 0$.

(III) Suppose $x_i = 0$ and $f_i(x) < 0$. Noticing that $f_i(y) < 0$ when y near x enough because of the continuity of f_i, according to Lemma 2, it is obtained that

$$\{\lim_{y\to x} \nabla G_i(y) \mid y \to x, y \in D_{G_i}, y_i = 0\} = \{-2\nabla f_i(x)\}. \tag{12}$$

Likewise to (11), it is obtained that

$$\begin{aligned}
&\{\lim_{y\to x} \nabla G_i(y) \mid y \to x, y \in D_{G_i}, y_i \neq 0\} \\
&= (\frac{x_i}{\sqrt{x_i^2 + f_i^2(x)}} - 1)e_i + (\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)\partial_B f_i(x) \\
&= \{-2\nabla f_i(x)\} \ (\textit{since } x = 0).
\end{aligned} \tag{13}$$

Combing (9) with (12) and (13) leads to $\partial_B G_i(x) = \{-2\nabla f_i(x)\}$. Evidently,

$$\{-2\nabla f_i(x)\} = (\frac{x_i}{\sqrt{x_i^2 + f_i^2(x)}} - 1)e_i + (\frac{f_i(x)}{\sqrt{x_i^2 + f_i^2(x)}} - 1)\partial_B f_i(x)$$

if $x_i = 0$. Therefore, (8) holds if $x_i = 0, f_i(x) < 0$. Thus, the proof of the theorem is completed. □

In what follows, we discuss the Clarke generalized gradient of $G_i(x)$ at the point where $(x_i, f(x_i)) = 0$.

Theorem 2. *Let $f_i : \Re^n \to \Re$ be locally Lipschitzian and directionally differentiable. If $(x_i, f_i(x)) = 0$, then*

$$\partial_{Cl} G_i(x) = \bigcup_{\xi^2+\eta^2=1} co(\xi - 1)e_i + (\eta - 1)\partial_{Cl} f_i(x). \tag{14}$$

Proof. Let $f : R^n \to \Re$ be locally Lipschzian. According to [1], the Clarke generalized gradient of f at x is expressed as the following

$$\partial_{Cl} f(x) = co\{\lim_{y\to x} \nabla f(y) \mid y \to x, y \in D_f, y \notin \Omega\},$$

where Ω is any set in $\Re^n$ whose measure is zero. Choosing $\Omega = \{y \in \Re^n \mid y_i = 0\}$ and by deducing, it follows that

$$\partial_{Cl} G_i(x) = co\{\lim_{y\to x} \nabla G_i(y) \mid y \to x, y \in D_{G_i}, y_i \neq 0\}$$

$$= co\Big\{ \lim_{y\to x}\Big(\Big(\frac{y_i}{\sqrt{y_i^2 + f_i^2(y)}} - 1\Big)e_i + \Big(\frac{f_i(y)}{\sqrt{y_i^2 + f_i^2(y)}} - 1\Big)\nabla f_i(y)\Big) \mid y \to x, y \in D_{f_i}, y_i \neq 0\Big\}$$

$$\subset \bigcup_{\xi^2+\eta^2=1} co\{\lim_{y\to x}((\eta - 1)e_i + (\xi - 1)\nabla f_i(y)) \mid y \to x, y \in D_{f_i}, y_i \neq 0\}$$

$$= \bigcup_{\xi^2+\eta^2=1} (\xi - 1)e_i + (\eta - 1)\partial_{Cl} f_i(x).$$

Thus, (14) holds. The proof of the theorem is completed. □

4. An application to Newton methods

Based on Theorems 1 and 2, we can use the Newton method (4) to solve the equation (2). In this method, we need compute the subdifferential of each function $f_i(x)$. If $f_i(x), i = 1, \ldots, n$ are semismooth, we can get the suplinear convergence of Newton method.

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