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FEM APPROACH TO ONE DIMENSIONAL UNSTEADY STATE TEMPERATURE DISTRIBUTION IN HUMAN DERMAL PARTS WITH QUADRATIC SHAPE FUNCTIONS

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ABSTRACT. This paper presents a Finite Element Method (FEM) application to thermal study of natural three layers of human dermal parts of varying properties. This paper carries out investigation of temperature distributions in these layers namely epidermis, dermis and under lying tissue layer. It is assumed that the outer skin is exposed to atmosphere and the loss of heat due to convection, radiation and evaporation of water have also been taken into account. The computations are carried out for one dimensional unsteady state case and the shape functions in dermal parts have been considered to be quadratic. A Finite Element scheme that uses the Crank-Nicolson method is used to solve the problem and the results computed have been exhibited graphically.

AMS Mathematics Subject Classification : 92C35
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1. Introduction

The skin of human body mainly consists of two natural layers - epidermis and dermis. Below the skin there is an extensive network of blood vessels, lymphatic, fat cells and nerve fibers of subcutaneous tissue that rest upon body core (Figure - 1). The population density of blood vessels varies for skin and subcutaneous region. The density of blood vessels is negligible in epidermis; whereas it increases gradually in dermis from epidermis towards subcutaneous tissue and is uniformly distributed in subcutaneous tissue. The flow of blood in blood vessels helps in maintaining uniform body core temperature irrespective of changes in environmental temperature. There are two more processes - metabolic heat generation, and sensible and insensible perspiration that help in maintaining uniform body core temperature.

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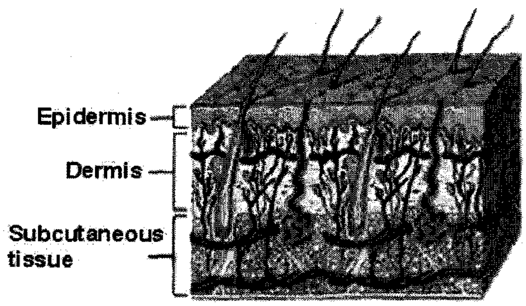

FIGURE 1. Skin and Subcutaneous tissue

Patterson [5] has experimentally determined temperature profile of the outer skin using radio-camera. Earlier Chao et al. [1] and Chao and Yang [2] considered two simple models and obtained temperature distribution curve in skin and subcutaneous tissue (SST) for certain fixed values of parameters. Later on Saxena et al. [8] used variational finite element approach with linear shape functions to find one dimensional unsteady state temperature distribution in SST assuming rate of blood flow and rate of metabolic heat generation as variables in the dermis part. Saxena et al. [11] used steady state temperature variation in SST with quadratic shape functions. In this paper we use variational finite element approach with quadratic shape functions to find one dimensional unsteady state temperature variation in SST taking into account the role of arterial blood temperature, rate of blood mass flow, rate of metabolic heat generation, and the tissue thermal conductivity and treating these as variables in dermis part.

2. Mathematical model

The partial differential equation for heat transfer in peripheral layers of human body due to Perl [6] is

$$\rho c \frac{\partial T}{\partial t} = \nabla . (K \nabla T) + \rho_b c_b (\phi_A T_A - \phi_v T_v) + S \tag{1}$$

where ρ = tissue density (g/cm^3) ; c = tissue specific heat (cal/g°C); T = tissue temperature (°C); K = tissue thermal conductivity (cal/cm-min°C); ρ_b = blood density (g/cm^3); c_b = blood specific heat (cal/g°C); ϕ_A = tissue perfusion due to arterial blood (/min); ϕ_v = tissue perfusion due to venous blood (/min); T_A = arterial blood temperature (°C); T_v = venous blood temperature (°C); S = metabolic heat generation rate (cal/cm^3-min).

For skin and subcutaneous tissue there is hardly any differences between ϕ_A and ϕ_v, so $\phi_A = \phi_v$. Also T_v is dominated by tissue temperature T. Hence $T_v \approx T$. So equation (1) can be rewritten as

$$\rho c \frac{\partial T}{\partial t} = \nabla . (K \nabla T) + M (T_A - T) + S \tag{2}$$

where $m_b = \rho_b \phi_A$ = Blood mass flow rate (g/cm^3-min), and $M = m_b c_b$.

Perl [6] used Equation (1) to investigate solution of some simple problems of infinite tissue medium considering the whole skin as a single layer and the parameters as constants. Cooper and Trezek [3] used it to investigate the solution of temperature distribution in brain tissues with negligible effect of blood flow and metabolic heat generation. Chao et al. [1] and Chao and Yang [2] applied finite difference method to steady and unsteady models taking all physiological parameters as constants. Later on, Saxena [7], Saxena and his coworkers [9-11] have used the above equation extensively for the investigation of thermoregulation with more realistic assumptions considering other important parameters. Based on the physiology appropriate functions have been chosen to represent the processes.

In one dimensional case the thickness of epidermis, dermis and subcutaneous tissue are taken as $l_1, l_2 - l_1$ and $l_3 - l_2$ respectively. While T_0, T_1, T_2 and T_3 represent the nodal temperatures respectively at the interfaces which are at distances $l_i (i = 0, 1, 2, 3)$ from the outer skin surface.

In the model, the following boundary conditions are considered:
(i) Outer skin surface ($x = 0$)

$$K \frac{\partial T}{\partial x}\bigg|_{\text{at } x = 0} = h\,(T - T_a) + LE \tag{3}$$

where x is measured from the outer surface towards the body core.
(ii) At the interface of subcutaneous tissue and the body core ($x = l_3$)

$$T_3 = T_b = 37°\text{C} \tag{4}$$

where h, T_a, L, E and T_b are respectively heat transfer coefficient, atmospheric temperature, latent heat of evaporation, rate of sweat evaporation and body core temperature. Blood mass flow rate and metabolic heat generation rate depend on density of blood vessels at different depths. It is reasonable to consider M and S negligible in epidermis, constant throughout in subcutaneous tissue, and linear function of the depth in various positions in dermis. We also make assumptions regarding temperature profiles in each layer. In the last two layers we take temperature profiles as second degree polynomials with unknown coefficients and in the first layer it is linear. This assumption ensures continuity of the heat flux at the interfaces. Accordingly we describe the complete layer wise formulation as given below.

(i) For epidermis ($0 < x < l_1$)

$$\begin{aligned} T &= T^{(1)} = A_1 + B_1 x + C_1 x^2 \text{ with } C_1 = 0; \\ T_A &= T_A^{(1)} = 0;\ K = K_1; \\ M &= M_1 = 0;\ S = S_1 = 0 \end{aligned} \tag{5a}$$

(ii) At interface - I ($x = l_1$)

$$
\begin{aligned}
&T = T_1;\ T_A = 0;\ K = K_1 = K_2;\\
&M = M_1 = M_2 = 0;\ S = S_1 = S_2 = 0;\\
&T^{(1)} = T^{(2)} = T_1;\\
&K_1 \frac{\partial T^{(1)}}{\partial x} = K_2 \frac{\partial T^{(2)}}{\partial x}
\end{aligned}
\tag{5b}
$$

(iii) For dermis ($l_1 < x < l_2$)

$$
\begin{aligned}
&T = T^{(2)} = A_2 + B_2 x + C_2 x^2;\\
&K = K_2 = \left(\frac{x - l_1}{l_2 - l_1}\right) K_3 + \left(\frac{l_2 - x}{l_2 - l_1}\right) K_1;\\
&M = M_2 = \left(\frac{x - l_1}{l_2 - l_1}\right) m;\\
&S = S_2 = \left(\frac{x - l_1}{l_2 - l_1}\right) s;\\
&T = T_A^{(2)} = \left(\frac{x - l_1}{l_2 - l_1}\right) T_b;
\end{aligned}
\tag{5c}
$$

(iv) At interface - II ($x = l_2$)

$$
\begin{aligned}
&T = T_2;\ T_A = T_b;\ K = K_2 = K_3;\\
&M = M_2 = M_3;\ S = S_2 = S_3;\\
&T^{(2)} = T^{(3)} = T_2;\\
&K_2 \frac{\partial T^{(2)}}{\partial x} = K_3 \frac{\partial T^{(3)}}{\partial x}
\end{aligned}
\tag{5d}
$$

(v) For subcutaneous tissue ($l_2 < x < l_3$)

$$
\begin{aligned}
&T = T^{(3)} = A_3 + B_3 x + C_3 x^2;\\
&T_A = T_A^{(3)} = T_b;\\
&K = K_3;\ M = M_3 = m;\\
&S = S_3 = s
\end{aligned}
\tag{5e}
$$

(vi) At inner boundary ($x = l_3$)

$$
\begin{aligned}
&T = T_3 = T_b;\ T_A = T_b;\ K = K_3;\\
&M = m;\ S = s
\end{aligned}
\tag{5f}
$$

The set of equations (2), (3), (4) and (5) describe the model equations in detail.

3. Solution of the Problem

The partial differential equation (2) together with the equation (3) when written for one dimensional unsteady state case and compared with Euler-Lagrange equation transforms into the following integral form for its optimum value [4].

$$I = \frac{1}{2}\int_0^{l_3}\left[K\left(\frac{dT}{dx}\right)^2 + M\left(T_A - T\right)^2 - 2ST + \rho c\frac{\partial T^2}{\partial t}\right]dx + \lambda\left[\frac{1}{2}h\left(T - T_a\right)^2 + LET\right] \quad (6)$$

We rewrite I separately for the three layers, i.e.,

$$I = \sum_{i=1}^{3} I_i \quad (7)$$

where

$$\lambda = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$I_i = \frac{1}{2}\int_{l_{i-1}}^{l_i}\left[K_i\left(\frac{dT^{(i)}}{dx}\right)^2 + M_i\left(T_A^{(i)} - T^{(i)}\right)^2 - 2S_iT^{(i)} + \rho c\frac{\partial T^{(i)^2}}{\partial t}\right]dx + \lambda\left[\frac{1}{2}h\left(T_0 - T_a\right)^2 + LET_0\right] \quad (8)$$

with $l_0 = 0$. The expression not including within the integral sign will appear for $i = 1$ at $x = 0$.

With the help of heat flux continuities at the interfaces and nodal temperatures at $x = l_i (i = 0, 1, 2, 3)$, the values of A_1, B_1 and C_1 in terms of nodal temperatures are obtained as T_0, $\frac{T_1 - T_0}{l_1}$ and 0 respectively, while the values of $A_k, B_k,$ and $C_k (k = 2, 3)$ in terms of nodal temperatures are obtained by the relation

$$AB = R \quad (9)$$

where

$$A = \begin{bmatrix} 1 & l_1 & l_1^2 & 0 & 0 & 0 \\ 1 & l_2 & l_2^2 & 0 & 0 & 0 \\ 0 & 1 & 2l_1 & 0 & 0 & 0 \\ 0 & 1 & 2l_2 & 0 & -1 & -2l_2 \\ 0 & 0 & 0 & 1 & l_2 & l_2^2 \\ 0 & 0 & 0 & 1 & l_3 & l_3^2 \end{bmatrix}; \quad B = \begin{bmatrix} A_2 \\ B_2 \\ C_2 \\ A_3 \\ B_3 \\ C_3 \end{bmatrix}; \quad R = \begin{bmatrix} T_1 \\ T_2 \\ \frac{T_1 - T_0}{l_1} \\ 0 \\ T_2 \\ T_b \end{bmatrix}$$

On solving (9), we get

$$A_2 = a_0T_0 + a_1T_1 + a_2T_2 + a_3T_b$$
$$B_2 = m_0T_0 + m_1T_1 + m_2T_2 + m_3T_b$$

$$\begin{aligned} C_2 &= n_0T_0 + n_1T_1 + n_2T_2 + n_3T_b \\ A_3 &= p_0T_0 + p_1T_1 + p_2T_2 + p_3T_b \\ B_3 &= q_0T_0 + q_1T_1 + q_2T_2 + q_3T_b \\ C_3 &= r_0T_0 + r_1T_1 + r_2T_2 + r_3T_b \end{aligned}$$

The coefficents of T_0, T_1, T_2 and T_b in all the cases are defined as in Appendix - I. Now using (4), (5) and (9) in (8) we obtain

$$\begin{aligned} I_1 &= N + N_0T_0 + N_{00}T_0^2 + N_{11}T_1^2 + N_{01}T_0T_1 + \frac{\rho cl_1}{6}\frac{d}{dt}\left(T_0^2 + T_1^2 + T_0T_1\right) \\ I_2 &= F + F_0T_0 + F_1T_1 + F_2T_2 + E_{00}T_0^2 + E_{11}T_1^2 + E_{22}T_2^2 + E_{01}T_0T_1 + E_{02}T_0T_2 \\ &\quad + E_{12}T_1T_2 + \frac{\rho c}{2}\frac{d}{dt}\Big(P_{00}T_0^2 + P_{11}T_1^2 + P_{22}T_2^2 + P_{01}T_0T_1 + P_{02}T_0T_2 \\ &\quad + P_{12}T_1T_2\Big) \\ I_3 &= M + M_0T_0 + M_1T_1 + M_2T_2 + L_{00}T_0^2 + L_{11}T_1^2 + L_{22}T_2^2 + L_{01}T_0T_1 \\ &\quad + L_{02}T_0T_2 + L_{12}T_1T_2 + \frac{\rho c}{2}\frac{d}{dt}(Q_0T_0 + Q_1T_1 + Q_2T_2 + Q_{00}T_0^2 + Q_{11}T_1^2 \\ &\quad + Q_{22}T_2^2 + Q_{01}T_0T_1 + Q_{02}T_0T_2 + Q_{12}T_1T_2) \end{aligned}$$

The coefficients of nodal temperatures and their combines and also N, F and M are all constants defined as in Appendix - II.

For optimizing I we differentiate it with respect to each nodal temperature and set the derivatives equal to zero, that is,

$$\frac{\partial T}{\partial T_i} = 0 \qquad (i = 0, 1, 2)$$

$T_3 = T_b$ is a constant. Thus we get the following system of equations

$$\begin{aligned} C_{11}\frac{dT_0}{dt} + C_{12}\frac{dT_1}{dt} + C_{13}\frac{dT_2}{dt} + K_{11}T_0 + K_{12}T_1 + K_{13}T_2 &= W_0 \\ C_{21}\frac{dT_0}{dt} + C_{22}\frac{dT_1}{dt} + C_{23}\frac{dT_2}{dt} + K_{21}T_0 + K_{22}T_1 + K_{23}T_2 &= W_1 \\ C_{31}\frac{dT_0}{dt} + C_{32}\frac{dT_1}{dt} + C_{33}\frac{dT_2}{dt} + K_{31}T_0 + K_{32}T_1 + K_{33}T_2 &= W_2 \end{aligned}$$

where C_{ij} and K_{ij} $(i, j = 1, 2, 3)$ and W_k $(k = 0, 1, 2)$ are defined as in Appendix - II.

The above system of equations in matrix form can be written as

$$C\dot{T} + PT = W \tag{10}$$

where

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}; \quad P = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}; \quad W = \begin{bmatrix} W_0 \\ W_1 \\ W_2 \end{bmatrix}$$

$$\dot{T} = \begin{bmatrix} \frac{dT_0}{dt} \\ \frac{dT_1}{dt} \\ \frac{dT_2}{dt} \end{bmatrix}; \quad \text{and} \quad T = \begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix}$$

To solve the system of ordinary differential equation (10) we use Crank-Nicolson method. This method moves the solution of the system ahead in time according to the relation

$$\left(C + \frac{\Delta t}{2}P\right)T^{(i+1)} = \left(C - \frac{\Delta t}{2}P\right)T^{(i)} + \Delta t W \tag{11}$$

where Δt is time interval, and $T^{(0)}$ is 3×1 matrix for initial nodal temperatures.

Generally, under normal condition the temperature decreases from body core towards skin surface. Hence we consider T_0, T_1, T_2 and T_3 in linear order towards body core at $t = 0$. Thus we assume the following initial condition for $T^{(0)}$

$$T(x, 0) = T_0 + rx \tag{12}$$

where $T_0 = 22.87°\text{C}$ and r is a constant whose value is determined during numerical computation taking $x = l_3$, the total thickness of SST region in which case $T(l_3, 0) = T_b = 37°\text{C}$.

4. Numerical results

To solve equations (11) we make use of the following values of the physical constants [3, 10, 11].

$$\begin{aligned} K_1 &= 0.030\,\text{cal/cm-min°C}; \quad K_3 = 0.060\,\text{cal/cm-min°C} \\ h &= 0.009\,\text{cal/cm}^2\text{-min°C} \\ L &= 579\,\text{cal/g}; \quad \rho = 1.05\,\text{g/cm}^3; \quad c = 0.83\,\text{cal/g} \end{aligned}$$

We can assign different values to the constants l_1, l_2 and l_3 depending on the sample of the skin and subcutaneous tissue under study. The set of values of l_1, l_2 and l_3 we considered here are as follows:

Set - I: $l_1 = 0.10$ cm; $l_2 = 0.35$ cm and $l_3 = 0.50$ cm

Set - II: $l_1 = 0.10$ cm; $l_2 = 0.40$ cm and $l_3 = 0.90$ cm

The numerical calculations have been made for the following three cases of atmospheric temperatures together with the respective values of s, m and E [3, 10, 11].

$(i) T_a = 15°\text{C}$; $s = 0.0357$ cal/cm^3-min; $m = m_b c_b = 0.003$ cal/cm^3-min°C; $E = 0$ g/cm^2-min

$(ii) T_a = 23°\text{C}$; $s = 0.018$ cal/cm^3-min; $m = m_b c_b = 0.018$ cal/cm^3-min°C; $E = 0, 0.48 \times 10^{-3}$ g/cm^2-min

$(iii) T_a = 33°\text{C}$; $s = 0.018$ cal/cm^3-min; $m = m_b c_b = 0.0315$ cal/cm^3-min°C; $E = 0.48 \times 10^{-3}, 0.96 \times 10^{-3}$, g/cm^2- min

In all cases of atmospheric temperatures it is assumed that skin has been exposed to the source temperatures for a time of 20 minutes. The system of equation (11) is then solved by subdividing this time of exposure in different time intervals of one tenth of a minute. Then curves have been plotted for $T^{(i)}(i = 1, 2, 3)$ versus time t at the distances of $x = 0.05$ cm, 0.25 cm and 0.45 cm respectively. Each of these values of x lies respectively in epidermis, dermis and subcutaneous tissue of both sample sets. The profiles of temperature distributions in all cases of atmospheric temperatures are shown in Figures - 2 to 7.

5. Discussion

From the graphs of temperature distribution profiles at different atmospheric temperatures, we observe that the curves for $T^{(i)}(i = 1, 2, 3)$ rise more quickly in Set - I and reach steady state earlier than in Set - II. It is also observed that steady state temperature for $T^{(i)}$ in Set - I is higher than in Set - II. These are due to SST region being thinner in Set - I than in Set - II.

In Figures - 4 to 7, it can be seen that the gap between the curves for $T^{(i)}$ are more for larger values of E and the values for $T^{(i)}$ for any atmospheric temperatures are more for smaller values of E. Thus, the rate of sweat evaporation plays a significant role for temperature distribution in SST region. We, furthermore, observed that steady state temperature $T^{(i)}$ of lower atmospheric temperatures is higher than higher atmospheric temperatures. This is because more heat moves from body core by blood flow towards the skin surface at lower atmospheric temperatures than higher atmospheric temperatures.

This paper will be useful for Biomedical Scientists for the following reasons:

(1) To analyze the impact of environmental conditions (temperature, humidity etc.) on human thermo-regulation.
(2) To prepare ground to understand effect of hot and cold climates.
(3) To visualize situations of abnormalities (disease etc.) in thermo-regulation.

Our results are comparable with those obtained by Saxena et al.[10] who considered only two layers, epidermis and dermis. The differences in the results whatsoever may be due to the extension up to subcutaneous tissue.

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Appendix - I

$$
\begin{aligned}
n_0 &= \frac{1}{l_1(l_2-l_1)}; \quad n_1 = -\frac{l_2}{l_1(l_2-l_1)^2}; \quad n_2 = \frac{1}{(l_2-l_1)^2}; \quad n_3 = 0;\\
m_0 &= -n_0(l_1+l_2); \quad m_1 = -n_1(l_1+l_2) + \frac{1}{(l_1-l_2)};\\
m_2 &= -n_2(l_1+l_2) + \frac{1}{(l_1-l_2)}; \quad m_3 = 0;\\
r_0 &= \frac{1}{l_1(l_2-l_3)}; \quad r_1 = \frac{l_1+l_2}{l_1(l_1-l_2)(l_2-l_3)}; \quad r_2 = \frac{l_1+l_2-2l_3}{(l_2-l_1)(l_2-l_3)^2};\\
r_3 &= \frac{1}{(l_2-l_3)^2};\\
q_0 &= -r_0(l_2+l_3); \quad q_1 = -r_1(l_2+l_3); \quad q_2 = \frac{1}{(l_2-l_3)} - r_2(l_2+l_3);
\end{aligned}
$$

$$q_3 = -2r_3l_2;$$
$$a_0 = n_0l_1l_2; \quad a_1 = n_1l_1l_2 + \frac{l_2}{(l_2-l_1)}; \quad a_2 = n_2l_1l_2 + \frac{l_1}{(l_1-l_2)}; \quad a_3 = 0;$$
$$p_0 = r_0l_2l_3; \quad p_1 = r_1l_2l_3; \quad p_2 = r_2l_2l_3 + \frac{l_3}{(l_3-l_2)}; \quad p_3 = r_3l_2l_3 + \frac{l_2}{(l_2-l_3)};$$

Appendix - II

$$C_{11} = \rho c\left(\frac{l_1}{3} + P_{00} + Q_{00}\right); C_{12} = \frac{\rho c}{2}\left(\frac{l_1}{3} + P_{01} + Q_{01}\right);$$
$$C_{13} = \frac{\rho c}{2}(P_{02}+Q_{02}); C_{21} = \frac{\rho c}{2}\left(\frac{l_1}{3} + P_{01} + Q_{01}\right);$$
$$C_{22} = \rho c\left(\frac{l_1}{3} + P_{11} + Q_{11}\right); C_{23} = \frac{\rho c}{2}(P_{12}+Q_{12});$$
$$C_{31} = \frac{\rho c}{2}(P_{02}+Q_{02}); \quad C_{32} = \frac{\rho c}{2}(P_{12}+Q_{12}); \quad C_{33} = \rho c(P_{22}+Q_{22});$$
$$K_{11} = 2(N_{00}+E_{00}+L_{00}); \quad K_{12} = N_{01}+E_{01}+L_{01}; \quad K_{13} = E_{02}+L_{02};$$
$$K_{21} = N_{01}+E_{01}+L_{01}; \quad K_{22} = 2(N_{11}+E_{11}+L_{11}); \quad K_{23} = E_{12}+L_{12};$$
$$K_{31} = E_{02}+L_{02}; \quad K_{32} = E_{12}+L_{12}; \quad K_{33} = 2(E_{22}+L_{22});$$
$$W_0 = -(N_0+F_0+M_0); \quad W_1 = -(F_1+M_1); \quad W_2 = -(F_2+M_2);$$
$$P_{ii} = a_i^2(l_2-l_1) + a_im_i(l_2^2-l_1^2) + \frac{1}{3}(m_i^2+2a_in_i)(l_2^3-l_1^3) + \frac{1}{2}m_in_i(l_2^4-l_1^4) + \frac{1}{5}n_i^2(l_2^5-l_1^5), \text{where } i = 0,1,2$$
$$P_{ij} = 2a_ia_j(l_2-l_1) + (a_im_j+a_jm_i)(l_2^2-l_1^2) + \frac{2}{3}(m_im_j+a_in_j+a_jn_i) \times(l_2^3-l_1^3) + \frac{1}{2}(m_in_j+m_jn_i)(l_2^4-l_1^4) + \frac{2}{5}n_in_j(l_2^5-l_1^5),\ i\neq j,\ i<j \text{ and } i,j = 0,1,2$$
$$Q_{ii} = p_i^2(l_3-l_2) + p_iq_i(l_3^2-l_2^2) + \frac{1}{3}(q_i^2+2p_ir_i)(l_3^3-l_2^3) + \frac{1}{2}q_ir_i(l_3^4-l_2^4) + \frac{1}{5}r_i^2(l_3^5-l_2^5), \text{where } i = 0,1,2$$
$$Q_{ij} = 2p_ip_j(l_3-l_2) + (p_iq_j+q_jp_i)(l_3^2-l_2^2) + \frac{2}{3}(q_iq_j+p_ir_j+p_jr_i)(l_3^3-l_2^3) + \frac{1}{2}(q_ir_j+q_jr_i)(l_3^4-l_2^4) + \frac{2}{5}r_ir_j(l_3^5-l_2^5),\ i\neq j,\ i<j \text{ and } i,j = 0,1,2,3$$
$$Q_0 = Q_{03}T_b; \quad Q_1 = Q_{13}T_b; \quad Q_2 = Q_{23}T_b;$$
$$N = \frac{1}{2}hT_a^2; N_0 = LE - hT_a; N_{00} = \frac{1}{2}\left(\frac{K_1}{l_1}+h\right); N_{11} = \frac{K_1}{2l_1}; N_{01} = -\frac{K_1}{l_1};$$
$$F = V_{20} + E_3T_b + E_{33}T_b^2; F_0 = E_0 + E_{03}T_b; F_1 = E_1 + E_{13}T_b;$$
$$F_2 = E_2 + E_{23}T_b;$$
$$M = W_{20} + L_3T_b + L_{33}T_b^2; M_0 = L_0 + L_{03}T_b; M_1 = L_1 + L_{13}T_b;$$
$$M_2 = L_2 + L_{23}T_b$$

$$E_i = a_iX_1 + m_iX_2 + n_iX3, \quad i = 0,1,2,3$$

$$E_{ii} = a_im_iV_{24} + m_in_iX_5 + a_in_iV_{26} + a_i^2V_{27} + m_i^2X_8 + n_i^2X_9, i = 0,1,2,3$$

$$E_{ij} = (a_im_j + m_ia_j)V_{24} + (m_in_j + n_im_j)X_5 + (a_in_j + n_ia_j)V_{26} + 2a_ia_jV_{27} + 2m_im_jX_8 + 2n_in_jX_9, i \neq j,\ i < j, \text{and}\, i,j = 0,1,2,3$$

$$X_i = V_{2i} + V_{3i},\ i = 1,2,3; \qquad X_j = V_{1j} + V_{2j},\ j = 5,8,9;$$

$$V_{15} = \frac{1}{3}\left[3(K_1l_2 - K_3l_1)(l_1 + l_2) + 2(K_3 - K_1)(l_2^2 + l_1l_2 + l_1^2)\right]$$

$$V_{18} = \frac{1}{4}\left[2(K_1l_2 - K_3l_1) + (K_3 - K_1)(l_2 + l_1)\right]$$

$$V_{19} = \frac{1}{6}\left[4(K_1l_2 - K_3l_1)(l_2^2 + l_1l_2 + l_1^2) + 3(K_3 - K_1)(l_1 + l_2)(l_2^2 + l_1^2)\right]$$

$$V_{20} = \frac{XT_b^2}{4(l_2 - l_1)}\left[(l_2^3 - l_1^3) + 3l_1l_2(l_1 - l_2)\right]; \ V_{21} = XT_b\left[-\frac{20}{3}l_1^2 - \frac{2}{3}l_2^2 + \frac{10}{3}l_1l_2\right]$$

$$V_{22} = XT_b\left[-\frac{1}{6}l_1^3 - \frac{1}{2}l_2^3 - \frac{1}{6}l_1^2l_2 + \frac{5}{6}l_1l_2^2\right]$$

$$V_{23} = \frac{XT_b}{(l_2 - l_1)}\left[\frac{2}{3}l_1^2(l_2^3 - l_1^3) - l_1(l_2^4 - l_1^4) + \frac{2}{5}(l_2^5 - l_1^5)\right]$$

$$V_{24} = X\left[-l_1(l_2^2 - l_1^2) + \frac{2}{3}(l_2^3 - l_1^3)\right]; \ V_{25} = X\left[\frac{1}{2}l_1(l_2^4 - l_1^4) - \frac{2}{5}(l_2^5 - l_1^5)\right];$$

$$V_{26} = X\left[\frac{2}{3}l_1l_2^3 - \frac{1}{2}l_2^4 - \frac{1}{6}l_1^4\right]; \ V_{27} = X\left[\frac{1}{2}(l_2^2 + l_1^2) - l_1l_2\right];$$

$$V_{28} = X\left[-\frac{1}{3}l_1(l_2^3 - l_1^3) + \frac{1}{4}(l_2^4 - l_1^4)\right]; \ V_{29} = X\left[-\frac{1}{5}l_1(l_2^5 - l_1^5) + \frac{1}{6}(l_2^6 - l_1^6)\right];$$

$$V_{31} = \frac{s(l_1 - l_2)}{2}; \ V_{32} = \frac{s(l_1^2 + l_1l_2 - 2l_2^2)}{6}; \ V_{33} = \frac{s}{12}\left[l_1^3 - 3l_2^3 + l_1l_2(l_1 + l_2)\right];$$

$$X = \frac{m}{2(l_2 - l_1)}; \ L_i = p_iY_1 + q_iY_2 + r_iY3, \quad i = 0,1,2,3;$$

$$L_{ii} = p_iq_iW_{24} + q_ir_iY_5 + p_ir_iW_{26} + p_i^2W_{27} + q_i^2Y_8 + r_i^2Y_9, \quad i = 0,1,2,3$$

$$L_{ij} = (p_iq_j + q_ip_j)W_{24} + (q_ir_j + r_iq_j)Y_5 + (p_ir_j + r_ip_j)W_{26} + 2p_ip_jW_{27} + 2q_iq_jY_8 + 2r_ir_jY_9, i \neq j,\ i < j, \text{and}\, i,j = 0,1,2,3$$

$$Y_i = W_{2i} + W_{3i},\ i = 1,2,3; \ Y_j = W_{1j} + W_{2j},\ j = 5,8,9; \ W_{15} = \frac{K_3(l_3 - l_2)}{2};$$

$$W_{18} = 2(l_3 + l_2)W_{15}; \ W_{19} = \frac{4}{3}(l_2^2 + l_2l_3 + l_3^2)W_{15}; \ W_{20} = \frac{m(l_3 - l_2)T_b^2}{2};$$

$$W_{21} = -mT_b(l_3 - l_2); \ W_{22} = -\frac{mT_b(l_3^2 - l_2^2)}{2}; \ W_{23} = -\frac{mT_b(l_3^2 - l_2^2)}{3};$$

$$W_{24} = \frac{m(l_3^2 - l_2^2)}{2};$$

$$W_{25} = \frac{m(l_3^4 - l_2^4)}{8}; \ W_{26} = \frac{m(l_3^3 - l_2^3)}{3}; \ W_{27} = \frac{m(l_3 - l_2)}{2}; \ W_{28} = \frac{m(l_3^3 - l_2^3)}{6};$$

$$W_{29} = \frac{m(l_3^5 - l_2^5)}{10}; \ W_{31} = -s(l_3 - l_2); \ W_{32} = -\frac{s(l_3^2 - l_2^2)}{2}; \ W_{33} = -\frac{s(l_3^3 - l_2^3)}{3}$$

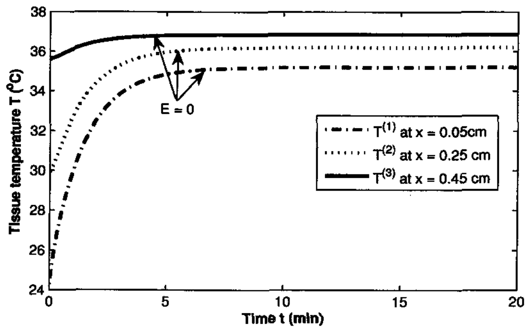

FIGURE 2. $T_a = 15°\text{C}, m = 0.003$ and $s = 0.0357$ for Set - I

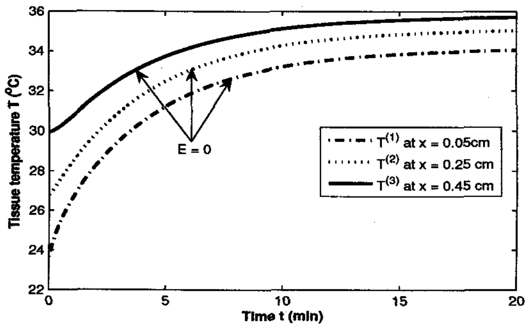

FIGURE 3. $T_a = 15°\text{C}, m = 0.003$ and $s = 0.0357$ for Set - II

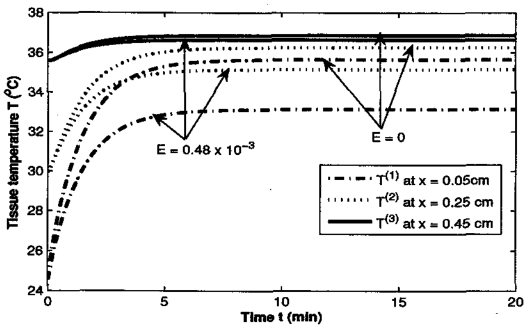

FIGURE 4. $T_a = 23°\text{C}, m = 0.018$ and $s = 0.018$ for Set - I

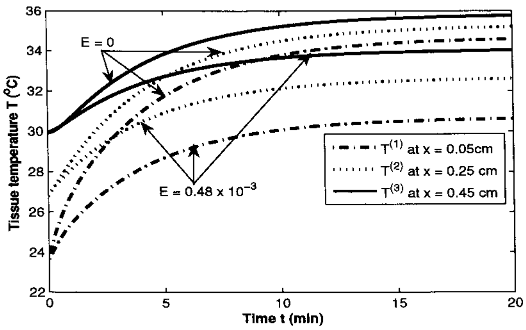

FIGURE 5. $T_a = 23°\mathrm{C}, m = 0.018$ and $s = 0.018$ for Set - II

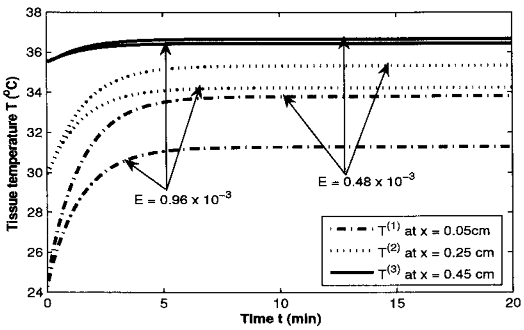

FIGURE 6. $T_a = 33°\mathrm{C}, m = 0.0315$ and $s = 0.018$ for Set - I

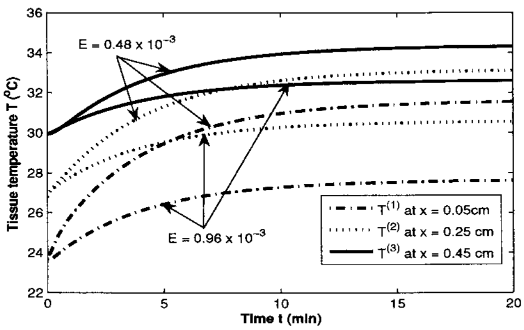

FIGURE 7. $T_a = 33°\mathrm{C}, m = 0.0315$ and $s = 0.018$ for Set - II