J. Appl. Math. & Informatics Vol. **27***(2009), No. 1 - 2, pp. 217 - 231*
Website: http://www.kcam.biz

SOME IDEALS OF PSEUDO BCI-ALGEBRAS

KYOUNG JA LEE AND CHUL HWAN PARK*

ABSTRACT. The notion of $*$-medial pseudo BCI-algebras is introduced, and its characterization is discussed. The concepts of associative pseudo ideals (resp. pseudo p-ideals, pseudo q-ideals and pseudo a-ideals) are introduced, and related properties are investigated. Conditions for a pseudo ideal to be a pseudo p-ideal (resp. pseudo q-ideal) are provided. A characterization of an associative pseudo ideal is given. We finally show that every pseudo BCI-homomorphic image and preimage of an associative pseudo ideal (resp. a pseudo p-ideal, a pseudo q-ideal and a pseudo a-ideal) is also an associative pseudo ideal (resp. a pseudo p-ideal, a pseudo q-ideal and a pseudo a-ideal).

AMS Mathematics Subject Classification : 06F35, 03G25.
Keywords and phrases : Pseudo BCI-algebra, (associative) pseudo ideal, pseudo p-ideal, pseudo q-ideal, pseudo a-ideal.

1. Introduction

G. Georgescu and A. Iorgulescu [2] introduced the notion of a pseudo BCK-algebra as an extended notion of BCK-algebras. In [3], Y. B. Jun, one of the present authors, gave a characterization of pseudo BCK-algebra, and provided conditions for a pseudo BCK-algebra to be $\wedge$-semi-lattice ordered (resp. $\cap$-semi-lattice ordered). Y. B. Jun et al. [5] introduced the notion of (positive implicative) pseudo-ideals in a pseudo-BCK algebra, and then they investigated some of their properties. In [1], W. A. Dudek and Y. B. Jun introduced the notion of pseudo BCIalgebras as an extension of BCI-algebras, and investigated some properties. Y. B. Jun et al. [4] introduced the concepts of pseudo-atoms, pseudo ideals and pseudo BCI-homomorphisms in pseudo BCI-algebras. They displayed characterizations of a pseudo ideal, and provided conditions for a subset to be a pseudo ideal. They also introduced the notion of a $\diamond$-medial pseudo BCI-algebra, and gave its characterization. They proved that every pseudo BCI-homomorphic image and preimage of a pseudo ideal is also a pseudo ideal. In

Received February 28, 2008. Accepted April 10, 2008. *Corresponding author.

[6], Y. L. Liu et al. extended the ideal and congruence theory to pseudo BCK-algebras, and investigated the connections between pseudo BCK-algebras and PD (GPD)-posets.

In this paper, we introduce the notion of $*$-medial pseudo BCI-algebras, and investigate its characterization. We also introduce the concepts of associative pseudo ideals (resp. pseudo p-ideals, pseudo q-ideals and pseudo a-ideals), and investigate related properties. We provide conditions for a pseudo ideal to be a pseudo p-ideal (resp. pseudo q-ideal). We give a characterization of an associative pseudo ideal. We show that every pseudo BCI-homomorphic image and preimage of an associative pseudo ideal (resp. a pseudo p-ideal, a pseudo q-ideal and a pseudo a-ideal) is also an associative pseudo ideal (resp. a pseudo p-ideal, a pseudo q-ideal and a pseudo a-ideal).

2. Preliminaries

A BCK/BCI-algebra is an important calss of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x,y,z\in X)\ (((x*y)*(x*z))*(z*y)=0)$,
(II) $(\forall x,y\in X)\ ((x*(x*y))*y=0)$,
(III) $(\forall x\in X)\ (x*x=0)$,
(IV) $(\forall x,y\in X)\ (x*y=0\ \&\ y*x=0\ \Rightarrow\ x=y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x\in X)\ (0*x=0)$,

then X is called a *BCK-algebra*. Any BCK-algebra X satisfies the following axioms:

(a1) $(\forall x\in X)\ (x*0=x)$,
(a2) $(\forall x,y,z\in X)\ (x\le y\ \Rightarrow\ x*z\le y*z,\ z*y\le z*x)$,
(a3) $(\forall x,y,z\in X)\ ((x*y)*z=(x*z)*y)$,
(a4) $(\forall x,y,z\in X)\ ((x*z)*(y*z)\le x*y)$

where $x\le y$ if and only if $x*y=0$.

A nonempty subset I of a BCI-algebra X is called an *ideal* of X if it satisfies:

$$0\in I \tag{1}$$

and

$$(\forall x,y\in X)\,(x*y\in I\ \&\ y\in I\ \Longrightarrow\ x\in I). \tag{2}$$

A nonempty subset I of a BCI-algebra X is called a *p-ideal* of X (see [8]) if it satisfies (1) and

$$(\forall x,y,z\in X)\,((x*z)*(y*z)\in I\ \&\ y\in I\ \Longrightarrow\ x\in I). \tag{3}$$

A nonempty subset I of a BCI-algebra X is called a *q-ideal* of X (see [7]) if it satisfies (1) and

$$(\forall x, y, z \in X)\,(x * (y * z) \in I \ \&\ y \in I \implies x * z \in I). \tag{4}$$

A nonempty subset I of a BCI-algebra X is called an *a-ideal* of X (see [7]) if it satisfies (1) and

$$(\forall x, y, z \in X)\,((x * z) * (0 * y) \in I \ \&\ z \in I \implies y * x \in I). \tag{5}$$

Definition 1. [2] A *pseudo BCK-algebra* is a structure $\mathfrak{X} := (X, \preceq, *, \diamond, 0)$, where "$\preceq$" is a binary relation on a set X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, verifying the axioms: for all $x, y, z \in X$,

$$(x * y) \diamond (x * z) \preceq z * y, \ \ (x \diamond y) * (x \diamond z) \preceq z \diamond y, \tag{6}$$

$$x * (x \diamond y) \preceq y, \ \ x \diamond (x * y) \preceq y, \tag{7}$$

$$x \preceq x, \tag{8}$$

$$0 \preceq x, \tag{9}$$

$$x \preceq y \ \&\ y \preceq x \implies x = y, \tag{10}$$

$$x \preceq y \iff x * y = 0 \iff x \diamond y = 0. \tag{11}$$

Definition 2. [1] A *pseudo BCI-algebra* is a structure $\mathfrak{X} := (X, \preceq, *, \diamond, 0)$, where "$\preceq$" is a binary relation on a set X, "$*$" and "$\diamond$" are binary operations on X and "0" is an element of X, verifying the axioms (6), (7), (8), (10) and (11).

Example 1. [4] Let $X = [0, \infty]$ and let $\leq$ be the usual order on X. Define binary operations "$*$" and "$\diamond$" on X by

$$x * y := \begin{cases} 0 & \text{if } x \leq y, \\ \dfrac{2x}{\pi} \arctan\left(\ln(\dfrac{x}{y})\right) & \text{if } y < x, \end{cases}$$

$$x \diamond y := \begin{cases} 0 & \text{if } x \leq y, \\ x e^{-\tan(\frac{\pi y}{2x})} & \text{if } y < x, \end{cases}$$

for all $x, y \in X$. Then $\mathfrak{X} := (X, \leq, *, \diamond, 0)$ is a pseudo BCK-algebra, and hence a pseudo BCI-algebra.

Proposition 1. [1, 4] *In a pseudo BCI-algebra $\mathfrak{X}$ the following holds:*

(b1) $x \preceq 0 \Rightarrow x = 0$.
(b2) $x \preceq y \Rightarrow z * y \preceq z * x, \ z \diamond y \preceq z \diamond x$.
(b3) $x \preceq y, \ y \preceq z \Rightarrow x \preceq z$.
(b4) $(x * y) \diamond z = (x \diamond z) * y$.
(b5) $x * y \preceq z \Leftrightarrow x \diamond z \preceq y$.
(b6) $(x * y) * (z * y) \preceq x * z, \ (x \diamond y) \diamond (z \diamond y) \preceq x \diamond z$.

(b7) $x \preceq y \Rightarrow x * z \preceq y * z,\ x \diamond z \preceq y \diamond z$.
(b8) $x * 0 = x = x \diamond 0$.
(b9) $x * \big(x \diamond (x * y)\big) = x * y,\ x \diamond \big(x * (x \diamond y)\big) = x \diamond y$.
(b10) $0 * (x \diamond y) \preceq y \diamond x$.
(b11) $0 \diamond (x * y) \preceq y * x$.
(b12) $0 * (x * y) = (0 \diamond x) \diamond (0 * y)$.
(b13) $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y)$.

3. Further properties of pseudo BCI-algebras

Proposition 2. *Let* $\mathfrak{X} := (X, \preceq, *, \diamond, 0)$ *be a pseudo BCI-algebra. Then we have*

$$(\forall x \in X)\,(0 * x = 0 \diamond x). \tag{12}$$

Proof. Putting $y = x$ and $z = 0$ in (6), we obtain $(x * x) \diamond (x * 0) \preceq 0 * x$ and $(x \diamond x) * (x \diamond 0) \preceq 0 \diamond x$ for all $x \in X$. It follows from (8) and (b8) that $0 \diamond x \preceq 0 * x$ and $0 * x \preceq 0 \diamond x$. Hence $0 * x = 0 \diamond x$ by (10). □

Definition 3. A pseudo BCI-algebra $\mathfrak{X}$ is said to be $*$-*medial* if it satisfies the following identity:

$$(\forall x, y, a, b \in X)\,((x \diamond y) * (a \diamond b) = (x \diamond a) * (y \diamond b)). \tag{13}$$

Proposition 3. *A pseudo BCI-algebra* $\mathfrak{X}$ *is* $*$-*medial if and only if it satisfies:*

$$(\forall x, y, z \in X)\,(x * (y \diamond z) = (x \diamond y) * (0 \diamond z)). \tag{14}$$

Proof. Assume that $\mathfrak{X}$ is $*$-medial. Putting $a = 0$ and $b = z$ in (14) and using (b8), we have

$$(x \diamond y) * (0 \diamond z) = (x \diamond 0) * (y \diamond z) = x * (y \diamond z).$$

Conversely, suppose that $\mathfrak{X}$ satisfies the condition (14). Using (b4), we have

$$\begin{aligned} (x \diamond y) * (a \diamond b) &= (x * (a \diamond b)) \diamond y \\ &= ((x \diamond a) * (0 \diamond b)) \diamond y \\ &= ((x \diamond a) \diamond y) * (0 \diamond b) \\ &= (x \diamond a) * (y \diamond b) \end{aligned}$$

for all $x, y, a, b \in X$. Therefore $\mathfrak{X}$ is $*$-medial. □

Proposition 4. *Every* $*$-*medial pseudo BCI-algebra* $\mathfrak{X}$ *satisfies the following identities.*

(i) $x \diamond y = 0 * (y \diamond x)$.
(ii) $0 * (0 \diamond x) = x$.
(iii) $x * (x \diamond y) = y$.

Proof. (i) For any $x, y \in X$, we have

$$\begin{aligned} x \diamond y &= (x \diamond y) * 0 = (x \diamond y) * (x \diamond x) \\ &= (x \diamond x) * (y \diamond x) = 0 * (y \diamond x). \end{aligned}$$

(ii) If we put $y = 0$ in (i), then we have (ii).
(iii) Using (ii), (8) and (b8), we get

$$x * (x \diamond y) = (x \diamond 0) * (x \diamond y) = (x \diamond x) * (0 \diamond y) = 0 * (0 \diamond y) = y.$$

This completes the proof. □

4. Pseudo ideals

In what follows, let $\mathfrak{X} := (X, \preceq, *, \diamond, 0)$ be a pseudo BCI-algebra unless otherwise specified.

For any nonempty subset J of X and any element y of X, we denote

$$*(y, J) := \{x \in X \mid x * y \in J\} \text{ and } \diamond(y, J) := \{x \in X \mid x \diamond y \in J\}.$$

Definition 4. [4] A nonempty subset J of $\mathfrak{X}$ is called a *pseudo ideal* of $\mathfrak{X}$ if it satisfies

(c1) $0 \in J$,
(c2) $(\forall y \in J)\ (*(y, J) \subseteq J \ \&\ \diamond(y, J) \subseteq J)$.

Proposition 5. *Let J be a pseudo ideal of $\mathfrak{X}$. Then*

$$(\forall x \in X)\,(x \in J \implies 0 * (0 \diamond x) \in J \ \&\ 0 \diamond (0 * x) \in J). \tag{15}$$

Proof. Let $x \in J$. Then

$$0 = (0 \diamond x) * (0 \diamond x) = (0 * (0 \diamond x)) \diamond x$$

and

$$0 = (0 * x) \diamond (0 * x) = (0 \diamond (0 * x)) * x$$

which imply that $0 * (0 \diamond x) \in \diamond(x, J) \subseteq J$ and $0 \diamond (0 * x) \in *(x, J) \subseteq J$. This completes the proof. □

Lemma 1. [4] *Let J be a pseudo ideal of $\mathfrak{X}$. If $x \in J$ and $y \preceq x$, then $y \in J$.*

Theorem 1. *Let J be a pseudo ideal of $\mathfrak{X}$ and let*

$$J^\sharp := \{x \in X \mid 0 * (0 \diamond x) \in J,\ 0 \diamond (0 * x) \in J\}.$$

Then $J^\sharp$ is a pseudo ideal of $\mathfrak{X}$ and $J \subseteq J^\sharp$.

Proof. Obviously, $0 \in J^\sharp$. For any $y \in J^\sharp$, let $a \in *(y, J^\sharp)$ and $b \in \diamond(y, J^\sharp)$. Then $a * y \in J^\sharp$ and $b \diamond y \in J^\sharp$, that is, $0 * (0 \diamond (a * y)) \in J$, $0 \diamond (0 * (a * y)) \in J$, $0 * (0 \diamond (b \diamond y)) \in J$, $0 \diamond (0 * (b \diamond y)) \in J$. Using (b12) and (b13), we have

$$(0 \diamond (0 * b)) \diamond (0 * (0 \diamond y)) = 0 * ((0 * b) * (0 \diamond y)) = 0 * (0 \diamond (b \diamond y)) \in J$$

and

$$(0 * (0 \diamond a)) * (0 \diamond (0 * y)) = 0 \diamond ((0 \diamond a) \diamond (0 * y)) = 0 \diamond (0 * (a * y)) \in J.$$

Since $0 * (0 \diamond y) \in J$ and $0 \diamond (0 * y) \in J$, it follows that

$$\begin{aligned} 0 \diamond (0 * b) &\in \diamond(0 * (0 \diamond y), J) \subseteq J, \\ 0 * (0 \diamond a) &\in *(0 \diamond (0 * y), J) \subseteq J. \end{aligned} \tag{16}$$

Now, since $0 \diamond (a * y) \preceq y * a$ and $0 * (b \diamond y) \preceq y \diamond b$, it follows from (b2) that

$$(0 \diamond y) \diamond (0 * a) = 0 * (y * a) \preceq 0 * (0 \diamond (a * y)) \in J$$

and

$$(0 * y) * (0 \diamond b) = 0 \diamond (y \diamond b) \preceq 0 \diamond (0 * (b \diamond y)) \in J.$$

Using Lemma 1, we get

$$(0 \diamond y) \diamond (0 * a) \in J, \ \ (0 * y) * (0 \diamond b) \in J. \tag{17}$$

Taking $y = 0$ in (17) implies that

$$0 \diamond (0 * a) \in J, \ \ 0 * (0 \diamond b) \in J. \tag{18}$$

Combining (16) and (18), we have $a \in J^\sharp$ and $b \in J^\sharp$. Hence $*(y, J^\sharp) \subseteq J^\sharp$ and $\diamond(y, J^\sharp) \subseteq J^\sharp$, that is, $J^\sharp$ is a pseudo ideal of $\mathfrak{X}$. By Proposition 5, we know that $J \subseteq J^\sharp$. This completes the proof. □

Definition 5. A nonempty subset J of $\mathfrak{X}$ is called a *pseudo p-ideal* of $\mathfrak{X}$ if it satisfies (c1) and

$$\begin{aligned} (x * z) \diamond (y * z) \in J \ \&\ y \in J &\Longrightarrow x \in J, \\ (x \diamond z) * (y \diamond z) \in J \ \&\ y \in J &\Longrightarrow x \in J \end{aligned} \tag{19}$$

for all $x, y, z \in X$.

Note that if $\mathfrak{X}$ is a pseudo BCI-algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then the notions of a pseudo p-ideal and a p-ideal coincide.

Theorem 2. *Every pseudo p-ideal of $\mathfrak{X}$ is a pseudo ideal of $\mathfrak{X}$.*

Proof. Let J be a pseudo p-ideal of $\mathfrak{X}$. For any $y \in J$, let $a \in *(y, J)$ and $b \in \diamond(y, J)$. Then

$$(a \diamond 0) * (y \diamond 0) = a * y \in J, \ (b * 0) \diamond (y * 0) = b \diamond y \in J.$$

It follows from (19) that $a \in J$ and $b \in J$. Hence $*(y, J) \subseteq J$ and $\diamond(y, J) \subseteq J$. Therefore J is a pseudo p-ideal of $\mathfrak{X}$. □

The converse of Theorem 2 is not true in general as seen in the following example.

Example 2. Consider the pseudo BCI-algebra $\mathfrak{X}$ which is described in Example 1. Note that $J := \{0\}$ is a pseudo ideal of $\mathfrak{X}$. But $J := \{0\}$ is not a pseudo p-ideal of $\mathfrak{X}$ since $(1 * 2) \diamond (0 * 2) = 0 \diamond (0 * 2) = 0 \in J$ and $(1 \diamond 2) * (0 \diamond 2) = 0 \in J$, but $1 \notin J$.

Proposition 6. *Let J be a pseudo p-ideal of $\mathfrak{X}$. Then we have*

$$\begin{aligned} 0 * (0 \diamond x) \in J &\Longrightarrow x \in J, \\ 0 \diamond (0 * x) \in J &\Longrightarrow x \in J \end{aligned} \tag{20}$$

for all $x \in X$.

Proof. Assume that $0 * (0 \diamond x) \in J$ and $0 \diamond (0 * x) \in J$ for all $x \in X$. Then

$$(x \diamond x) * (0 \diamond x) = 0 * (0 \diamond x) \in J, \ (x * x) \diamond (0 * x) = 0 \diamond (0 * x) \in J.$$

Using (19), we have $x \in J$. This completes the proof. □

Combining Propositions 5 and 6, we have the following corollary.

Corollary 1. *Let J be a pseudo p-ideal of $\mathfrak{X}$. Then we have*

$$\begin{aligned} 0 * (0 \diamond x) \in J &\Longleftrightarrow x \in J, \\ 0 \diamond (0 * x) \in J &\Longleftrightarrow x \in J \end{aligned} \tag{21}$$

for all $x \in X$.

We give a condition for a pseudo ideal to be a pseudo p-ideal.

Theorem 3. *Let J be a pseudo ideal of $\mathfrak{X}$ that satisfies the following assertions:*

$$\begin{aligned} (x * z) \diamond (y * z) \in J &\Longrightarrow x \diamond y \in J, \\ (x \diamond z) * (y \diamond z) \in J &\Longrightarrow x * y \in J \end{aligned} \tag{22}$$

for all $x, y, z \in X$. Then J is a pseudo p-ideal of $\mathfrak{X}$.

Proof. Let J be a pseudo ideal of $\mathfrak{X}$ that satisfies (22). Let $x, z \in X$ and $y \in J$ be such that $(x * z) \diamond (y * z) \in J$ and $(x \diamond z) * (y \diamond z) \in J$. It follows from (22) that $x \diamond y \in J$ and $x * y \in J$. Hence $x \in \diamond(y, J) \subseteq J$ and $x \in *(y, J) \subseteq J$. Therefore J is a pseudo p-ideal of $\mathfrak{X}$. □

Definition 6. A nonempty subset J of $\mathfrak{X}$ is called an *associative pseudo ideal* of $\mathfrak{X}$ if it satisfies (c1) and

$$\begin{array}{l}(x*y)\diamond z\in J\ \&\ y\diamond z\in J\Longrightarrow x\in J,\\(x\diamond y)*z\in J\ \&\ y*z\in J\Longrightarrow x\in J\end{array}\tag{23}$$

for all $x,y,z\in X$.

Theorem 4. *A nonempty subset J of $\mathfrak{X}$ is an associative pseudo ideal of $\mathfrak{X}$ if and only if it satisfies (c1) and*

$$\begin{array}{l}(x*y)\diamond y\in J\Longrightarrow x\in J,\\(x\diamond y)*y\in J\Longrightarrow x\in J\end{array}\tag{24}$$

for all $x,y\in X$.

Proof. Assume that J is an associative pseudo ideal of $\mathfrak{X}$. Let $x,y\in X$ be such that $(x*y)\diamond y\in J$ and $(x\diamond y)*y\in J$. Since $y\diamond y=0=y*y$, it follows from (c1) and (23) that $x\in J$. Conversely, let J be a nonempty subset of $\mathfrak{X}$ satisfying (c1) and (24). Let $x,y,z\in X$ be such that $(x*y)\diamond z\in J$, $y\diamond z\in J$, $(x\diamond y)*z\in J$ and $y*z\in J$. If we take $z=y$, then $(x*y)\diamond y\in J$ and $(x\diamond y)*y\in J$. By (24), we have $x\in J$. Hence $\mathfrak{X}$ is associative. □

Theorem 5. *Every associative pseudo ideal of $\mathfrak{X}$ is a pseudo ideal of $\mathfrak{X}$.*

Proof. Let J be an associative pseudo ideal of $\mathfrak{X}$. For any $y\in J$, let $x\in *(y,J)$ and $a\in\diamond(y,J)$. Then $(x*y)\diamond 0=x*y\in J$ and $(a\diamond y)*0=a\diamond y\in J$. Since $y\diamond 0=y\in J$ and $y*0=y\in J$, it follows from (23) that $x\in J$ and $a\in J$. Hence $*(y,J)\subseteq J$ and $\diamond(y,J)\subseteq J$. Therefore J is a pseudo ideal of $\mathfrak{X}$. □

The converse of Theorem 5 is not true in general as seen in the following example.

Example 3. Consider the pseudo BCI-algebra $\mathfrak{X}$ which is described in Example 1. We know that $J:=\{0\}$ is a pseudo ideal of $\mathfrak{X}$. But J is not an associative pseudo ideal of $\mathfrak{X}$ since $(1*2)\diamond 2=0\diamond 2=0\in J$, $(1\diamond 2)*2=0*2=0\in J$ and $2*2=2\diamond 2=0\in J$, but $1\notin J$.

Proposition 7. *Every associative pseudo ideal J of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{array}{l}x*y\in J\ \&\ x\in J\Longrightarrow y\in J,\\x\diamond y\in J\ \&\ x\in J\Longrightarrow y\in J\end{array}\tag{25}$$

for all $x,y\in X$.

Proof. Let $x, y \in X$ be such that $x \in J$, $x*y \in J$ and $x \diamond y \in J$. Then $0*(0\diamond x) \in J$ and $0 \diamond (0 * x) \in J$ by Proposition 5. Hence

$$((0 \diamond x) \diamond (0 \diamond x)) * (0 \diamond x) = 0 * (0 \diamond x) \in J,$$
$$((0 * x) * (0 * x)) \diamond (0 * x) = 0 \diamond (0 * x) \in J.$$

Using (24), we get $0 * x \in J$ and $0 \diamond x \in J$. Then

$$(y \diamond x) * y = (y * y) \diamond x = 0 \diamond x \in J,$$
$$(y * x) \diamond y = (y \diamond y) * x = 0 * x \in J.$$

Since $x * y \in J$ and $x \diamond y \in J$, it follows from (24) that $y \in J$. This completes the proof. □

Definition 7. A nonempty subset J of $\mathfrak{X}$ is called a *pseudo q-ideal* of $\mathfrak{X}$ if it satisfies (c1) and

$$\begin{aligned} x * (y \diamond z) \in J \ \& \ y \in J \Longrightarrow x * z \in J, \\ x \diamond (y * z) \in J \ \& \ y \in J \Longrightarrow x \diamond z \in J \end{aligned} \tag{26}$$

for all $x, y, z \in X$.

Note that if $\mathfrak{X}$ is a pseudo BCI-algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then the notions of a pseudo q-ideal and a q-ideal coincide.

Example 4. Consider the pseudo BCI-algebra $\mathfrak{X}$ which is described in Example 1. Then $J := \{0\}$ is a pseudo q-ideal of $\mathfrak{X}$.

Theorem 6. *Every pseudo q-ideal of $\mathfrak{X}$ is a pseudo ideal of $\mathfrak{X}$.*

Proof. Let J be a pseudo q-ideal of $\mathfrak{X}$. Taking $z = 0$ in (26) and using (b8), we have

$$\begin{aligned} x * y \in J \ \& \ y \in J \Longrightarrow x \in J, \\ x \diamond y \in J \ \& \ y \in J \Longrightarrow x \in J \end{aligned}$$

for all $x, y \in X$. This means that $*(y, J) \subseteq J$ and $\diamond(y, J) \subseteq J$ for all $y \in J$. Hence J is a pseudo ideal of $\mathfrak{X}$. □

Proposition 8. *Every pseudo q-ideal J of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{aligned} x * (0 \diamond y) \in J \Longrightarrow x * y \in J, \\ x \diamond (0 * y) \in J \Longrightarrow x \diamond y \in J \end{aligned} \tag{27}$$

for all $x, y \in X$.

Proof. Let $x, y \in X$ be such that $x * (0 \diamond y) \in J$ and $x \diamond (0 * y) \in J$. Since $0 \in J$, it follows from (26) that $x * y \in J$ and $x \diamond y \in J$. □

Proposition 9. *Every pseudo q-ideal J of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{aligned} x * (y \diamond z) \in J &\Longrightarrow (x * y) * z \in J, \\ x \diamond (y * z) \in J &\Longrightarrow (x \diamond y) \diamond z \in J \end{aligned} \tag{28}$$

for all $x, y, z \in X$.

Proof. Suppose that $x * (y \diamond z) \in J$ and $x \diamond (y * z) \in J$ for all $x, y, z \in X$. Then

$$\begin{aligned} &((x * y) * (0 \diamond z)) \diamond (x * (y \diamond z)) \\ &\quad = ((x * y) \diamond (x * (y \diamond z))) * (0 \diamond z) \\ &\quad \preceq ((y \diamond z) * y) * (0 \diamond z) \\ &\quad = ((y * y) \diamond z) * (0 \diamond z) \\ &\quad = (0 \diamond z) * (0 \diamond z) = 0 \in J \end{aligned}$$

and

$$\begin{aligned} &((x \diamond y) \diamond (0 * z)) * (x \diamond (y * z)) \\ &\quad = ((x \diamond y) * (x \diamond (y * z))) \diamond (0 * z) \\ &\quad \preceq ((y * z) \diamond y) \diamond (0 * z) \\ &\quad = ((y \diamond y) * z) \diamond (0 * z) \\ &\quad = (0 * z) \diamond (0 * z) = 0 \in J. \end{aligned}$$

Using Lemma 1, we get

$$((x * y) * (0 \diamond z)) \diamond (x * (y \diamond z)) \in J$$

and

$$((x \diamond y) \diamond (0 * z)) * (x \diamond (y * z)) \in J.$$

Hence

$$(x * y) * (0 \diamond z) \in \diamond(x * (y \diamond z), J) \subseteq J$$

and

$$(x \diamond y) \diamond (0 * z) \in *(x \diamond (y * z), J) \subseteq J.$$

It follows from Proposition 8 that $(x * y) * z \in J$ and $(x \diamond y) \diamond z \in J$. □

We provide conditions for a pseudo ideal to be a pseudo q-ideal.

Theorem 7. *If a pseudo ideal J of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{aligned} x * (y \diamond z) \in J &\Longrightarrow (x \diamond y) * z \in J, \\ x \diamond (y * z) \in J &\Longrightarrow (x * y) \diamond z \in J \end{aligned} \tag{29}$$

for all $x, y, z \in X$, then J is a pseudo q-ideal of $\mathfrak{X}$.

Proof. Let $x, y, z \in X$ be such that $y \in J$, $x * (y \diamond z) \in J$ and $x \diamond (y * z) \in J$. Applying (b4) and (29), we have

$$(x * z) \diamond y = (x \diamond y) * z \in J \text{ and } (x \diamond z) * y = (x * y) \diamond z \in J.$$

Hence $x * z \in \diamond(y, J) \subseteq J$ and $x \diamond z \in *(y, J) \subseteq J$. Therefore J is a pseudo q-ideal of $\mathfrak{X}$. □

Theorem 8. *Let J be a pseudo ideal of $\mathfrak{X}$ which satisfies:*

$$(\forall x, y \in X)\,(x \in J \Longrightarrow x * y \in J \;\&\; x \diamond y \in J). \tag{30}$$

Then J is a pseudo q-ideal of $\mathfrak{X}$.

Proof. Let $x, y, z \in X$ be such that $y \in J$, $x * (y \diamond z) \in J$ and $x \diamond (y * z) \in J$. Using (30) and (b4), we have $y * z \in J$, $y \diamond z \in J$,

$$(x \diamond z) * (y \diamond z) = (x * (y \diamond z)) \diamond z \in J$$

and

$$(x * z) \diamond (y * z) = (x \diamond (y * z)) * z \in J.$$

Hence $x \diamond z \in *(y \diamond z, J) \subseteq J$ and $x * z \in \diamond(y * z, J) \subseteq J$. Therefore J is a pseudo q-ideal of $\mathfrak{X}$. □

Theorem 9. *Let J be a pseudo ideal of $\mathfrak{X}$ that satisfies* (28) *and*

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y \;\&\; (x \diamond y) \diamond z = (x \diamond z) \diamond y). \tag{31}$$

Then J is a pseudo q-ideal of $\mathfrak{X}$.

Proof. Let $x, y, z \in X$ be such that $y \in J$, $x * (y \diamond z) \in J$ and $x \diamond (y * z) \in J$. Using (28) and (31), we obtain $(x*z)*y = (x*y)*z \in J$ and $(x \diamond z) \diamond y = (x \diamond y) \diamond z \in J$. Hence $x * z \in *(y, J) \subseteq J$ and $x \diamond z \in \diamond(y, J) \subseteq J$. Therefore J is a pseudo q-ideal of $\mathfrak{X}$. □

Definition 8. A nonempty subset J of $\mathfrak{X}$ is called a *pseudo a-ideal* of $\mathfrak{X}$ if it satisfies (c1) and

$$\begin{array}{l} (x * y) \diamond (0 * z) \in J \;\&\; y \in J \Longrightarrow z \diamond x \in J, \\ (x \diamond y) * (0 \diamond z) \in J \;\&\; y \in J \Longrightarrow z * x \in J \end{array} \tag{32}$$

for all $x, y, z \in X$.

Note that if $\mathfrak{X}$ is a pseudo BCI-algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then the notions of a pseudo a-ideal and an a-ideal coincide.

Theorem 10. *Every pseudo a-ideal of $\mathfrak{X}$ is a pseudo ideal of $\mathfrak{X}$.*

Proof. For any $y \in J$, let $x \in *(y, J)$ and $w \in \diamond(y, J)$. Then $(x * y) \diamond (0 * 0) = x * y \in J$ and $(w \diamond y) * (0 \diamond 0) = w \diamond y \in J$. It follows from (32) that

$$0 \diamond x \in J \text{ and } 0 * w \in J. \tag{33}$$

Putting $z = y = 0$ in (32), we have

$$x \in J \Longrightarrow 0 \diamond x \in J \;\&\; 0 * x \in J. \tag{34}$$

Combining (33) and (34), we get $(0\diamond 0)*(0\diamond x) = 0*(0\diamond x) \in J$ and $(0*0)\diamond(0*w) = 0\diamond(0*w) \in J$. Using (32), we obtain $x = x*0 \in J$ and $w = w\diamond 0 \in J$. Hence $*(y,J) \subseteq J$ and $\diamond(y,J) \subseteq J$. Therefore J is a pseudo ideal of $\mathfrak{X}$. □

Proposition 10. *Every pseudo a-ideal of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{aligned}(x*z)\diamond(0*y) \in J &\Longrightarrow y\diamond(x*z) \in J,\\ (x\diamond z)*(0\diamond y) \in J &\Longrightarrow y*(x\diamond z) \in J\end{aligned} \tag{35}$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ be such that $(x*z)\diamond(0*y) \in J$ and $(x\diamond z)*(0\diamond y) \in J$. Using (b4), we obtain

$$((x*z)*((x*z)\diamond(0*y)))\diamond(0*y) = ((x*z)\diamond(0*y))*((x*z)\diamond(0*y)) = 0 \in J$$

and

$$((x\diamond z)\diamond((x\diamond z)*(0\diamond y)))*(0\diamond y) = ((x\diamond z)*(0\diamond y))\diamond((x\diamond z)*(0\diamond y)) = 0 \in J.$$

It follows from (32) that $y\diamond(x*z) \in J$ and $y*(x\diamond z) \in J$. □

Taking $z = 0$ in (35) and using (b8), we have the following corollary.

Corollary 2. *Every pseudo a-ideal of $\mathfrak{X}$ satisfies the following assertions:*

$$\begin{aligned}x\diamond(0*y) \in J &\Longrightarrow y\diamond x \in J,\\ x*(0\diamond y) \in J &\Longrightarrow y*x \in J\end{aligned} \tag{36}$$

for all $x, y \in X$.

Definition 9. [4] Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo BCI-algebras. A mapping $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *pseudo BCI-homomorphism* if $f(x*y) = f(x)*f(y)$ and $f(x\diamond y) = f(x)\diamond f(y)$ for all $x, y \in X$.

Proposition 11. [4] *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a pseudo BCI-homomorphism from a pseudo BCI-algebra $\mathfrak{X}$ to a pseudo BCI-algebra $\mathfrak{Y}$. Then*

(i) *if J is a pseudo ideal of $\mathfrak{Y}$, then $f^{-1}(J)$ is a pseudo ideal of $\mathfrak{X}$.*
(ii) *if f is surjective and I is a pseudo ideal of $\mathfrak{X}$, then $f(I)$ is a pseudo ideal of $\mathfrak{Y}$.*

Theorem 11. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a pseudo BCI-homomorphism from a pseudo BCI-algebra $\mathfrak{X}$ to a pseudo BCI-algebra $\mathfrak{Y}$. Then*

(i) *if J is an associative pseudo ideal of $\mathfrak{Y}$, then $f^{-1}(J)$ is an associative pseudo ideal of $\mathfrak{X}$.*
(ii) *if J is a pseudo p-ideal of $\mathfrak{Y}$, then $f^{-1}(J)$ is a pseudo p-ideal of $\mathfrak{X}$.*
(iii) *if J is a pseudo q-ideal of $\mathfrak{Y}$, then $f^{-1}(J)$ is a pseudo q-ideal of $\mathfrak{X}$.*

(iv) *if J is a pseudo a-ideal of $\mathfrak{Y}$, then $f^{-1}(J)$ is a pseudo a-ideal of $\mathfrak{X}$.*
(v) *if f is bijective and I is an associative pseudo ideal of $\mathfrak{X}$, then $f(I)$ is an associative pseudo ideal of $\mathfrak{Y}$.*
(vi) *if f is bijective and I is a pseudo p-ideal of $\mathfrak{X}$, then $f(I)$ is a pseudo p-ideal of $\mathfrak{Y}$.*
(vii) *if f is bijective and I is a pseudo q-ideal of $\mathfrak{X}$, then $f(I)$ is a pseudo q-ideal of $\mathfrak{Y}$.*
(viii) *if f is bijective and I is a pseudo a-ideal of $\mathfrak{X}$, then $f(I)$ is a pseudo a-ideal of $\mathfrak{Y}$.*

Proof. (i) Assume that J is an associative pseudo ideal of $\mathfrak{Y}$. Let $x, y, z \in X$ be such that $(x * y) \diamond z \in f^{-1}(J)$ and $y \diamond z \in f^{-1}(J)$. Then

$$(f(x) * f(y)) \diamond f(z) = f((x * y) \diamond z) \in J \text{ and } f(y) \diamond f(z) = f(y \diamond z) \in J.$$

Since J is an associative pseudo ideal of $\mathfrak{Y}$, it follows from (23) that $f(x) \in J$. Hence $x \in f^{-1}(J)$. Similarly, if $(x \diamond y) * z \in f^{-1}(J)$ and $y * z \in f^{-1}(J)$, then $x \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is an associative pseudo ideal of $\mathfrak{X}$.

(ii) Suppose that J is a pseudo p-ideal of $\mathfrak{Y}$ and let $x, y, z \in X$ be such that $y \in f^{-1}(J)$, $(x*z)\diamond(y*z) \in f^{-1}(J)$ and $(x\diamond z)*(y\diamond z) \in f^{-1}(J)$. Then $f(y) \in J$ and

$$(f(x) * f(z)) \diamond (f(y) * f(z)) = f((x * z) \diamond (y * z)) \in J,$$
$$(f(x) \diamond f(z)) * (f(y) \diamond f(z)) = f((x \diamond z) * (y \diamond z)) \in J.$$

It follows from (19) that $f(x) \in J$ so that $x \in f^{-1}(J)$. Hence $f^{-1}(J)$ is a pseudo p-ideal of $\mathfrak{X}$.

(iii) Suppose that J is a pseudo q-ideal of $\mathfrak{Y}$ and let $x, y, z \in X$ be such that $y \in f^{-1}(J)$, $x * (y \diamond z) \in f^{-1}(J)$ and $x \diamond (y * z) \in f^{-1}(J)$. Then $f(y) \in J$ and

$$f(x) * (f(y) \diamond f(z)) = f(x * (y \diamond z)) \in J,$$
$$f(x) \diamond (f(y) * f(z)) = f(x \diamond (y * z)) \in J.$$

It follows from (26) that $f(x*z) = f(x)*f(z) \in J$ and $f(x\diamond z) = f(x)\diamond f(z) \in J$ so that $x * z \in f^{-1}(J)$ and $x \diamond z \in f^{-1}(J)$. Hence $f^{-1}(J)$ is a pseudo q-ideal of $\mathfrak{X}$.

(iv) Assume that J is a pseudo a-ideal of $\mathfrak{Y}$. Let $x, y, z \in X$ be such that $y \in f^{-1}(J)$, $(x * y) \diamond (0 * z) \in f^{-1}(J)$ and $(x \diamond y) * (0 \diamond z) \in f^{-1}(J)$. Then $f(y) \in J$ and

$$(f(x) * f(y)) \diamond (0 * f(z)) = f((x * y) \diamond (0 * z)) \in J,$$
$$(f(x) \diamond f(y)) * (0 \diamond f(z)) = f((x \diamond y) * (0 \diamond z)) \in J.$$

Using (32), we get $f(z \diamond x) = f(z) \diamond f(x) \in J$ and $f(z * x) = f(z) * f(x) \in J$. Hence $z \diamond x \in f^{-1}(J)$ and $z * x \in f^{-1}(J)$. Therefore $f^{-1}(J)$ is a pseudo a-ideal of $\mathfrak{X}$.

Now, suppose that f is bijective. Let $a, b, c \in Y$. Then $f(x_a) = a$, $f(x_b) = b$ and $f(x_c) = c$ for some $x_a, x_b, x_c \in X$. Assume that I is an associative pseudo ideal of $\mathfrak{X}$. Let $(a * b) \diamond c \in f(I)$ and $b \diamond c \in f(I)$. Then there exist $x, y \in I$ such that $f(x) = (a * b) \diamond c$ and $f(y) = b \diamond c$. It follows that

$$f((x_a * x_b) \diamond x_c) = (f(x_a) * f(x_b)) \diamond f(x_c) = (a * b) \diamond c = f(x) \in f(I)$$

and

$$f(x_b \diamond x_c) = f(x_b) \diamond f(x_c) = b \diamond c = f(y) \in f(I).$$

Hence $(x_a * x_b) \diamond x_c \in I$ and $x_b \diamond x_c \in I$, which imply from (23) that $x_a \in I$. Similarly, if $(a \diamond b) * c \in f(I)$ and $b * c \in f(I)$, then $a \in f(I)$. Therefore $f(I)$ is an associative pseudo ideal of $\mathfrak{Y}$. Suppose that I is a pseudo p-ideal of $\mathfrak{X}$. let $b \in f(I)$, $(a * c) \diamond (b * c) \in f(I)$ and $(a \diamond c) * (b \diamond c) \in f(I)$. Then there exist $x, x_\diamond, x_* \in I$ such that $f(x) = b$, $f(x_\diamond) = (a*c)\diamond(b*c)$ and $f(x_*) = (a\diamond c)*(b\diamond c)$. It follows that $b = f(x) \in f(I)$ and

$$\begin{aligned} f((x_a * x_c) \diamond (x * x_c)) &= (f(x_a) * f(x_c)) \diamond (f(x) * f(x_c)) \\ &= (a * c) \diamond (b * c) \;=\; f(x_\diamond) \in f(I), \end{aligned}$$

$$\begin{aligned} f((x_a \diamond x_c) * (x \diamond x_c)) &= (f(x_a) \diamond f(x_c)) * (f(x) \diamond f(x_c)) \\ &= (a \diamond c) * (b \diamond c) \;=\; f(x_*) \in f(I). \end{aligned}$$

Hence $(x_a * x_c) \diamond (x * x_c) \in I$ and $(x_a \diamond x_c) * (x \diamond x_c) \in I$, which imply from (19) that $x_a \in I$. Thus $a = f(x_a) \in f(I)$, and so $f(I)$ is a pseudo p-ideal of $\mathfrak{Y}$. Assume that I is a pseudo q-ideal of $\mathfrak{X}$. Let $b \in f(I)$, $a * (b \diamond c) \in f(I)$ and $a \diamond (b * c) \in f(I)$. Then $f(x) = b$, $f(x_*) = a * (b \diamond c)$ and $f(x_\diamond) = a \diamond (b * c)$ for some $x, x_*, x_\diamond \in I$. It follows that

$$f(x_a * (x \diamond x_c)) = f(x_a) * (f(x) \diamond f(x_c)) = a * (b \diamond c) = f(x_*) \in f(I)$$

and

$$f(x_a \diamond (x * x_c)) = f(x_a) \diamond (f(x) * f(x_c)) = a \diamond (b * c) = f(x_\diamond) \in f(I).$$

Hence $x_a \diamond (x * x_c) \in I$ and $x_a * (x \diamond x_c) \in I$. Using (26), we have $x_a * x_c \in I$ and $x_a \diamond x_c \in I$, and so

$$a * c = f(x_a) * f(x_c) = f(x_a * x_c) \in f(I)$$

and

$$a \diamond c = f(x_a) \diamond f(x_c) = f(x_a \diamond x_c) \in f(I).$$

Consequently, $f(I)$ is a pseudo q-ideal of $\mathfrak{Y}$. Finally, suppose that I is a pseudo a-ideal of $\mathfrak{X}$. Let $b \in f(I)$, $(a * b) \diamond (0 * c) \in f(I)$ and $(a \diamond b) * (0 \diamond c) \in f(I)$. Then there exist $y, y_\diamond, y_* \in I$ such that $f(y) = b$, $f(y_\diamond) = (a * b) \diamond (0 * c)$ and $f(y_*) = (a \diamond b) * (0 \diamond c)$. Hence

$$\begin{aligned} f((x_a * y) \diamond (0 * x_c)) &= (f(x_a) * f(y)) \diamond (f(0) * f(x_c)) \\ &= (a * b) \diamond (0 * c) \;=\; f(y_\diamond) \in f(I) \end{aligned}$$

and

$$\begin{aligned} f((x_a \diamond y) * (0 \diamond x_c)) &= (f(x_a) \diamond f(y)) * (f(0) \diamond f(x_c)) \\ &= (a \diamond b) * (0 \diamond c) \;=\; f(y_*) \in f(I). \end{aligned}$$

It follows that $(x_a * y) \diamond (0 * x_c) \in I$ and $(x_a \diamond y) * (0 \diamond x_c) \in I$. Since I is a pseudo a-ideal of $\mathfrak{X}$, we have $x_c \diamond x_a \in I$ and $x_c * x_a \in I$ by (32). Therefore

$$c \diamond a = f(x_c) \diamond f(x_a) = f(x_c \diamond x_a) \in f(I)$$

and

$$c * a = f(x_c) * f(x_a) = f(x_c * x_a) \in f(I).$$

Consequently, $f(I)$ is a pseudo a-ideal of $\mathfrak{Y}$. □

REFERENCES

1. W. A. Dudek and Y. B. Jun, *Pseudo BCI-algebras,* East Asian Math. J. (to appear).
2. G. Georgescu and A. Iorgulescu, *Pseudo-BCK algebras: an extension of BCK algebras,* Combinatorics, computability and logic (Constanţa, 2001), 97–114, Springer Ser. Discrete Math. Theor. Comput. Sci. Springer, London, 2001.
3. Y. B. Jun, *Characterizations of pseudo-BCK algebras,* Sci. Math. Jpn. **57** (2003), no. 2, 265-270.
4. Y. B. Jun, H. S. Kim and J. Neggers, *On pseudo-BCI ideals of pseudo BCI-algebras,* Mat. Vesnik, **58** (2006), 39–46.
5. Y. B. Jun, M. Kondo and K. H. Kim, *Pseudo-ideals of pseudo-BCK algebras,* Sci. Math. Jpn. **58** (2003), no. 1, 93–97.
6. Y. L. Liu, S. Y. Liu and Y. Xu, *Pseudo-BCK algebras and PD-posets,* Soft Comput. **11** (2007), 91–101.
7. Y. L. Liu, J. Meng, X. H. Zhang and Z. C. Yue, *q-ideals and a-ideals in BCI-algebras,* Southeast Asian Bull. Math. **24** (2000), 243–253.
8. X. H. Zhang, H. Jiang and S. A. Bhatti, *On p-ideals of a BCI-algebra,* Punjab Univ. J. Math. **27** (1994), 121–128.

Kyoung Ja Lee received her Ph.D degree from Yonsei University, Korea, in 2000. From 2001 to 2002, she was a Post-Doc researcher at Korea Universty, Korea. She is currently a faculty member of the Hannam University in Daejeon, Korea. Her research interests are in the areas of Fuzzy algebraic structure, BCK/BCI/*d*-algebraiac structure, Homological algebraic structure, and Representation theory.

Department of Mathematics Education, Hannam University, Daejeon 306-791, Korea
e-mail: `kjlee@hnu.kr`

Chul Hwan Park received his B.S., M.S. and Ph.D. degree from the Department of Mathematics of University of Ulsan,Korea, in 1986,1988 and 1997 respectively. From 1997 to 1998, he was a researcher at the Institute of Basic Science,The Kyungpook National Universty,Korea(supported by KOSEF). He is currently a full time lecture at the Department of Mathematics in University of Ulsan,Korea since 2005. His research interests are in the areas of Fuzzy Algebraic Structure,BCK-algebra,Quantum Structure ,semigroup and Commutative ring.

Department of Mathematics, University of Ulsan, Ulsan 680-749, Korea
e-mail: `skyrosemary@gmail.com`