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MIXED TYPE MULTIOBJECTIVE VARIATIONAL PROBLEMS WITH HIGHER ORDER DERIVATIVES

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ABSTRACT. A mixed type dual for multiobjective variational problem involving higher order derivatives is formulated and various duality results under generalized invexity are established. Special cases are generated and it is also pointed out that our results can be viewed as a dynamic generalization of existing results in the static programming.

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1. Introduction

Calculus of variation, whose importance is fast growing in science and engineering, basically concerns with finding an optimal of a definite integral involving a certain function subject to fixed point boundary conditions. Hanson [5] pointed out that some of the duality results of mathematical programming have been analogous in variational calculus. Exploring this relationship between mathematical programming and the classical calculus of variations, Mond and Hanson [10] formulated a contrained variational problem as a mathematical programming problem and using Valentine's [12] optimality conditions for the same, presented its Wolfe type dual variational problem for validating various duality results under convexity. Later Bector, Chandra and Husain [1] studied Mond-Weir type duality for the problem of [10] for weakening its convexity requirement.

Husain and Jabeen [6] studied a wider class of variational problems in which the arc function is twice differentiable by extending the notion of invexity given in [11]. They obtained Fritz-John as well as Karush-Kuhn-Tucker necessary optimality conditions and an application of Karush-Kuhn-Tucker optimality conditions formulated Wolfe and Mond-Weir type models in [7] and [8] respectively.

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In this research we propose, in the spirit of Xu [13], a mixed type dual for a wider class of variational problems involving higher order derivatives is constructed with a view to unify the duality results of [7] and [8], under invexity and generalized invexity conditions. It is pointed out that our results can be considered as dynamic generalizations of the results of Xu [13].

2. Pre-requisites and primal problem

Consider the real interval $I = [a, b]$, and the continuously differentiable function $\phi : I \times R^n \times R^n \times R^n \to R$, where x is twice differentiable with its first and second order derivatives $\dot{x}$ and $\ddot{x}$ respectively. If $x = (x^1, x^2, \ldots, x^n)^T$, the gradient vectors of f with respect to $x, \dot{x}$ and $\ddot{x}$ respectively denoted by

$$\phi_x = \left[\frac{\partial\phi}{\partial x^1}, \ldots, \frac{\partial\phi}{\partial x^n}\right]^T, \phi_{\dot{x}} = \left[\frac{\partial\phi}{\partial \dot{x}^1}, \ldots, \frac{\partial\phi}{\partial \dot{x}^n}\right]^T, \phi_{\ddot{x}} = \left[\frac{\partial\phi}{\partial \ddot{x}^1}, \ldots, \frac{\partial\phi}{\partial \ddot{x}^n}\right]^T.$$

Invexity was introduced for functions in variational problems by Mond, Chandra and Husain [11] while Mond and Smart [9] defined invexity for functionals instead of functions. The authors in [6] introduced the extended forms of definitions of invexity and various generalized invexity for functional in variational problems involving higher order derivatives, which are mentioned below for ready reference.

Definition 1 (*Invexity*). If there exists vector function $\eta(t, \dot{u}, \ddot{u}, x, \dot{x}, \ddot{x}) \in R^n$ with $\eta = 0$ and $x(t) = u(t), t \in I$ and $D\eta = 0$ for $\dot{x}(t) = \dot{u}(t), t \in I$ such that for a scalar function $\phi(t, x, \dot{x}, \ddot{x})$, the functional $\Phi(x, \dot{x}, \ddot{x}) = \int_I \phi(t, x, \dot{x}, \ddot{x})dt$ satisfies

$$\begin{aligned}&\Phi(x, \dot{u}, \ddot{u}) - \Phi(x, \dot{x}, \ddot{x})\\ &\geqq \int_I \{\eta^T \phi_x(t, x, \dot{x}, \ddot{x}) + (D\eta)^T \phi_{\dot{x}}(t, x, \dot{x}, \ddot{x}) + (D^2\eta)^T \phi_{\ddot{x}}(t, x, \dot{x}, \ddot{x})\}dt,\end{aligned}$$

Φ is said to be invex in $x, \dot{x}$ and $\ddot{x}$ on I with respect to η.

Definition 2 (*Pseudoinvexity*). Φ is said to be pseudoinvex in $x, \dot{x}$ and $\ddot{x}$ with respect to η if

$$\int_I \{\eta^T \phi_x(t, x, \dot{x}, \ddot{x}) + (D\eta)^T \phi_{\dot{x}}(t, x, \dot{x}, \ddot{x}) + (D^2\eta)^T \phi_{\ddot{x}}(t, x, \dot{x}, \ddot{x})\}dt \geqq 0$$

implies $\Phi\,(x, \dot{u}, \ddot{u}) \geqq \Phi\,(x, \dot{x}, \ddot{x})$.

Definition 3 (*Quasi-invex*). The functional Φ is said to quasi-invex in $\dot{x}$ and $\ddot{x}$ with respect to η if

$$\Phi(x, \dot{u}, \ddot{u}) \leqq \Phi(x, \dot{x}, \ddot{x})$$

implies

$$\int_I \{\eta^T \phi_x(t, x, \dot{x}, \ddot{x}) + (D\eta)^T \phi_{\dot{x}}(t, x, \dot{x}, \ddot{x}) + (D^2\eta)^T \phi_{\ddot{x}}(t, x, \dot{x}, \ddot{x})\}dt \leqq 0.$$

The following conventions for vectors in R^n will be used throughout the research.

$$x < y, \quad \Leftrightarrow \quad x_i < y_i, \quad i = 1, 2, \ldots, n.$$

$$x \leqq y, \quad \Leftrightarrow \quad x_i \leqq y_i, \quad i = 1, 2, \ldots, n.$$

$$x \leq y, \quad \Leftrightarrow \quad x_i \leq y_i, \quad i = 1, 2, \ldots, n, \text{ but } x \neq y$$
$$x \nleq y, \text{ is the negation of } x \leq y$$

In [7, 8], the following multiobjective variational problems involving higher order derivatives are considered.

$$\text{(VPE)} \quad \text{Minimize}\left(\int_I f^1(t,x,\dot{x},\ddot{x})dt, \int_I f^2(t,x,\dot{x},\ddot{x})dt, \ldots, \int_I f^p(t,x,\dot{x},\ddot{x})dt\right)$$

Subject to

$$x(a) = 0 = x(b)\,, \quad \dot{x}(a) = 0 = \dot{x}(b)\,,$$
$$g(t,x,\dot{x},\dot{x}) \leqq 0, \quad t \in I,$$
$$h(t,x,\dot{x},\dot{x}) = 0, \quad t \in I\,.$$

$$\text{(VP)} \quad \text{Minimize}\left(\int_I f^1(t,x,\dot{x},\ddot{x})dt, \ldots, \int_I f^p(t,x,\dot{x},\ddot{x})dt\right)$$

Subject to

$$x(a) = 0 = x(b)\,, \quad \dot{x}(a) = 0 = \dot{x}(b)\,,$$
$$g(t,x,\dot{x},\ddot{x}) \leqq 0, \quad t \in I\,,$$

where

(1) $I = [a,b]$ is a real interval,
(2) $f^i : I \times R^n \times R^n \times R^n \to R$, $(i = 1, 2, \ldots, p)$, $g : I \times R^n \times R^n \times R^n \to R^m$ and $h : I \times R^n \times R^n \times R^n \to R^k$ are continuously differentiable functions, and
(3) X designates the space of piecewise functions $x : I \to R^n$ possessing derivatives $\dot{x}$ and $\ddot{x}$ with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)ds,$$

where α is given a boundary value; thus $D \equiv \frac{d}{dt}$ except at discontinuities.

In the following theorem, $C(I, R^m)$ denotes the space of continuous functions $\phi : I \to R^k$ with the uniform norm $\|\phi\| = \sup|\phi|_{t\in I}$. The partial derivatives of g and h are $m \times n$ and $k \times n$ matrices respectively; superscript T denotes the matrix transpose.

We require the following definition of efficient solution for our further analysis.

Definition 4 (*Efficient Solution*). A feasible solution $\bar{x}$ is efficient for (VPE) if there exist no other feasible x for (VP) such that for some $i \in P = \{1, 2, \ldots, p\}$,

$$\int_I f^i\left(t, x, \dot{x}, \ddot{x}\right) dt < \int_I f^i\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) dt$$

and

$$\int_I f^j\left(t, x, \dot{x}, \ddot{x}\right) dt \leqq \int_I f^i\left(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}\right) dt \quad \text{for all } j \in P, j \neq i.$$

In relation to (VPE), we introduce the following set of problems $(\bar{\mathrm{P}}_{\mathrm{r}})$ for each $r = 1, 2, \ldots, p$ in the spirit of [4], with a single objective,

$$(\mathrm{P_r}) \qquad \text{Minimize} \int_I f^r(t, x, \dot{x}, \ddot{x})dt$$

Subject to

$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b),$$
$$g(t, x, \dot{x}, \ddot{x}) \leqq 0, \quad t \in I,$$
$$h(t, x, \dot{x}, \ddot{x}) = 0, \quad t \in I,$$
$$\int_I f^i(t, x, \dot{x}, \ddot{x})dt \leqq \int_I f^i(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}})dt, \quad i = 1, 2, \ldots, p, \ i \neq r.$$

The following lemma can be proved on the lines of Chankong and Haimes [4].

Lemma 1. *x^* is an efficient solution of* (VPE) *if and only if $\bar{x}$ is an optimal solution of $(\bar{\mathrm{P}}_{\mathrm{r}})$ for each $r = 1, 2, \ldots, p$.*

Consider the following single objective variational problem considered in [6].

$$(\mathrm{P_0}) \qquad \textit{Minimize} \int_I \phi(t, x, \dot{x}, \ddot{x})dt$$

Subject to

$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b),$$
$$g(t, x, \dot{x}, \ddot{x}) \leqq, \quad t \in I,$$
$$h(t, x, \dot{x}, \ddot{x}) = 0, \quad t \in I,$$

where $\phi : I \times R^n \times R^n \times R^n \to R$.

The authors in [7] established the Fritz John necessary and Karush-Kuhn-Tucker type optimality conditions for the above problem (VPE) which are quoted in the form of the following proposition for easy reference.

Proposition 1 (*Fritz John necessary optimality conditions*). *Let $\bar{x}$ be an efficient solution of* (VPE) *and $h_x(x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot))$ maps X into the subspace of $C(I, R^k)$. Then there exist $\bar{\lambda} \in R^k$ and the piecewise smooth $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^k$, such that*

$$(\bar{\lambda} f_x + \bar{y}(t)^T g_x + \bar{z}(t)^T h_x) - D(\bar{\lambda} f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}})$$
$$+ D^2(\bar{\lambda} f_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}} + \bar{z}(t)^T h_{\ddot{x}}) = 0, \quad t \in I$$
$$\bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0, \quad t \in I$$
$$(\bar{\lambda}, \bar{y}(t)) \geqq 0, \quad (\bar{\lambda}, \bar{y}(t), \bar{z}(t)) \neq 0, \quad t \in I.$$

The above proposition will be used to establish converse duality theorem.

Proposition 2 (*Karush-Kuhn-Tucker Conditions*). *Let $\bar{x}$ be an efficient solution for* (VPE) *which is assumed to be normal for $(\bar{\mathrm{P}})_{\mathrm{r}}$ for each $r = 1, 2, \ldots, p$. Let the constraints of $(\bar{\mathrm{P}}_{\mathrm{r}})$ satisfy the Slater's Constraint Qualification* [3] *for each $r = 1, 2, \ldots, p$. Then there exist $\bar{\lambda}^T \in R^k_+$, $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^k$,* *such that the following relations hold for all $t \in I$,*

$$(\bar{\lambda}^T f_x + \bar{y}(t)^T g_x + \bar{z}(t)^T h_x) - D(\bar{\lambda}^T f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}})$$
$$+ D^2(\bar{\lambda}^T f_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}} + \bar{z}(t)^T h_{\ddot{x}}) = 0, t \in I,$$
$$\bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0, \quad \bar{\lambda} > 0, y(t) \geqq 0, t \in I.$$

In [7, 8], the authors formulated Wolfe and Mond-Weir type duals to the above variational problem (VP) and proved various duality results under invexity and generalized invexity conditions:

(WD) $\quad$ Maximize $\left(\int_I (f^1(t, u, \dot{u}, \ddot{u})dt + y(t)^T g(t, u, \dot{u}, \ddot{u}))\, dt, \ldots, \int_I (f^p(t, u, \dot{u}, \ddot{u})dt + y(t)^T g(t, u, \dot{u}, \ddot{u}))dt \right)$

Subject to

$$u(a) = 0 = u(b)\,, \quad \dot{u}(a) = 0 = \dot{u}(b)\,,$$
$$(\lambda f_u + y(t)^T g_u) - D(\lambda f_{\dot{u}} + y(t)^T g_{\dot{u}}) + D^2(\lambda f_{\ddot{u}} + y(t)^T g_{\ddot{u}}) = 0,$$
$$y(t) \geqq 0, \quad t \in I\,, \quad \lambda \in \Lambda^+\,,$$

where $\Lambda^+ = \{\lambda \in R^p | \lambda > 0, \lambda^T e = 1, e = (1, 1, \ldots, 1)^T\}$

(M-WD) $\quad$ Maximize $\left(\int_I f^1(t, u, \dot{u}, \ddot{u})dt, \ldots, \int_I f^p(t, u, \dot{u}, \ddot{u})dt \right)$

Subject to

$$u(a) = 0 = u(b)\,, \quad \dot{u}(a) = 0 = \dot{u}(b)\,,$$
$$(\lambda f_u + y(t)^T g_u) - D(\lambda f_{\dot{u}} + y(t)^T g_{\dot{u}}) + D^2(\lambda f_{\ddot{u}} + y(t)^T g_{\ddot{u}}) = 0,$$
$$\int_I y\,(t)^T g\,(t, u, \dot{u}, \ddot{u})\, dt \geqq 0\,, \quad y\,(t) \geqq 0, t \in I\,, \quad \lambda > 0\,.$$

3. Mixed type multiobjective duality

Following is the mixed type dual for multiobjective variational problem.

(Mix VD) $\quad$ Maximize $\left(\int_I \left(f^1(t, u, \dot{u}, \ddot{u})dt + \sum_{j \in I_\circ} y^j\,(t)^T g^j\,(t, u, \dot{u}, \ddot{u}) \right) dt, \ldots, \int_I \left(f^p(t, u, \dot{u}, \ddot{u})dt + \sum_{j \in I_\circ} y^j\,(t)^T g^j\,(t, u, \dot{u}, \ddot{u}) \right) dt \right)$

Subject to

$$x(a) = 0 = x(b)\,, \tag{3.1}$$
$$\dot{x}(a) = 0 = \dot{x}(b)\,, \tag{3.2}$$
$$(\lambda f_u + y(t)^T g_u) - D(\lambda f_{\dot{u}} + y(t)^T g_{\dot{u}}) + D^2(\lambda f_{\ddot{u}} + y(t)^T g_{\ddot{u}}) = 0,\ t \in I\,, \tag{3.3}$$

$$\sum_{j\in I_\alpha}\int_I y^j(t)^T g^j(t,u,\dot{u},\ddot{u})dt \geqq 0,\ \alpha=1,2,\ldots,r, \tag{3.4}$$

$$y(t)\geqq 0,\quad t\in I, \tag{3.5}$$

$$\lambda\in\Lambda^+ \tag{3.6}$$

where $I_\alpha \subseteq M=\{1,2,\ldots,m\}$, $\alpha=1,2,\ldots,r$ with $\bigcup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \bigcap I_\beta=\phi$, if $\alpha\neq\beta$. If $I_\circ = M$, then (Mix VD) becomes (WD) considered in [7]. If $I_\circ=\phi$ for $I_\alpha = M$ (for some $\alpha\in\{1,2,\ldots,r\}$, then the (Mix VD) becomes the problem (M-WD) considered in [8].

Theorem 1 (*Weak duality*). *Let $x\in X$ for feasible* (VP) *and (u,y,λ) be feasible for* (Mix VD). *If, for all feasible (x,u,y,λ)*

$$\int_I (\lambda^T f(t,u,\dot{u},\ddot{u})dt + \sum_{j\in I_\circ} y^j(t)^T g^j(t,u,\dot{u},\ddot{u}))dt$$

is pseudoinvex and

$$\sum_{j\in I_\circ}\int_I y^j(t) g^j(t,u,\dot{u},\ddot{u})dt,\quad \alpha=1,2,\ldots,r$$

is quasi-invex with respect to same η, then,

$$\int_I f(t,x,\dot{x},\ddot{x})dt \nleq \int_I (f(t,u,\dot{u},\ddot{u})dt + \sum_{j\in I_\circ} y^j(t)^T g^j(t,u,\dot{u},\ddot{u})e)dt$$

Proof. The relations $g(t,x,\dot{x},\dot{x})\leqq 0$ and $y(t)\geqq 0, t\in I$ imply

$$\sum_{j\in I_\circ}\int_I y^j(t)g^j(t,x,\dot{x},\ddot{x})dt \leqq \sum_{j\in I_\circ}\int_I y^j(t)g^j(t,u,\dot{u},\ddot{u})dt$$

This because of the quasi-invexity of $\sum_{j\in I_\circ}\int_I y^j(t)g^j(t,u,\dot{u},\ddot{u})dt$, $\alpha=1,2,\ldots,r$, implies,

$$\begin{aligned} 0 \ \geqq\ & \sum_{j\in I_\circ}\int_I \{\eta^T y^j(t)g^j(t,u,\dot{u},\ddot{u})dt + (D\eta)^T y^j(t) g^j_{\dot{u}}(t,u,\dot{u},\ddot{u}) \\ & + (D^2\eta)^T y^j(t) g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u})\}dt,\quad \alpha=1,2,\ldots,r \end{aligned}$$

This by integrating by parts gives,

$$\begin{aligned} 0 \ \geqq\ & \sum_{j\in I_\alpha}\Bigg[\int_I \eta^T y^j(t)g^j(t,u,\dot{u},\ddot{u})dt - \eta^T y^j(t)g^j_{\dot{u}}(t,u,\dot{u},\ddot{u})\Big|_{t=a}^{t=b} \\ & - \int_I \eta^T D(y^j(t)g^j_{\dot{u}}(t,u,\dot{u},\ddot{u}))dt + (D\eta)^T y^j(t)g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u})\Big|_{t=a}^{t=b} \\ & - \int_I \eta^T D(y^j(t)g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u}))dt\Bigg],\quad \alpha=1,2,\ldots,r \end{aligned}$$

Integrating by parts and using the boundary conditions which at $t = a$, $t = b$ give $D\eta = 0 = \eta$, from this we have,

$$\begin{aligned} 0 \geqq & \sum_{j\in I_\alpha} \Bigg[\int_I \eta^T y^j(t) g^j(t,u,\dot{u},\ddot{u})dt - \int_I \eta^T D(y^j(t) g^j_{\dot{u}}(t,u,\dot{u},\ddot{u}))dt \\ & -\eta^T D(y^j(t) g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u}))\Bigg|_{t=a}^{t=b} + \int_I \eta^T D^2(y^j(t) g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u}))dt \Bigg] \end{aligned}$$

again using the boundary conditions which at $t = a$, $t = b$ give $D\eta = 0 = \eta$

$$\begin{aligned} 0 \geqq & \sum_{j\in M\setminus I_\circ} \Bigg[\int_I \eta^T \{ y^j(t) g^j(t,u,\dot{u},\ddot{u})dt - D(y^j(t) g^j_{\dot{u}}(t,u,\dot{u},\ddot{u})) \\ & -D^2(y^j(t) g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u}))\} dt \Bigg] \end{aligned} \tag{3.7}$$

Using equations (3) and (7) we have

$$\begin{aligned} 0 \leqq & \int_I \eta^T \Bigg\{ (\lambda^T f_u(t,u,\dot{u},\ddot{u})) + \sum_{j\in I_\circ} y^j(t) g^j_{\dot{u}}(t,u,\dot{u},\ddot{u}) \\ & -D\Bigg(\lambda^T f_u(t,u,\dot{u},\ddot{u}) + \sum_{j\in I_\circ} y^j(t) g^j_{\dot{u}}(t,u,\dot{u},\ddot{u}) \Bigg) dt \\ & +D^2 \Bigg(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + \sum_{j\in I_\circ} y^j(t) g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \Bigg) \Bigg\} dt \end{aligned}$$

This, on integrating by parts, implies

$$\begin{aligned} 0 \leqq & \int_I \Bigg[\eta^T \Bigg((\lambda^T f_u(t,u,\dot{u},\ddot{u})) + \sum_{j\in I_\circ} y^j(t) g^j_u(t,u,\dot{u},\ddot{u}) \Bigg) \\ & +(D\eta)^T \Bigg(\lambda^T f_{\dot{u}}(t,u,\dot{u},\ddot{u}) + \sum_{j\in I_\circ} y^j(t) g^j_{\dot{u}}(t,u,\dot{u},\ddot{u}) \Bigg) \\ & +(D^2\eta)^T \Bigg(\lambda^T f_{\ddot{u}}(t,u,\dot{u},\ddot{u}) + \sum_{j\in I_\circ} y^j(t) g^j_{\ddot{u}}(t,u,\dot{u},\ddot{u}) \Bigg) \Bigg] dt \geqq 0 \end{aligned}$$

This, because of pseudoinvexity of $\int_I \left\{ \lambda^T f(t,.,.,.) + \sum_{j\in I_\circ} y^j(t) g(t,.,.,.) \right\}$ dt with respect to η implies

$$\begin{aligned} & \int_I \Bigg\{ \lambda^T f(t,x,\dot{x},\ddot{x}) + \sum_{j\in I_\circ} y^j(t) g^j(t,x,\dot{x},\ddot{x}) \Bigg\} dt \\ \geqq & \int_I \Bigg\{ \lambda^T f(t,u,\dot{u},\ddot{u}) + \sum_{j\in I_\circ} y^j(t) g^j(t,u,\dot{u},\ddot{u}) \Bigg\} dt \end{aligned}$$

Because $y(t) \geqq 0$, $t \in I$ and $g(t,x,\dot{x},\dot{x}) \leqq 0$, $t \in I$, it follows

$$\int_I \lambda^T f(t,x,\dot{x},\ddot{x})dt \geqq \int_I \left\{\lambda^T f(t,u,\dot{u},\ddot{u}) + \sum_{j\in I_o} y^j(t) g^j(t,u,\dot{u},\ddot{u})\right\}dt$$

Since $\lambda > 0$ and $\lambda^T e = 1$, this yields,

$$\lambda^T \int_I f(t,x,\dot{x},\ddot{x})dt \geqq \lambda^T \int_I \left\{f(t,u,\dot{u},\ddot{u}) + \sum_{j\in I_o} y^j(t) g^j(t,u,\dot{u},\ddot{u})e\right\}dt$$

This implies,

$$\int_I f(t,x,\dot{x},\ddot{x})dt \nleq \int_I \left\{f(t,u,\dot{u},\ddot{u}) + \sum_{j\in I_o} y^j(t) g^j(t,u,\dot{u},\ddot{u})e\right\}dt$$

□

Theorem 2 (*Strong Duality*). *Let $\bar{x}$ be a feasible solution for* (VP) *and assume that*

(*i*) *$\bar{x}$ is an efficient solution of* (VP), *and*

(*ii*) *for at least one i, $i \in P$, $\bar{x}$ satisfies a regularity condition for* [3] *for $P_i(\bar{x})$*
Then there exist $\bar{\lambda} \in R^p$, $\bar{y} \in R^m$ such that $(\bar{x},\bar{y},\bar{\lambda})$ is efficient for (Mix VD).

Further if the assumptions of Theorem 1 are satisfied, then $(\bar{x},\bar{y},\bar{\lambda})$ is efficient for (Mix VD).

Proof. Since $\bar{x}$ is an optimal solution for (VP) and is normal, then by Proposition 2, there exists piecewise smooth $\bar{y}: I \to R^m$ such that

$$\begin{aligned}
&\bar{\lambda} f_x(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) + \bar{y}(t)^T g_x(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) \\
&\quad -D(\bar{\lambda} f_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})) \\
&\quad +D^2(\bar{\lambda} f_{\ddot{x}}(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) + \bar{y}(t)^T g_{\ddot{x}}(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})) = 0, \quad t \in I &(3.8)\\
&\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) = 0, \quad t \in I &(3.9)\\
&\bar{y}(t) \geqq 0, \quad t \in I &(3.10)\\
&\bar{\lambda} > 0 &(3.11)
\end{aligned}$$

The relation $\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) = 0$, $t \in I$ implies

$$\sum_{j\in I_o} \bar{y}^j(t)^T g^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) = 0, \quad t \in I \quad \text{and} \tag{3.12}$$

$$\sum_{j\in I_\alpha} \bar{y}^j(t)^T g^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) = 0, \ t \in I, \ \alpha = 1,2,\ldots,r, \tag{3.13}$$

giving $\displaystyle\sum_{j\in I_\alpha} \int_I \bar{y}^j(t)^T g^j(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}}) = 0, \quad t \in I, \ \alpha = 1,2,\ldots,r$

From (8), (10) and (11), it follows that $(\bar{x}, \bar{y}, \bar{\lambda})$ is feasible for (Mix VD). Also

$$\int_I \Big\{ f^i(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) + \sum_{j \in I_0} y^j(t) g^j(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) \Big\} dt = \int_I f^i(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) dt, \quad i = 1, \ldots, p$$

That is, the objective values of (VP) and (Mix VD) are equal. The efficiency of $(\bar{x}, \bar{y}, \bar{\lambda})$ follows from Theorem 1.

As in [10], by employing the chain rule in calculus, it can be easily seen that the expression $(\lambda^T f_x + y(t)^T g_x) - D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) + D^2(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}})$, may be regarded as a function θ of variables $t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}$ and λ, where $\dddot{x} = D^3 x$ and $\ddot{y} = D^2 y$. That is, we can write

$$\begin{aligned} \theta(t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \lambda) \quad = \quad & (\lambda^T f_x + y(t)^T g_x) - D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) \\ & + D^2(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}) \end{aligned}$$

In order to prove converse duality between (VP) and (Mix VD), the space X is now replaced by a smaller space X_2 of piecewise smooth thrice differentiable function $x : I \to R^n$ with the norm $\|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty + \|D^3 x\|_\infty$. The problem (Mix VD) may now be briefly written as,

$$\text{Minimize} - \Bigg(\int_I \Big(f^1(t, x, \dot{x}, \ddot{x}) dt + \sum_{j \in I_0} y^j(t)^T g^j(t, x, \dot{x}, \ddot{x}) \Big) dt,$$
$$\ldots, \int_I \Big(f^p(t, x, \dot{x}, \ddot{x}) dt + \sum_{j \in I_0} y^j(t)^T g^j(t, x, \dot{x}, \ddot{x}) \Big) dt \Bigg)$$

Subject to

$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b),$$
$$\theta(t, x, \dot{x}, \ddot{x}, y, \dot{y}, \ddot{y}) = 0, \quad t \in I,$$
$$\sum_{j \in I_\alpha} \int_I y^j(t)^T g^j(t, u, \dot{u}, \ddot{u}) dt \geqq 0, \quad \alpha = 1, 2, \ldots, r,$$
$$y(t) \geqq 0, \quad t \in I, \quad \lambda > 0,$$

where $\theta(\cdot) = \lambda f_u + y(t)^T g_u - D(\lambda f_{\dot{u}} + y(t)^T g_{\dot{u}}) + D^2(\lambda f_{\ddot{u}} + y(t)^T g_{\ddot{u}})$, $t \in I$.

Consider $\theta(t, x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot), \dddot{x}(\cdot), y(\cdot), \dot{y}(\cdot), \ddot{y}(\cdot), \lambda) = 0$ as defining a mapping $\psi : X_2 \times Y \times R^p \to B$ where Y is a space of piecewise twice differentiable functions and B is the Banach space. In order to apply Theorem 1 [7] to the problem (Mix D), the infinite dimensional inequality must be restricted. In the following theorem, we use ψ' to represent the Frèchèt derivative $[\psi_x(x, y, \lambda), \psi_y(x, y, \lambda), \psi_\lambda(x, y, \lambda)]$.

Theorem 3 (*Converse Duality*). *Let* $(\bar{x}, \bar{y}, \bar{\lambda})$ *be an efficient solution of (Mix VD). Assume that*

H_1 : *The Frèchèt derivative* ψ' *has a (weak*) closed range,*

H_2 : f *and* g *are twice continuously differentiable,*

H_3 : $f_x^i + \sum\limits_{j\in I_\circ} y_j(t)^T g_x^j - D\left(f_{\dot{x}}^i + \sum\limits_{j\in I_\circ} y_j(t)^T g_{\dot{x}}^j\right) + D^2\left(f_{\ddot{x}}^i + \sum\limits_{j\in I_\circ} y_j(t)^T g_{\ddot{x}}^j\right)$, *$t \in I$, $i = 1, \dots, p$ are linearly independent, and*

H_4 : $(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}} - D^3\beta(t)^T\theta_{\dddot{x}})\beta(t) = 0$
$\Rightarrow \quad \beta(t) = 0,\ t \in I.$

Further, if the hypotheses of Theorem 1 *are satisfied, then $\bar{x}$ is an efficient solution of* (Mix VD).

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda})$ with ψ' has closed (weak*) range is an efficient solution, by (Mix VD), there exist Lagrange multipliers $\tau \in R^p$, piecewise smooth $\beta : I \to R^n, \gamma \in R$ for each of r constraints, $\eta \in R^p$ and piecewise smooth $\mu(t) : I \to R^m$, satisfying the Fritz-John conditions

$$\begin{aligned}
&\left(\tau^T f_x + (\alpha^T e)\sum_{j\in I_\circ} y^j(t)^T g_x^j\right) - D\left(\tau^T f_{\dot{x}} + (\alpha^T e)\sum_{j\in I_\circ} y^j(t)^T g_{\dot{x}}^j\right)\\
&\quad + D^2\left(\tau^T f_{\ddot{x}} + (\alpha^T e)\sum_{j\in I_\circ} y^j(t)^T g_{\ddot{x}}^j\right) + \beta(t)^T\theta_x\\
&\quad - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}} - D^3\beta(t)^T\theta_{\dddot{x}}\\
&\quad - \gamma\sum_{\alpha=1}^{r}\sum_{j\in I_\alpha}[y^j(t)^T g_x^j - Dy^j(t)^T g_{\dot{x}}^j + D^2 y^j(t)^T g_{\ddot{x}}^j] = 0, \quad t \in I
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&-(\tau^T e)g^i + \beta(t)^T\theta_{y^j} - D\beta(t)^T\theta_{\dot{y}^j} + D^2\beta(t)^T\theta_{\ddot{y}^j} - \mu^j(t) = 0,\\
&\qquad j \in I_\circ, t \in I
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\gamma g^i + \beta(t)^T\theta_{y^j} - D\beta(t)^T\theta_{\dot{y}^j} + D^2\beta(t)^T\theta_{\ddot{y}^j} - \mu^j(t) = 0, j \in I_\alpha\\
&\qquad t \in I, \alpha = 1, \dots, r
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\left[\left(f_x^i + \sum_{j\in I_\circ} y^j(t)^T g_x^j\right) - D\left(f_{\dot{x}}^i + \sum_{j\in I_\circ} y^j(t)^T g_{\dot{x}}^j\right)\right.\\
&\quad \left. + D^2\left(f_{\ddot{x}}^i + \sum_{j\in I_\circ} y^j(t)^T g_{\ddot{x}}^j\right)\right]\beta(t) - \eta^i = 0
\end{aligned} \tag{3.17}$$

$$\mu(t)^T y(t) = 0, \quad t \in I, \tag{3.18}$$

$$\eta^T\lambda = 0 \tag{3.19}$$

$$\sum_{j\in I_\alpha}\int_I y^j(t)^T g^j(t, u, \dot{u}, \ddot{u})dt = 0, \alpha = 1, 2, \dots, r \tag{3.20}$$

$$(\tau, \gamma, \mu(t), \eta) \geqq 0, \quad t \in I, \tag{3.21}$$

$$(\tau, \beta(t), \gamma, \mu(t), \eta) \neq 0, \quad t \in I. \tag{3.22}$$

Since $\lambda > 0$, (19) implies $\eta = 0$. Consequently (17) yields

$$\left[\left(f_x^i + \sum_{j\in I_o} y^j(t)^T g_x^j\right) - D\left(f_{\dot{x}}^i + \sum_{j\in I_o} y^j(t)^T g_{\dot{x}}^j\right)\right.$$

$$+D^2\Big(\tau^T f_{\ddot{x}} + \sum_{j\in I_o} y^j(t)^T g^j_{\ddot{x}}\Big)\Big]\beta(t) = 0, \quad t \in I \tag{3.23}$$

Using the equality constraint of (Mix VD) in (14) we have,

$$-\sum_{i=1}^{p}(\tau^i - \gamma\lambda^i)\Big[\Big(f_x + \sum_{j\in I_o} y^j(t)^T g^j_x\Big) - D\Big(f_{\dot{x}} + \sum_{j\in I_o} y^j(t)^T g^j_{\dot{x}}\Big)$$
$$+D^2\Big(f_{\ddot{x}} + \sum_{j\in I_o} y^j(t)^T g^j_{\ddot{x}}\Big)\Big]\beta(t) + \beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}}$$
$$+D^2\beta(t)^T\theta_{\ddot{x}} - D^3\beta(t)^T\theta_{\dddot{x}} = 0, \quad t \in I \tag{3.24}$$

Post-multiplying (24) by $\beta(t)$, we have

$$(\beta(t)^T\theta_x - D\beta(t)^T\theta_{\dot{x}} + D^2\beta(t)^T\theta_{\ddot{x}} - D^3\beta(t)^T\theta_{\dddot{x}})\beta(t) = 0, \quad t \in I$$

This, because of the hypothesis (H_4) gives

$$\beta(t) = 0, \quad t \in I \tag{3.25}$$

Using (25) in (24), we have

$$-\sum_{i=1}^{p}(\tau^i - \gamma\lambda^i)\Big[\Big(f_x + \sum_{j\in I_o} y^j(t)^T g^j_x\Big)$$
$$-D\Big(f_{\dot{x}} + \sum_{j\in I_o} y^j(t)^T g^j_{\dot{x}}\Big) + D^2\Big(f_{\ddot{x}} + \sum_{j\in I_o} y^j(t)^T g^j_{\ddot{x}}\Big)\Big] = 0, \quad t \in I$$

This because of the linear independence stated in (H_3) gives

$$(\tau^i - \gamma\lambda^i) = 0, \qquad i = 1, 2, \ldots, p \tag{3.26}$$

If possible, let $\gamma = 0$. Then (26) implies $\tau = 0$. The equations (15) and (16) imply that $\mu(t) = 0$, $t \in I$. Then $(\tau, \beta(t), \gamma, \mu(t), \eta) = 0$, contradicting $\tau > 0$.

From (15) and (16) it follows,

$$g^j = -\frac{\mu^j(t)}{(\tau^T e)}, \qquad j \in I_o,\ t \in I$$
$$g^j = -\frac{\mu^j(t)}{\gamma}, \qquad j \in I_\alpha,\ \alpha = 1, 2, \ldots, r,\ t \in I$$

This implies $g \leqq 0$, also in view of (18) $y^T g = 0$. From $y^T g = 0$, it implies $\sum_{j\in J_o} y^j(t) g^j = 0, t \in I$

$$\int_I \Big(f^i(t, x, \dot{x}, \ddot{x}) + \sum_{j\in J_o} y^j(t) g^j(t, x, \dot{x}, \ddot{x})\Big) dt = \int_I f^i(t, x, \dot{x}, \ddot{x}) dt$$

This along with the application of Theorem 1 establishes the efficiency of $\bar{x}$ for (VP). □

4. Related nonlinear problems

If f and g do not explicitly depend on t, the variational problem considered in the preceding section reduces to the following static problems similar to those by Xu [13] and Mond and Zhang [14].

(NP) Minimize $f(x)$

Subject to

$$g(x) \leqq 0$$

(Mix ND)
$$\text{Maximize} \int_I \left(f^1(u) + \sum_{j \in J_0} y^j g^j(u), \ldots, f^p(u) + \sum_{j \in J_0} y^j g^j(u) \right)$$

Subject to

$$\nabla(\lambda^T f(u) + y^T g(u)) = 0$$

$$\sum_{j \in J_\alpha} y^j g^j(u) \geqq 0, \quad \alpha = 1, \ \ldots, r$$

$$y \geqq 0, \quad \lambda \in \Lambda^+.$$

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