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GENERAL HÖLDER TYPE INEQUALITIES ON THE FUNCTIONS OF $G\kappa G\phi$-BOUNDED VARIATIONS

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ABSTRACT. For $G\phi$-sequences ϕ_i and κ-functions $\kappa_i (i = 1,2,3)$ we obtain the most general Hölder type inequalities on the functions of $G\kappa G\phi$-bounded variations.

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In his studies on generalized functions of bounded variation and their application to the theory of harmonic functions, D. S. Cyphert([1]) viewed κ-*function* κ as a rescaling of lengths of subintervals in $[a, b]$ such that the length of $[a, b]$ is 1 if $\kappa(1) = 1$. We are requiring that κ has the following properties on a closed interval $[0, 1]$;

(1) κ is continuous with $\kappa(0) = 0$ and $\kappa(1) = 1$,
(2) κ is concave and strictly increasing, and
(3) $\lim\limits_{x\to 0^+} (\kappa(x)/x) = \infty$.

We shall say that $\kappa_i (i = 1,2,3)$ satisfy the Δ_κ-*condition* (briefly $\kappa_i \in \Delta_\kappa (i = 1,2,3)$) if κ-functions κ_1, κ_2 and κ_3 satisfy $\kappa_1^{-1}(x)\kappa_2^{-1}(x) \ge \kappa_3^{-1}(x)$ for $x \ge 0$.

Let $\phi = \{\phi_n\}$ be a sequence of monotone nondecreasing convex functions defined on the nonnegative real numbers such that $\phi_n(0) = 0$ and $\phi_n(x) > 0$ for $x > 0$ and $n = 1, 2, \cdots$. We shall say that ϕ is a $G\phi$-*sequence*, in symbol

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$\phi \in G\phi S$ if $\phi_n(x) \geq \phi_{n+1}(x)$ for all n and x and in addition if $\sum_n \phi_n(x)$ diverges for $x > 0$ [4].

Note that if $\phi = \{\phi_n\}$ is a ϕ-sequence, in symbol $\phi \in \phi S$, then $\phi = \{\phi_n\}$ is a $G\phi$-sequence, that is, $\phi S \subset G\phi S$ in the sence of D.S. Cyphert[1].

Let us define $\psi_n(y) = \sup\left\{x|y| - \phi_n(x) : x \geq 0\right\} = \int_0^y \varphi_n^{-1}(s)ds, y \geq 0$. Then $\psi = \{\psi_n\}$ is called as the *complementary function* to ϕ_n and (ϕ_n, ψ_n) the *complementary pair*.

In sequel we denote by $A \cdot B$ the collection of all products $f \cdot g$ for any functions $f \in A$ and $g \in B$, and define $R_{\phi_i}(t) = \dfrac{\phi_i(t)}{t}$ and $S_{\phi_i}(t) = \dfrac{\psi_i(t)}{t}$ for the corresponding complementary pair (ϕ_i, ψ_i) of a $G\phi$-sequence $\phi = \{\phi_n\}$.

Note that $R_{\phi_i}(t)$ and $S_{\phi_i}(t)$ are substitutes for the useful expressions t^{p-1}, $t^{1/(p-1)}$ in the L^p-spaces, but not mutually complementary inverse, in general.

A $G\phi$-sequence $\phi = \{\phi_n\}$ is said to satisfy the Δ_2-*condition*, in symbol $\phi \in \Delta_2$ if there is a constant $c > 0$ and $u_0 \geq 0$ such that $2R_{\phi_n}(2x) \leq cR_{\phi_n}(x)$, for all $x \geq u_0$, and the Δ'-*condition*, in symbol $\phi \in \Delta'(c)$ if there is a constant $c > 0$ such that $R_{\phi_n}(xy) \leq cR_{\phi_n}(x)R_{\phi_n}(y)$, $x, y \geq x_0 \geq 0$.

For $\phi_1 = \{\phi_{1n}\}$, $\phi_2 = \{\phi_{2n}\} \in G\phi S$, ϕ_1 is *stronger than* ϕ_2, $\phi_1 \succ \phi_2$ in symbols, if $R_{\phi_{2n}}(x) \leq aR_{\phi_{1n}}(ax), x \geq x_0 \geq 0$ for some $a > 0$ and x_0 (depending on a). We can define equivalence of pair of $G\phi$-sequences: $\phi_1 \sim \phi_2$ iff $\phi_1 \succ \phi_2$ and $\phi_2 \succ \phi_1$: there exist numbers $0 < a \leq b < \infty, x_0 \geq 0$ such that $aR_{\phi_{1n}}(ax) \leq R_{\phi_{2n}}(x) \leq bR_{\phi_{1n}}(bx), x \geq x_0$.

Let $\phi = \{\phi_n\} \in G\phi S$ be defined on $[0, \infty)$ for all n. The *average function* $A(\phi_n)$ of ϕ_n is the function given by $A(\phi_n)(x) = \dfrac{1}{x}\int_0^x \phi_n(\tau)d\tau$ for all $x > 0$ and $A(\phi_n)(0) = 0$.

A function f is of κ-*bounded variation* on $[a, b]$ if there exists a positive constant c such that, for every collection $\{I_n\}$ of nonoverlapping subintervals of $[a, b]$, $\sum |f(I_n)| \leq c \sum \kappa(|I_n|/(b-a))$, where $I_n = [x_n, y_n]$ and $|I_n| = y_n - x_n$.

M. J. Schramm [4] generalized the above idea by considering a sequence of increasing convex functions $\phi = \{\phi_n\}$ defined on $[0, \infty)$; f is of ϕ-*bounded variation* on $[a, b]$ if $V_\phi(f; a, b) = V_\phi(f) = \sup\left(\sum \phi_n(|f(I_n)|)\right)$ is finite.

A function f is said to be of $\kappa G\phi$-*bounded variation* on $[a, b]$ if there exists a positive constant c such that for any appropriate collection $\{I_n\}$ of nonoverlapping subintervals of $[a, b]$, $\sum \phi_n(|f(I_n)|) \leq c \sum \kappa(|I_n|/(b-a))$, where $[a, b] = \bigcup I_n$ and $|I_n|$ is the length of I_n.

The total variation of f over $[a,b]$ is defined by

$$\kappa V_{G\phi}(f) = \kappa V_{G\phi}(f : a, b) = \sup\Big(\sum \phi_n(|f(I_n)|)\Big)/\sum \kappa(|I_n|/(b-a))),$$

where the supremum is taken over all collections $\{I_n\}$ of nonoverlapping subintervals in $[a,b]$. We denote by $\kappa G\phi BV$ the collection of all functions of $\kappa G\phi$-bounded variation on $[a,b]$.

A function f is said to be of generalized *$\kappa G\phi$-bounded variation* on $[a,b]$, in symbols $f \in G\kappa G\phi BV[a,b]$ if there exists a positive constant c such that for any sequences $\{I_n\}$ of intervals in A,

$$\sum \phi_n(|f(I_n)|) \leq c\sum \kappa(|I_n|/(b-a)).$$

where $[a,b] = \bigcup I_n$ and $|I_n|$ is the length of I_n. The total $G\kappa G\phi$-variation of f over $[a,b]$ is defined by

$$\kappa V_{G\phi}(f, A) = \sup\Big(\sum \phi_n(|f(I_n)|)\Big)/\sum \kappa(|I_n|/(b-a))),$$

where the supremum is taken over all collections $\{I_n\}$ of intervals in A[2].

Let $G\kappa G\phi BV_0 = \Big\{f \in G\kappa G\phi BV[a,b] \mid f(a) = 0\Big\}$. For any f in $G\kappa G\phi BV_0$, let us define the norm as in the Orlicz spaces;

$$|||f||| = |||f|||_{G\kappa G\phi} = inf\Big\{c > 0; G\kappa V_{G\phi}(f) \leq 1\Big\}$$

Then $(G\kappa G\phi BV_0, |||\cdot|||)$ is a Bannach space and $G\kappa G\phi BV$ may be a Bannach space with the norm $|||f - f(a)||| + \mid f(a) \mid$.

From now on we will consider $G\phi$-sequences $\phi_i = \{\phi_{in}\}$ and κ-functions $\kappa_i \in \Delta_\kappa$ for $i = 1,2,3$.

Theorem 1. *(a) If $\phi_i = \{\phi_{in}\} \in G\phi S$ $(i = 1,2,3)$ satisfy the inequality*

$$R_{R_{\phi_{1n}^{-1}}}(x)R_{R_{\phi_{2n}^{-1}}}(x) \leq \alpha R_{R_{\phi_{3n}^{-1}}}(x) \tag{1}$$

for all n and $x \geq 0$, then we have, for all $f \in G\kappa_1 G\phi_1 BV_0$ and $g \in G\kappa_2 G\phi_2 BV_0$,

$$G\kappa_1 G\phi_1 BV_0 \cdot G\kappa_2 G\phi_2 BV_0 \subset G\kappa_3 G\phi_3 BV_0$$

and

$$|||fg|||_{G\kappa_3 G\phi_3} \leq 2\alpha |||f|||_{G\kappa_1 G\phi_1} |||g|||_{G\kappa_2 G\phi_2}.$$

(b) If the inequality (1) in part (a) is replaced by

$$\frac{1}{\alpha} \leq R_{\phi_{1n}}(x)R_{(\phi_{3n}^{-1}\circ\phi_{2n})}(y) + R_{\phi_{2n}}(y)R_{(\phi_{3n}^{-1}\circ\phi_{1n})}(x) \tag{2}$$

then, for all $f \in G\kappa_1 G\phi_1 BV_0$ *and* $g \in G\kappa_2 G\phi_2 BV_0$,

$$G\kappa_1 G\phi_1 BV_0 \cdot G\kappa_2 G\phi_2 BV_0 \subset G\kappa_3 G\phi_3 BV_0$$

and

$$|||fg|||_{G\kappa_3 G\phi_3} \leq 4\alpha |||f|||_{G\kappa_1 G\phi_1} |||g|||_{G\kappa_2 G\phi_2}.$$

Proof. (a)By the convexity of ϕ_{in}, since the inequality (1) implies the inequality(2), the part (a) holds.

(b)By the inequality (2), since

$$\frac{\sum \phi_{3n}(|f(I_n)g(I_n)|/4\alpha(1+\varepsilon)^2)}{\sum \kappa_3(|I_n|/(b-a))} \leq \frac{1}{2}(V_{G\phi_1}(f) + V_{G\phi_2}(g)) \leq 1,$$

we have $\kappa_3 V_{G\phi_3}(fg/4(1+\varepsilon)^2) \leq 1$, which implies $|||fg|||_{\kappa_3 G\phi_3} \leq 4(1+\varepsilon)^2$, and hence the theorem follows by letting $\varepsilon \to 0$. □

Corollary 2. *For* $\phi_i = \{\phi_{in}\} \in G\phi S$ $(i = 1, 2)$ *with* $\int_0^1 R_{\phi_{1n}^{-1}}(t)R_{\phi_{2n}^{-1}}(t)dt < \infty$, *if we let* $\phi_{3n}^{-1}(x) = \alpha \int_0^x R_{\phi_{1n}^{-1}}(t)R_{\phi_{2n}^{-1}}(t)dt$ *for some constant* α, *then* $\phi_3 = \{\phi_{3n}\} \in G\phi S$, *and, for all* $f \in G\kappa_1 G\phi_1 BV_0$ *and* $g \in G\kappa_2 G\phi_2 BV_0$,

$$G\kappa_1 G\phi_1 BV_0 \cdot G\kappa_2 G\phi_2 BV_0 \subset G\kappa_3 G\phi_3 BV_0$$

and

$$|||fg|||_{G\kappa_3 G\phi_3} \leq 2\alpha |||f|||_{G\kappa_1 G\phi_1} |||g|||_{G\kappa_2 G\phi_2}.$$

Proof. Since $R_{\phi_{in}^{-1}}(x)$ $(i = 1, 2)$ are nonincreasing, it follow that $xR_{\phi_{3n}^{-1}}(x)$ is concave and $\phi_{3n}^{-1}(0) = 0$. By the inequality (1), this is proved. □

Lemma 3. *For* $\phi_i = \{\phi_{in}\} \in G\phi S(i = 1, 2, 3)$, *the followings are equivalent; for nonnegative* $x, y, z \geq 0$,

(a) $\alpha_1 xyR_{\phi_{3n}}(\alpha_1 xy) \leq xR_{\phi_{1n}}(x) + yR_{\phi_{2n}}(y)$ *for some* $\alpha > 0$;

(b) $\lim_{x\to\infty} sup\Big(xR_{\phi_{1n}^{-1}}(x)R_{\phi_{2n}^{-1}}(x)/R_{\phi_{3n}^{-1}}(x)\Big) < \infty$;

(c) $\alpha_2 xyz \leq xR_{\phi_{1n}}(x) + yR_{\phi_{2n}}(y) + zS_{\phi_{3n}}(z)$;

(*d*) $\alpha_3 yz S_{\phi_{1n}}(\alpha_3 yz) \leq yR_{\phi_{2n}}(y) + zS_{\phi_{3n}}(z)$;
(*e*) $\alpha_4 xz S_{\phi_{2n}}(\alpha_4 xz) \leq xR_{\phi_{1n}}(x) + zS_{\phi_{3n}}(z)$,
where $\alpha_i > 0$ *are some constants and* (ϕ_{in}, ψ_{in}) *the coresponding complementary pairs.*

Proof. Assume that (a) holds. Since $x \leq (\phi_{in}^{-1} \circ \phi_{in})(x)$, if we let $x = uR_{\phi_{1n}^{-1}}(u)$ and $y = uR_{\phi_{2n}^{-1}}(u)$, then (a) implies (b).

Conversely, if (b) holds, then there are $\alpha_1 > 0$ and $u_0 \geq 0$ such that

$$R_{\phi_{1n}^{-1}}(u)R_{\phi_{2n}^{-1}}(u) \leq \frac{u}{\alpha_1}R_{\phi_{3n}^{-1}}(u), u \geq u_0,$$

and letting $x = uR_{\phi_{1n}^{-1}}(u)$ and $y = uR_{\phi_{2n}^{-1}}(u)$ for x, $y \geq max(u_0 R_{\phi_{1n}^{-1}}(u_0)$, $u_0 R_{\phi_{2n}^{-1}}(u_0))$, this shows that (b) implies (a).

By the property of $G\phi S$, (a) iff (c).

For $x, y, z \geq x_2 \geq 0$, we have the folldwing inequalities;

$$\alpha_2 yz S_{\phi_{1n}}(\alpha_2 yz) = \alpha_2 xyz - xR_{\phi_{1n}}(x) \leq yR_{\phi_{2n}}(y) + zS_{\phi_{3n}}(z),$$

$$\alpha_2 xyz - xyzR_{\phi_{1n}}(x) \leq \alpha_2 yzR_{\psi_{1n}}(\alpha_2 yz) \leq yR_{\phi_{2n}}(y) + zR_{\psi_{3n}}(z),$$

which implies that (c) iff (d). Similarily (c) iff (e). □

By Lemma 3, we have the following Theorem 4;

Theorem 4. *For* $\kappa_i \in \Delta_\kappa$ *and* $\phi_i = \{\phi_{in}\} \in G\phi S (i = 1, 2, 3)$, *suppose that one of* (*a*) $\sim$ (*e*) *in Lemma 3 is satisfied. Then for the coresponding complementary pairs* (ϕ_{in}, ψ_{in}) *there is a constant* α_i *such that*

$$G\kappa_1 G\phi_1 BV_0 \cdot G\kappa_2 G\phi_2 BV_0 \subset G\kappa_3 G\phi_3 BV_0$$

and

$$G\kappa_3 V_{G\phi_3}(fg) \leq \frac{1}{\alpha_i} G\kappa_1 V_{G\phi_1}(f) G\kappa_2 V_{G\phi_2}(g)$$

for any $f \in G\kappa_1 G\phi_1 BV_0$ *and* $g \in G\kappa_2 G\phi_2 BV_0$.

Theorem 5. *For* $\kappa_i \in \Delta_\kappa$, $\phi_i = \{\phi_{in}\} \in G\phi S (i = 1, 2, 3)$ *and the coresponding complementary pairs* (ϕ_{in}, ψ_{in}), *suppose that one of the following conditions is satisfied:*

(*i*) *there is a complementary pair* (ϕ_{4n}, ψ_{4n}) *such that*

$$\phi_{1n} \succ \phi_{3n} \circ \phi_{4n} \quad and \quad \phi_{2n} \succ \phi_{3n} \circ \psi_{4n},$$

(ii) for the above (ϕ_{4n}, ψ_{4n}), *if* $\phi_{3n} \in \Delta'$, *suppose that*

$$\phi_{1n} \succ \phi_{4n} \circ \phi_{3n} \quad and \quad \phi_{2n} \succ \psi_{4n} \circ \phi_{3n},$$

Then there is a constant $\alpha \geq 0$ *such that, for any* $f \in G\kappa_1 G\phi_1 BV_0$ *and* $g \in G\kappa_2 G\phi_2 BV_0$,

$$G\kappa_1 G\phi_1 BV_0 \cdot G\kappa_2 G\phi_2 BV_0 \subset G\kappa_3 G\phi_3 BV_0$$

and

$$G\kappa_3 V_{G\phi_3}(fg) \leq \frac{1}{\alpha} G\kappa_1 V_{G\phi_1}(f) G\kappa_2 V_{G\phi_2}(g)$$

Proof. By (i) and the definition of $\succ$, we have, for some $\beta > 0$,

$$2\beta x R_{(\phi_{3n} \circ \phi_{4n})}(2\beta x) \leq x R_{\phi_{1n}}(x)$$

and

$$2\beta y R_{(\phi_{3n} \circ \psi_{4n})}(2\beta y) \leq y R_{\phi_{2n}}(y), x, y \geq x_0 \geq 0.$$

Hence if we let $\alpha = \beta^2$, then

$$\begin{aligned} \alpha xy R_{\phi_{3n}}(\alpha xy) &\leq \beta x R_{(\phi_{3n} \circ \phi_{4n})}(2\beta x) + \beta y R_{(\phi_{3n} \circ \psi_{4n})}(2\beta y) \\ &\leq x R_{\phi_{1n}}(x) + y R_{\phi_{2n}}(y), \end{aligned}$$

which is (a) of Lemma 3.

Next let (ii) be true. Then, for $x, y \geq x_0 \geq 0$,

$$\begin{aligned} cxy R_{\phi_{3n}}(cxy) &\leq x R_{\phi_{3n}}(x) y R_{\psi_{3n}}(y) \\ &\leq bx R_{\phi_{1n}}(bx) + by R_{\phi_{2n}}(by), x, y \geq x_0 \geq 0, \end{aligned}$$

for some $b, c > 0$. If we let $\alpha = \frac{c}{b^2}, u = bx, v = by$ and $u_0 = bx_0$, then

$$\begin{aligned} \alpha uv R_{\phi_{3n}}(\alpha uv) &= \alpha b^2 xy R_{\phi_{3n}}(\alpha b^2 xy) \\ &\leq \frac{u}{b} R_{\phi_{3n}}(\frac{u}{b}) \frac{v}{b} R_{\phi_{3n}}(\frac{v}{b}) \\ &\leq u R_{\phi_{1n}}(u) + v R_{\phi_{2n}}(v), \end{aligned}$$

which reduces to (a) of Lemma 3. So the result holds. □

Lemma 6. *For $\phi_1 = \{\phi_{1n}\} \in G\phi S$, let $\phi_{2n}(t) = \int_0^t R_{\phi_{1n}}(u)du$, $\phi_{3n}(t) = \int_0^t R_{\phi_{2n}}(\tau)d\tau$ and $\phi_{in}(0) = 0$ for $i = 1, 2, 3$. Then $R_{\phi_{in}}$ and $S_{\phi_{in}}$ are strictly increasing continuous functions in $G\phi S$ with continuous derivatives which map $[0, \infty)$ onto itself and satisfy the followings; for any $t \geq 0$,*

$$R_{\phi_{in}}(t) \leq 2R_{\phi_{in}}(2t) \tag{3}$$

$$S_{\phi_{in}}(R_{\phi_{in}}(t)) \leq t \leq 2S_{\phi_{in}}(2R_{\phi_{in}}(t)) \qquad (4),$$

$$\phi_1 \sim \phi_2, \phi_2 \sim \phi_3 \qquad (5),$$

and the complementary version;

$$S_{\phi_{in}}(t) \leq 2S_{\phi_{in}}(2t) \qquad (3')$$

$$R_{\phi_{in}}(S_{\phi_{in}}(t)) \leq t \leq 2R_{\phi_{in}}(2S_{\phi_{in}}(t)) \qquad (4'),$$

$$\psi_1 \sim \psi_2, \psi_2 \sim \psi_3 \qquad (5').$$

Proof. Note that $1 \leq tR^{-1}_{\phi_{in}}(t)S^{-1}_{\phi_{in}}(t) \leq 2$, $t \geq 0$. Substituting $t \to tR_{\phi_{in}}(t)$, we get

$$R_{\phi_{in}}(t) \leq \psi_{in}^{-1}(\phi_{in}(t)) \leq 2R_{\phi_{in}}(t).$$

Hence

$$R_{\phi_{in}}(t) \leq 2R_{\phi_{in}}(t) \leq 2R_{\phi_{in}}(2t)$$

and

$$\psi_{in}(R_{\phi_{in}}(t)) \leq \phi_{in}(t) \leq \psi_{in}(2R_{\phi_{in}}(t)) \qquad (*).$$

Dividing $(*)$ with $R_{\phi_{in}}(t)$ or with $2R_{\phi_{in}}(t)$, we have

$$S_{\phi_{in}}(R_{\phi_{in}}(t)) \leq (\phi_{in}(t)/R_{\phi_{in}}(t)) \leq t \leq 2S_{\phi_{in}}(2R_{\phi_{in}}(t)).$$

By the continuity of ϕ_{1n} and definitions of $G\phi S$-sequences, we have

$$\lim_{x\to 0} R_{\phi_{2n}}(t) = \lim_{x\to 0} \phi'_{2n}(t) = \lim_{x\to 0} R_{\phi_{1n}}(t) = 0$$

and

$$\lim_{x\to\infty} R_{\phi_{2n}}(t) = \lim_{x\to\infty} R_{\phi_{1n}}(t) = \infty.$$

Since $\phi'_{2n}(t) = R_{\phi_{1n}}(t) \geq 0$, ϕ_{2n} is strictly increasing and $R_{\phi_{1n}}(t_1) < R_{\phi_{2n}}(t_2)$ if $t_1 < t_2$, which implies that ϕ_{2n} is convex on $[0, \infty)$.

Since $R_{\phi_{in}}(t) \le \psi_{in}(t) = S_{\psi_{in}}(t)$, each statement of the Lemma has its complementary version; Substituting $t \to \psi_{in}(t)$ and dividing with t,

$$S_{\phi_{in}}(t) \le \phi_{in}^{-1}(\psi_{in}(t)) \le S_{\phi_{in}}(t) \to \phi_{in}(S_{\phi_{in}}(t)) \le \psi_{in}(t) \le \phi_{in}(2S_{\phi_{in}}(t))$$

and

$$R_{\phi_{in}}(t) \le 2R_{\phi_{in}}(2t).$$

Also, $R_{\phi_{in}}(S_{\phi_{in}}(t)) \le t \le 2R_{\phi_{in}}(2S_{\phi_{in}}(t))$. By the simillar way, other cases are proved □

Note that if $\phi = \{\phi_{in}\}$ is a $G\phi S$-function, then $R_{\phi_{in}}$ and $S_{\phi_{in}}$ are increasing continuous functions in $G\phi S$ with continuous derivatives which map $[0, \infty)$ onto itself and satisfy for any $t \ge 0$ the above inequalities;(3) $\sim$ (5) and (3′) $\sim$ (5′).

Theorem 7. *For $\phi_1 = \{\phi_{1n}\} \in G\phi S$, let $\phi_{2n}(t) = \int_0^t R_{\phi_{1n}}(u)du$, $\phi_{3n}(t) = \int_0^t R_{\phi_{2n}}(\tau)d\tau$ and $\phi_{in}(0) = 0 (i = 1, 2, 3)$, then we have the followings;*

(i) for a κ-function $\kappa_i (i = 1, 2, 3)$ with $\kappa_1 \le \kappa_2 \le \kappa_3$,

$$G\kappa_1 G\phi_1 BV_0 \subset G\kappa_2 G\phi_1 BV_0 \subset G\kappa_2 G\phi_2 BV_0 \subset G\kappa_3 G\phi_2 BV_0 \subset G\kappa_3 G\phi_3 BV_0$$

and

$$G\kappa_1 G\psi_1 BV_0 \subset G\kappa_2 G\psi_1 BV_0 \subset G\kappa_2 G\psi_2 BV_0 \subset G\kappa_3 G\psi_2 BV_0 \subset G\kappa_3 G\psi_3 BV_0$$

(ii) for $\kappa_i \in \Delta_\kappa$,

$$G\kappa_1 G\phi_1 BV_0 \subset G\kappa_3 G\phi_1 BV_0 \subset G\kappa_3 G\phi_2 BV_0$$

and

$$G\kappa_2 G\psi_1 BV_0 \subset G\kappa_3 G\psi_1 BV_0 \subset G\kappa_3 G\psi_2 BV_0$$

Proof. By (5) and (5') of Lemma 3, these are proved. □

REFERENCES

1. D. S. Cyphert, *Generalized functions of bounded variation and their application to the theory of harmonic functions*, Dissertation in Math. Vanderbilt Univ., Tennesse, U.S.A., 1982.
2. S. K. Kim and J. Kim, *Functions of $\kappa\phi$-bounded variations*, Bull. Korean Math. Soc. 23 No.2 (1986) 171-175.
3. Pamela B. Pierce and Daniel Waterman, *On the invariance of classes ϕBV and ΛBV under composition*, Preprint..
4. M. J. Schramm, *Functions of ϕ-bounded variation and Riemann-Stieltjes integration*, Trans. Amer. Math. Soc. **287** (1985) 49-63.

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