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THE PERTURBATION FOR THE DRAZIN INVERSE

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ABSTRACT. A representation for the Drazin inverse of an arbitrary square matrix in terms of the eigenprojection was established by Rothblum [*SIAM J. Appl. Math.*, 31(1976):646-648]. In this paper perturbation results based on the representation for the Darzin inverse $A^D = (A - X)^{-1}(I - X)$ are developed. Norm estimates of $\|(A+E)^D - A^D\|_2/\|A^D\|_2$ and $\|(A+E)^\# - A^D\|_2/\|A^D\|_2$ are derived when $\|E\|_2$ is small.

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1. Introduction

Let A be an square complex matrix. The index of A, written ind(A), is the small nonnegative integer such that rank(A^{k+1}) =rank(A^k). The Drazin inverse [2] of A, denoted by A^D, is the unique matrix Z satisfying the following relations

$$ZAZ = Z,\ AZ = ZA,\ A^{k+1}Z = A^k, \tag{1}$$

where k =ind(A). Particularly when ind$(A) = 1$, the matrix Z satisfying (1) is called the group inverse of A and denoted by $Z = A^\#$.

The Drazin inverse is very useful since various applications such that singular differential difference equations, Markov chains, cryptography, iterative methods, and multibody system dynamics were found in the literature [3, 4, 5, 6, 7, 8, 9]. G. W. Stewart [10] stated as early as in 1969 the continuity properties of the generalized inverse under certain circumstances. In 1975, Campbell and Meyer [11] also established the continuity properties of Darzin inverse and formulated a difficult problem to establish norm estimates.

It is well known that in general the Drazin inverse is unstable with respect to perturbation. However, under some specific perturbation E, the closeness of the matrices $(A + E)^D$ and A^D can be proved and the explicit error bound can be obtained from [12, 13]. A perturbation theory based on the Jordan canonical

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form of A was developed by Y. Wei [14]. Norm estimates of $\|(A+E)^{\#}-A^{D}\|$ were derived when A^D and $A+E$ were of same rank and $\|E\|$ was small.

The representation of the Drazin inverse in terms of its eigenprojection $X = I - AA^D$ was established in [1]:

$$A^D = (A-X)^{-1}(I-X) = (I-X)(A-X)^{-1}.$$

In this paper we present explicit bounds for $\|(A+E)^D - A^D\|_2/\|A^D\|_2$ and $\|(A+E)^{\#} - A^D\|_2/\|A^D\|_2$ in terms of A, $A^D = (A-X)^{-1}(I-X) = (I-X)(A-X)^{-1}$, and $E = B - A$.

The main notations of this paper are as follows. C^n is an n-dimensional complex vector space. $C_r^{n\times n}$ represents a set of all $n\times n$ complex matrices with rank r, while rank(A) denotes the rank of A. $X = I - AA^D$ and $Y = I - BB^D$ stand for the eigenprojections A and B, respectively. $B^{\#} = (A+E)^{\#}$ denotes the group inverse of B.

2. The case $ind(B) \neq 1$

In this section we develop the perturbation theory which is based on the representation for the Drazin inverse $A^D = (A-X)^{-1}(I-X)$ by establishing a new norm estimate for $\|(A+E)^D - A^D\|_2/\|A^D\|_2$.

Lemma 2.1 [15]. *Suppose $A \in C^{n\times n}$ with* $\mathrm{ind}(A) = k$ *and* $\mathrm{rank}(A^k) = r$. *Then there exists a unique matrix X such that*

$$A^kX = 0,\ XA^k = 0,\ X^2 = X,\ rank(X) = n - r$$

and a unique matrix Z such that

$$rank\begin{bmatrix} A & I-X \\ I-X & Z \end{bmatrix} = rank(A).$$

The matrix Z is the Drazin inverse A^D of A. Further, we have

$$X = I - AA^D = I - A^DA.$$

Lemma 2.2 [16]. *Suppose $\|F\|_2 < 1$. Then $I+F$ is nonsingular and $\|(I+F)^{-1}\|_2 \le \dfrac{1}{1-\|F\|_2}$.*

Applying Lemma 2.1 and Lemma 2.2, we obtain a lower bounding for $\|A^D\|_2$ as follows.

Theorem 2.3. *Suppose $A^D = (A-X)^{-1}(I-X)$ with ind(A)=k. Then*

$$\|A^D\|_2 \ge \frac{1-\sum\limits_{i=0}^{k-1}\|A\|_2^{i+1}\|X\|_2}{\|A\|_2}.$$

Proof. By Lemma 2.1 and $XA^D = A^DX = 0$, we have

$$\begin{aligned}(A - X)(A^D - A^{k-1}X - \cdots - AX - X) &= AA^D + X + (X - I)A^{k-1}X \\ &\quad + \cdots + (X - I)AX \\ &= AA^D + X = I.\end{aligned}$$

That is,

$$(A - X)^{-1} = A^D - A^{k-1}X - A^{k-2}X - \cdots - AX - X. \tag{2}$$

By the representation of the Drazin inverse $A^D = (A - X)^{-1}(I - X)$, where X is the eigenprojection of A such that $A^D = (A - X)^{-1}AA^D$, we have

$$\begin{aligned}\|A^D\|_2 &\leq \|(A - X)^{-1}\|_2\|A\|_2\|A^D\|_2 \\ &\leq (\|A^D\|_2 + \sum_{i=0}^{k-1}\|A\|_2^i\|X\|_2)\|A\|_2\|A^D\|_2.\end{aligned}$$

Hence, $\left(\|A^D\|_2 + \sum_{i=0}^{k-1}\|A\|_2^i\|X\|_2\right)\|A\|_2 \geq 1$. Therefore

$$\|A^D\|_2 \geq \frac{1 - \sum_{i=0}^{k-1}\|A\|_2^{i+1}\|X\|_2}{\|A\|_2}.$$

□

Using the representation of the Drazin inverse and the expansion of $(B-Y)^{-1}$, we have an upper bound for $\|B^D - A^D\|_2/\|A^D\|_2$ as follows.

Theorem 2.4. *Let $B = A + E$ with ind$(B) = l$ and $\|A^D\|_2\|E\|_2 < 1$, where $E = AA^DE$. Then*

$$\frac{\|B^D - A^D\|_2}{\|A^D\|_2} \leq \frac{(1 + \sum_{i=0}^{l-1}\|B\|_2^{i+1})(1 - \|B\|_2\|B^D\|_2) - \|A^D\|_2\|E\|_2}{1 - \|A^D\|_2\|E\|_2}.$$

Proof. By the assumption $E = AA^DE$, we have $B = A(I + A^DE)$. Thus

$$B^D = (I + A^DE)^{-1}A^D.$$

Using the representation of the Drazin inverse and (2), we obtain

$$\begin{aligned}B^D &= (B - Y)^{-1}(I - Y) \\ &= (B^D - B^{l-1}Y - \cdots - BY - Y)(I - Y) \\ &= (B^D - B^{l-1}Y - \cdots - BY - Y)B(I + A^DE)^{-1}A^D.\end{aligned}$$

Now

$$B^D - A^D = [(B^D - B^{l-1}Y - \cdots - BY - Y)B(I + A^DE)^{-1} - I]A^D.$$

Taking the norm of both sides gives

$$\begin{aligned}\|B^D - A^D\|_2 &\leq [(\|B^D\|_2 + \sum_{i=0}^{l-1} \|B\|_2^i \|Y\|_2)\|B\|_2 \|(I + A^D E)^{-1}\|_2 + 1]\|A^D\|_2 \\ &\leq \frac{(1 + \sum_{i=0}^{l-1} \|B\|_2^{i+1})(1 - \|B\|_2 \|B^D\|_2) - \|A^D\|_2 \|E\|_2}{1 - \|A^D\|_2 \|E\|_2} \|A^D\|_2.\end{aligned}$$

Therefore

$$\frac{\|B^D - A^D\|_2}{\|A^D\|_2} \leq \frac{(1 + \sum_{i=0}^{l-1} \|B\|_2^{i+1})(1 - \|B\|_2 \|B^D\|_2) - \|A^D\|_2 \|E\|_2}{1 - \|A^D\|_2 \|E\|_2}.$$

□

Actually, the bound exists a drawback, that is, sometimes B^D is very sensitive to a small perturbation E so that the bound is also very sensitive. By the equation $B^D - A^D = -B^D E A^D + B^D X - Y A^D$, we can also obtain the following bound to deal with the drawback.

Theorem 2.5. *Suppose* A, $B = A + E \in C^{n\times n}$, $\|A^D\|_2 \|E\|_2 < 1$, *where* $E = AA^D E$. *Then*

$$\frac{\|B^D - A^D\|_2}{\|A^D\|_2} \leq \frac{2 + 2\|A\|_2 \|A^D\|_2 + \|A^D\|_2 \|E\|_2}{1 - \|A^D\|_2 \|E\|_2}. \tag{3}$$

Proof. By the assumption, we have $B^D = (I + A^D E)^{-1} A^D$ and

$$B^D - A^D = -B^D E A^D + B^D X + Y A^D,$$

where

$$X = I - AA^D \text{ and } Y = I - BB^D$$

are the eigenprojections of A and B, respectively. Taking the norm of both sides gives

$$\begin{aligned}\|B^D - A^D\|_2 &\leq \|B^D\|_2 \|E\|_2 \|A^D\|_2 + \|B^D\|_2 \|X\|_2 + \|Y\|_2 \|A^D\|_2 \\ &\leq \|(I + A^D E)^{-1}\|_2 \|A^D\|_2 \|E\|_2 \|A^D\|_2 \\ &\quad + \|(I + A^D E)^{-1}\|_2 \|A^D\|_2 \|X\|_2 + \|Y\|_2 \|A^D\|_2 \\ &\leq \frac{\|A^D\|_2^2}{1 - \|A^D\|_2 \|E\|_2} \|E\|_2 + \frac{\|A^D\|_2}{1 - \|A^D\|_2 \|E\|_2} \|X\|_2 + \|Y\|_2 \|A^D\|_2 \\ &\leq \frac{\|A^D\|_2}{1 - \|A^D\|_2 \|E\|_2} (2 + 2\|A\|_2 \|A^D\|_2 + \|A^D\|_2 \|E\|_2).\end{aligned}$$

Therefore

$$\frac{\|B^D - A^D\|_2}{\|A^D\|_2} \leq \frac{2 + 2\|A\|_2 \|A^D\|_2 + \|A^D\|_2 \|E\|_2}{1 - \|A^D\|_2 \|E\|_2}.$$

□

Comparing the result of Theorem 2.4 and that of Theorem 2.5 we notice that B^D vanishes in the inequality (3) so that the drawback also disappears.

3. The case $ind(B) = 1$

When the index of matrix B equals to 1, we can obtain the following equation

$$BB^{\#}B = B.$$

Applying this property, in this section we develop the perturbation theory which is based on the representation for the Drazin inverse $A^D = (A - X)^{-1}(I - X)$ by establishing a new norm estimate for $\|(A + E)^{\#} - A^D\|_2/\|A^D\|_2$. We first recall the following lemma presented by Y. Wei [14].

Lemma 3.1 [14]. *Let $B = A + E$ with* $\mathrm{ind}(A) = k$. *Then*

$$ind(B) = 1 \quad and \quad B^{\#} = (I + A^D E)^{-1} A^D = A^D (I + E A^D)^{-1}$$

if and only if

$$rank(B) = rank(A^k) \quad and \quad AA^D E = EAA^D = A - A^2 A^D + E.$$

By Lemma 2.1 and the representation of the Drazin inverse, $A^D = (A^{k+1})^D A^k$ in [1], we have the following theorem.

Theorem 3.2. *Let $B = A + E$ with* $\mathrm{ind}(A) = k$, $\mathrm{rank}(B) = rank(A^k)$ *and* $AA^D E = EAA^D = A - A^2 A^D + E$. *If* $\|A^D\|_2\|E\|_2 < 1$, *then*

$$\|B^{\#}\|_2 \leq \frac{\|A^D\|_2}{1 - \|A^D\|_2\|E\|_2}(\|A\|_2\|A^D\|_2)^k.$$

The proof is conspicuous that we ignore it here.

Theorem 3.3. *Let $B = A + E$ with* $\mathrm{ind}(A) = k$, $\mathrm{rank}(B) = rank(A^k)$ *and* $AA^D E = EAA^D = A - A^2 A^D + E$. *If* $\|A^D\|_2\|E\|_2 < 1$, *then*

$$\frac{\|B^{\#} - A^D\|_2}{\|A^D\|_2} < \frac{(2 + \|A^D\|_2\|A\|_2)(1 + \|A\|_2) + (1 - \|A^D\|_2)\|E\|_2}{(1 - \|A^D\|_2\|E\|_2)^2}.$$

Proof. For $\mathrm{ind}(B) = 1$, $BB^{\#}B = B$ holds. By Lemma 2.1, we have

$$(B - Y)(B^{\#} - Y) = BB^{\#} - BY - YB^{\#} + Y^2 = BB^{\#} + Y = I.$$

That is, $(B - Y)^{-1} = B^{\#} - Y$. Now

$$\begin{aligned} B^{\#} &= (B - Y)^{-1}(I - Y) = (B^{\#} - Y)(I - Y) \\ &= (B^{\#} - I + BB^{\#})BB^{\#} \\ &= [(A + E + I)(I + A^D E)^{-1} A^D - I](A + E)(I + A^D E)^{-1} A^D. \end{aligned}$$

Hence,

$$B^{\#} - A^D = [(A + E + I)(I + A^D E)^{-1} A^D - I](A + E)(I + A^D E)^{-1} A^D - A^D.$$

Taking the norm of both sides gives

$$\|B^{\#} - A^D\|_2 \le \left[\frac{(\|A\|_2 + \|E\|_2 + 1)\|A^D\|_2}{1 - \|A^D\|_2\|E\|_2} + 1\right] \frac{(\|A\|_2 + \|E\|_2)\|A^D\|_2}{1 - \|A^D\|_2\|E\|_2} + \|A^D\|_2.$$

Therefore

$$\begin{aligned}\frac{\|B^{\#} - A^D\|_2}{\|A^D\|_2} &\le \left[\frac{(\|A\|_2 + \|E\|_2 + 1)\|A^D\|_2}{1 - \|A^D\|_2\|E\|_2} + 1\right] \frac{\|A\|_2 + \|E\|_2}{1 - \|A^D\|_2\|E\|_2} + 1 \\ &= \frac{1 + \|A^D\|_2^2\|E\|_2^2 - \|A^D\|_2\|E\|_2 + \|A\|_2}{(1 - \|A^D\|_2\|E\|_2)^2} \\ &\quad + \frac{\|A^D\|_2\|A\|_2 + \|A^D\|_2\|A\|_2^2 + \|A^D\|_2\|A\|_2\|E\|_2 + \|E\|_2}{(1 - \|A^D\|_2\|E\|_2)^2} \\ &< \frac{2 - \|A^D\|_2\|E\|_2 + 2\|A\|_2}{(1 - \|A^D\|_2\|E\|_2)^2} \\ &\quad + \frac{\|A^D\|_2\|A\|_2 + \|A^D\|_2\|A\|_2^2 + \|E\|_2}{(1 - \|A^D\|_2\|E\|_2)^2} \\ &= \frac{(2 + \|A^D\|_2\|A\|_2)(1 + \|A\|_2) + (1 - \|A^D\|_2)\|E\|_2}{(1 - \|A^D\|_2\|E\|_2)^2}.\end{aligned}$$

□

Based on $A^D = A^k(A^{2k+1})^+A^k$, G. Rong [12] gave the bounds of $\|B^D - A^D\|_2/\|A^D\|_2$:

$$\frac{\|B^D - A^D\|_2}{\|A^D\|_2} \le C(A)\frac{\|E\|_2}{\|A\|_2} + o(\|E\|_2),$$

where $C(A) = \left[2\sum_{i=0}^{k-1} \|(A^D)^{i+1}\|_2\|A^i\|_2(1 + \|A\|_2\|A^D\|_2) + \|A^D\|_2\right]\|A\|_2$.

Based on the Jordan canonical form of A, Y. Wei [14] established the bound of $\|B^{\#} - A^D\|_2/\|A^D\|_2$:

$$\begin{aligned}\frac{\|B^{\#} - A^D\|_2}{\|A^D\|_2} &\le \frac{(1 - \|A^D\|_2\|E\|_2)(\|A^D\|_2\|E\|_2 + \|AA^D\|_2)}{(1 - \|A^D\|_2\|E\|_2 - \|A^D\|_2\|E\|_2\|AA^D\|_2)^2} \\ &\quad + \frac{1 - \|A^D\|_2\|E\|_2 + \|A^D\|_2\|E\|_2\|AA^D\|_2}{1 - \|A^D\|_2\|E\|_2 - \|A^D\|_2\|E\|_2\|AA^D\|_2}\|AA^D\|_2.\end{aligned}$$

The results of this paper are based on the representation of the Drazin inverse $A^D = (A - X)^{-1}(I - X)$, where X is the eigenprojection of A. Based on the different representations of the Drazin inverse, it is obvious that the different bounds are obtained.

Actually, the similar bounds as those of this paper can be also presented by applying an arbitrary consistent matrix norm which $\|I_i\| = 1$ is satisfied. However, we omit here.

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REFERENCES

1. U. G. Rothblum, *A representation of the Drazin inverse and characterizations of the index*, SIAM J. Appl. Math. **31** (1976), 646-648.
2. M. P. Darzin, *Pesudoinverses in associative rings and semigroup*, Amer. Math. Monthly. **65** (1958), 506-514.
3. S. L. Campbell, C. D. Meyer, Jr. and N. J. Rose, *Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients*, SIAM J. Appl. Math. **31** (1976), 411-425.
4. R. E. Hartwig and J. Levine, *Applications of the Drazin inverse to the hill cryptographic system*, Part III, Cryptologia **5** (1975), 443-464.
5. C. D. Meyer and Jr., *The role of the group generalized inverse in the theory of finite Markov chains*, SIAM Rev. **17** (1975), 443-464.
6. C. D. Meyer, Jr. and R. J. Plemmons, *Convergent powers of a matrix with applications to iterative methods for singular linear systems*, SIAM J. Numer. Anal. **14** (1977), 699-705.
7. U. G. Rothblum, *Multiplicative Markov decision chains*, Ph.D. dissertation, Stanford University, Stanford, CA, 1974.
8. B. Simeon, C. Fuhrer and P. Rentrop, *The Drazin inverse in multibody system dynamics*, Numer. Math. **64** (1993), 521-539.
9. G. R. Wang, *A cramer rule for finding the solution of a class of singular equations*, Linear Algebra Appl. **116** (1989), 27-34.
10. G. W. Stewart, *On the continuity of the generalizied inverse*, SIAM J. Appl. Math. **17** (1969), 33-45.
11. S. L. Campbell and C. D. Meyer Jr., *Continuity properties of the Drazin pesudoinverse*, Linear Algebra Appl. **10** (1975), 77-83.
12. G. Rong, *The error bound of the perturbation of the Drazin inverse*, Linear Algebra Appl. **47** (1982), 159-168.
13. Y. Wei and G. Wang, *The perturbation theory for the Drazin inverse and its applications*, Linear Algebra Appl. **258** (1997), 179-186.
14. Y. Wei, *Perturbation bound of the Drazin inverse*, Appl. Math. Comput. **125** (2002), 231-244.
15. Y. Wei, *A characterization and representation of the Drazin inverse*, SIAM J. Matrix Anal. Appl. **17** (1996), 744-747.
16. A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Wiley, New York, 1974.

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