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FRAME AND LATTICE SAMPLING THEOREM FOR SUBSPACES OF $L^2(R)$

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ABSTRACT. In this paper, a necessary and sufficient condition for lattice sampling theorem to hold for frame in subspaces of $L^2(R)$ is established. In addition, we obtain the formula of lattice sampling function in frequency space. Furthermore, by discussing the parameters in Theorem 3.1, some corresponding corollaries are derived.

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1. Introduction

The shannon sampling theorem has been extensively referred to in signal and image processing literature. It has many applications and generalizations. From wavelet transform point of view, the theorem provides the sinc wavelets. Walter extended the sinc wavelets to general wavelet subspaces [1]. From then on, many results have been derived [2-4]. Chen and Itoh [5] studied the necessary and sufficient conditions for the sampling theorem to hold for Riesz basis in shift invariant subspaces. Furthermore, Zhao and Liu in [6] generalized their result to hold for frame in shift invariant subspaces, deriving the following result [6, Theorem 2]:

Suppose that $\{\varphi(t-k)\}_{k\in Z}$ is a frame for the subspace $V_{1,0}(\varphi)$, such that the sampling sequence at the integers $\{\varphi(n)\}_{n\in Z} \in l^2$. Then, there exists a function $s(t) \in V_{1,0}(\varphi)$ such that

$$g(t) = \sum_{n\in Z} g(n)s(t-n), \forall g(t) \in V_{1,0}(\varphi) \tag{1}$$

holds in the $L^2(R)$ sense if and only if the following condition:

$$\frac{1}{\hat{\varphi}^*(\omega)}\chi_{E_\varphi} \in L^2[0,2\pi] \tag{2}$$

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holds. In this case

$$\hat{s}(\omega) = \begin{cases} \frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)}, & \omega \in E_\varphi \\ 0, & \omega \notin E_\varphi \end{cases} \tag{3}$$

holds for *a.e.* $\omega \in R$, where

$$E_\varphi = \{\omega \in R | G_\varphi(\omega) > 0\}, \tag{4}$$

$$G_\varphi(\omega) = (\sum_{k\in Z} |\hat{\varphi}(\omega + 2\pi k)|^2)^{\frac{1}{2}}, \tag{5}$$

$$\hat{\varphi}^*(\omega) = \sum_{n\in Z} \varphi(n) e^{-in\omega}, \tag{6}$$

$$\chi_{E_\varphi} = \begin{cases} 1, & t \in E_\varphi \\ 0, & t \notin E_\varphi. \end{cases} \tag{7}$$

All their results just base on the sequence of sampling at the integers(briefly: integer sampling sequence). In this paper, we generalize the integer sampling sequence to the lattice $aZ + b$, $a > 0$, $b \in R$ sampling sequence and get a necessary and sufficient condition for the lattice sampling theorem to hold for frame $\{\phi(\cdot - an)\}_{n\in Z,\ a>0}$ in subspaces of $L^2(R)$. Furthermore, the formula of lattice sampling function in frequency space is obtained and, by discussing the parameter in Theorem 1, several very useful corollaries are derived. This is an improvement of Zhao and Liu's result.

An outline of the paper is as follows: Section 2 contains some necessary definitions and lemmas. In section 3, we prove a necessary and sufficient condition for the lattice sampling theorem to hold for frame in subspaces of $L^2(R)$ and present some corresponding corollaries. Finally, an example is given to prove our theory.

2. Preliminary

We now introduce some definitions used in this correspondence.

A collection of elements $\{\mu_j\}_{j\in J}$ in a Hilbert space H is called a *frame* if there exist constants A and B, $0 < A \le B < \infty$, such that

$$A\|f\|^2 \le \sum_{j\in J} |\langle f, \mu_j\rangle|^2 \le B\|f\|^2, \quad \forall f \in H. \tag{8}$$

Let A_0 be the supremum of all such numbers A and B_0 be the infimum of all such numbers B, then A_0 and B_0 are called the frame bounds of the frame $\{\mu_j\}_{j\in J}$. When $A_0 = B_0$ we say that the frame is tight. When $A_0 = B_0 = 1$ we say that the frame is normalized tight. Any orthonormal basis in a Hilbert space is a normalized tight frame but not vice versa.

Definition 1. Let $t_b : L^2(R) \to L^2(R)$ be an unitary operator. For $\forall h \in H$, if $t_b h(x) = h(x - b)$ $b \in R$, then we call t_b translation operator.

It is clearly that t_b^{-1} exists and $t_b^{-1} h(x) = h(x + b)$.

Definition 2. Let $D_a: L^2(R) \to L^2(R)$ be an unitary operator. For $\forall h \in H$, if $D_a h(x) = h(\frac{x}{a}),\ a > 0,\ a \in R$, then we call D_a dilation operator.

It is easy to know that D_b^{-1} exists and $D_b^{-1}\varphi(x) = \varphi(ax)$. Let $T = t_b D_a$. We define $V_{a,b} = V_{a,b}(\phi) = T(V_{1,0}(\varphi))$ and $V_{1,0} = V_{1,0}(\varphi)$, where $\phi = T\varphi$. Throughout this article, $a > 0,\ b \in R$ hold.

Definition 3. Let $\{\phi(x - an)\}_{n\in Z}$ be a frame of $V_{a,b}$, where $V_{a,b} \subset L^2(R)$. If there exists function $S \in V_{a,b}$ such that $f(x) = \sum_{n\in Z} f(an + b)S(x - an), \forall f \in V_{a,b}$ holds, then we call $V_{a,b}$ the sampling subspace and S the sampling function.

Similar to (4)-(7), we can write

$$E_\phi = \{\omega \in R | G_\phi(\omega) > 0\}, \tag{9}$$

$$G_\phi(\omega) = (\sum_{k\in Z} |\hat{\phi}(\omega + \frac{2\pi k}{a})|^2)^{\frac{1}{2}}, \tag{10}$$

$$\hat{\phi}^*(\omega) = \sum_{n\in Z} \phi(an + b)e^{-ina\omega}, \tag{11}$$

$$\chi_{E_\phi} = \begin{cases} 1, & t \in E_\phi \\ 0, & t \notin E_\phi. \end{cases} \tag{12}$$

For proving Theorem 1, we need discussing the properties of the operator T mentioned above.

Lemma 1. *If $T = t_b D_a$, then the following statements hold:*

1. T is a bounded linear operator.

2. T^{-1} exists, and $T^{-1} = D_a^{-1} t_b^{-1}$.

3. $t_{ab} D_a = D_a t_b$, $t_n D_a^{-1} = D_a^{-1} t_{an}$.

Proof. 1.1 For $\forall g \in V_{1,0}$, we have

$$\begin{aligned} \frac{\|Tg\|}{\|g\|} &= \frac{\{\int_R |g(\frac{x-b}{a})|^2 dx\}^{\frac{1}{2}}}{\{\int_R |g(x)|^2 dx\}^{\frac{1}{2}}} \\ &= \frac{\{a\int_R |g(\frac{x-b}{a})|^2 d\frac{x-b}{a}\}^{\frac{1}{2}}}{\{\int_R |g(x)|^2 dx\}^{\frac{1}{2}}} \\ &= a^{\frac{1}{2}}. \end{aligned} \tag{13}$$

Hence T is a bounded operator. For $\forall g,\ h \in V_{1,0}$, we have $T(g + h) = f(\frac{x-b}{a}) + h(\frac{x-b}{a}) = Tg + Th$. Then T is a bounded linear operator.

1.2 From above results, it is obvious that T^{-1} exists. For $\forall g \in V_{1,0}$, the following equations hold:

$$\begin{aligned} TD_a^{-1}t_b^{-1}g(x) = t_b D_a D_a^{-1} t_b^{-1} g(x) = t_b D_a g(ax + b) = g(x), \\ D_a^{-1}t_b^{-1}Tg(x) = D_a^{-1}t_b^{-1}t_b D_a g(x) = D_a^{-1}t_b^{-1}g(\tfrac{x-b}{a}) = g(x). \end{aligned} \tag{14}$$

Hence $T^{-1} = t_b D_a$.

1.3 For $\forall g \in V_{1,0}$, we have

$$\begin{aligned} t_{ab}D_a g(x) &= g(\frac{x-ab}{a}) = D_a t_b g(x), \\ D_a^{-1} t_{an} g(x) &= g(ax-an) = t_n D_a^{-1} g(x). \end{aligned} \tag{15}$$

So $t_{ab}D_a = D_a t_b$, $D_a^{-1}t_{an} = t_n D_a^{-1}$. □

We also need the following lemmas.

Lemma 2. *If $\{\varphi(t-n)\}_{n\in Z}$ is the frame of $V_{1,0} \subset L^2(R)$ and $V_{a,b} = T(V_{1,0})$, then the operator T maps the frame of $V_{1,0}$ into the frame of $V_{a,b}$.*

Proof. Let $\{\varphi(x-n)\}_{n\in Z}$ be the frame of $V_{1,0}$. By the definition of frame, if $\forall g \in V_{1,0}$, then there must exist constant numbers A, B, $B \geq A > 0$ such that

$$A\|g\|^2 \leq \sum |\langle g, \varphi_n\rangle|^2 \leq B\|g\|^2. \tag{16}$$

By $T = t_b D_a$ and Lemma 1, there must exist $g \in V_{1,0}$ such that $f = Tg, \forall f \in V_{a,b}$. Hence

$$\begin{aligned} \|f\|^2 &= \int_R |Tg(x)|^2 dx \\ &= \int_R a|g(\frac{x-b}{a})|^2 d\frac{x-b}{a} \\ &= a\|g\|^2. \end{aligned} \tag{17}$$

Let $\phi = T\varphi$. Then

$$\begin{aligned} \sum_{n\in Z} |\langle f, \phi_{an}\rangle|^2 &= \sum_{n\in Z} |\langle t_b D_a f, t_{an} t_b D_a \varphi\rangle|^2 \\ &= \sum_{n\in Z} \left| \int_R g(\frac{x-b}{a})\overline{\varphi}(\frac{x-b-an}{a})dx \right|^2 \\ &= a^2 \sum_{n\in Z} \left| \int_R g(x')\overline{\varphi}(x'-n)dx' \right|^2 \\ &= a^2 \sum_{n\in Z} |\langle g, \varphi_n\rangle|^2. \end{aligned} \tag{18}$$

Using (16) and (18), we have

$$Aa\|f\|^2 = Aa^2\|g\|^2 \leq \sum_{n\in Z} |\langle f, \phi_{an}\rangle|^2 \leq Ba^2\|g\|^2 = Ba\|f\|^2. \tag{19}$$

According to the definition of frame, $\{\phi(x-an)\}_{n\in Z}$ is the frame of subspace $V_{a,b}$. This completes the proof of Lemma 2. □

If replacing the operator T in Lemma 2 with T^{-1}, we get

Lemma 3. *If $\{\phi(x-an)\}_{n\in Z}$ is the frame of $V_{a,b} \subset L^2(R)$ and $V_{a,b} = T(V_{1,0})$, then the operator T^{-1} maps the frame of $V_{a,b}$ into the frame of $V_{1,0}$.*

Since the proof of Lemma 3 is similar to the proof of Lemma 2, we leave it to the reader.

3. Main result

In this section, we prove a necessary and sufficient condition for lattice sampling theorem to hold for frame in subspaces of $L^2(R)$ and present some corresponding corollaries.

Theorem 1. *Suppose that $\{\phi_n(x-an)\}_{n\in Z}$ is a frame for the subspace $V_{a,b}$, such that the lattice sampling sequence $\{\phi(an+b)\}_{n\in Z} \in l^2$. Then, there exists a function $S \in V_{a,b}$ such that*

$$f(x) = \sum_{n\in Z} f(an+b)S(x-an), \ \forall f \in V_{a,b} \tag{20}$$

holds in the $L^2(R)$ sense if and only if the following condition:

$$\frac{1}{\hat{\phi}^*(\omega)}\chi_{E_\phi} \in L^2[0, \frac{2\pi}{a}] \tag{21}$$

holds. In this case

$$\hat{S}(\omega) = \begin{cases} \frac{\hat{\phi}(\omega)}{\hat{\phi}^*(\omega)} & \omega \in E_\phi \\ 0 & \omega \notin E_\phi \end{cases} \tag{22}$$

holds for a.e. $\omega \in R$.

Proof. Necessity: Suppose that $V_{a,b}$ is a sampling subspace of $L^2(R)$, and there exists $S \in V_{a,b}$ such that $f(x) = \sum_{n\in Z} f(an+b)S(x-an)$. If $\{\phi_n(x-an)\}_{n\in Z}$ is the frame of $V_{a,b}$, by Lemma 3, for $\forall g \in V_{1,0}$, there must exist $f \in V_{a,b}$ such that

$$\begin{aligned} g(x) &= \sum_{n\in Z} f(an+b)T^{-1}S(x-an) \\ &= \sum_{n\in Z} TT^{-1}f(an+b)T^{-1}S(x-an) \\ &= \sum_{n\in Z} g(n)D_a^{-1}t_b^{-1}t_{an}S(x) \\ &= \sum_{n\in Z} g(n)s(x-n). \end{aligned} \tag{23}$$

holds. So $s \in V_{1,0}$ is a sampling function in $V_{1,0}$ and (2) and (3) hold[6, Theorem 2]. Since $S(x) = T(s(x)) = s(\frac{x-b}{a})$, we have

$$\begin{aligned} \hat{S}(\omega) &= \int_R S(x)e^{-i\omega x}dx \\ &= \int_R s(\frac{x-b}{a})e^{-i\omega x}dx \\ &= ae^{-ib\omega}\int_R s(\frac{x-b}{a})e^{-ia\omega\frac{x-b}{a}}d\frac{x-b}{a} \\ &= ae^{-ib\omega}\hat{s}(a\omega). \end{aligned} \tag{24}$$

Repeating the above process, we get

$$\hat{\phi}(\omega) = ae^{-ib\omega}\hat{\varphi}(a\omega). \tag{25}$$

By (5), (10) and (32), then

$$\begin{aligned} G_\varphi(a\omega) &= \left(\sum_{k\in Z} |\hat{\varphi}(a\omega+2k\pi)|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{k\in Z} \frac{1}{a^2}\left|a\hat{\varphi}\left(a\left(\omega+\frac{2k\pi}{a}\right)\right)e^{-ib\omega}\right|^2\right)^{\frac{1}{2}} \\ &= \frac{1}{a}G_\phi(\omega). \end{aligned} \tag{26}$$

Hence

$$E_\phi = \{\omega \in R | G_\phi(\omega) > 0\} = \{\omega \in R | G_\varphi(a\omega) > 0\} \tag{27}$$

holds. By

$$\begin{aligned} \hat{\varphi}^*(a\omega) &= \sum_{n\in Z} \varphi(n)e^{-ina\omega} \\ &= \sum_{n\in Z} T^{-1}T\varphi(n)e^{-ina\omega} \\ &= \sum_{n\in Z} T^{-1}\phi(n)e^{-ina\omega} \\ &= \hat{\phi}^*(\omega) \end{aligned} \tag{28}$$

and

$$\frac{1}{\hat{\varphi}^*(\omega)}\chi_{E_\varphi}(\omega) \in L^2[0,2\pi] \Leftrightarrow \frac{1}{\hat{\varphi}^*(a\omega)}\chi_{E_\varphi}(a\omega) \in L^2[0,\frac{2\pi}{a}], \tag{29}$$

(21) is true.

According to (3), (24), (25), (27) and (28), we have (22). Necessity is proved.

Sufficiency: Suppose that (21), (22) hold, the operator T^{-1} maps the subspace $V_{a,b}$ into the subspace $V_{1,0}$. Let $S \in V_{a,b}$, then $\exists\ s \in V_{1,0}$ such that

$$s(x) = T^{-1}(S(x)) = S(ax+b). \tag{30}$$

Similar to the argument of (24) and (25), we have

$$\hat{s}(\omega) = \frac{1}{a}e^{i\frac{b}{a}\omega}\hat{S}\left(\frac{\omega}{a}\right) \tag{31}$$

and

$$\hat{\varphi}(\omega) = \frac{1}{a}e^{i\frac{b}{a}\omega}\hat{\phi}\left(\frac{\omega}{a}\right). \tag{32}$$

By (32), (5) and (10), then

$$\begin{aligned} G_\phi\left(\tfrac{\omega}{a}\right) &= \left(\sum_{k\in Z}\left|\hat{\phi}\left(\frac{\omega}{a}+\frac{2k\pi}{a}\right)\right|^2\right)^{\frac{1}{2}} \\ &= \left(a^2\sum_{k\in z}\left|\frac{1}{a}\hat{\phi}\left(\frac{\omega+2k\pi}{a}\right)e^{-i\frac{b}{a}\omega}\right|^2\right)^{\frac{1}{2}} \\ &= aG_\varphi(\omega). \end{aligned} \tag{33}$$

Since $a > 0$, we have

$$aG_\varphi(\omega) > 0 \Leftrightarrow G_\varphi(\omega) > 0. \tag{34}$$

From the equations (34), (4) and (9), we can get the following result

$$E_\phi = \left\{\omega \in R | G_\phi\left(\frac{\omega}{a}\right) > 0\right\} = \{\omega \in R | G_\varphi(\omega) > 0\} = E_\varphi. \tag{35}$$

Due to

$$\frac{1}{\hat{\phi}^*(\omega)}\chi_{E_\phi}(\omega) \in L^2[0, \frac{2\pi}{a}] \Leftrightarrow \frac{1}{\hat{\phi}^*(\frac{\omega}{a})}\chi_{E_\phi}(\frac{\omega}{a}) \in L^2[0, 2\pi], \tag{36}$$

$$\begin{aligned} \hat{\phi}^*\left(\frac{\omega}{a}\right) &= \sum_{n\in Z}\phi(an+b)e^{-ina\frac{\omega}{a}} \\ &= \sum_{n\in Z}T\varphi(an+b)e^{-in\omega} \\ &= \sum_{n\in Z}\varphi(n)e^{-in\omega} \\ &= \hat{\varphi}^*(\omega) \end{aligned} \tag{37}$$

and (35), (2) holds.

Again by (22), (31), (32) and (37), we conclude that

$$\hat{s}(\omega) = \frac{1}{a}e^{i\frac{b}{a}\omega}\hat{S}(\frac{\omega}{a}) = \begin{cases} \frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)} & \omega \in E_\varphi \\ 0 & \omega \notin E_\varphi \end{cases} \quad a.e\ \omega \in R \tag{38}$$

holds. If (2) and (38) hold[6, Theorem 2], then there must exists a sampling function $s \in V_{1,0}$ such that $g(x) = \sum_{n\in Z} g(n)s(x-n)$. Hence $\forall f \in V_{a,b}$, we have $g \in V_{1,0}$ such that $f = Tg$ and

$$\begin{aligned} f(x) &= \sum_{n\in Z} g(n)Ts(x-n) \\ &= \sum_{n\in Z} T^{-1}f(n)Ts(x-n) \\ &= \sum_{n\in Z} f(an+b)t_{an}Ts(x) \\ &= \sum_{n\in Z} f(an+b)S(x-an). \end{aligned} \tag{39}$$

From above results, we derive that the function S is an sampling function of $V_{a,b}$. Sufficiency is proved. □

Because the parameters a, b are very important in Theorem 1, we now discuss them and get some very useful corollaries. If $a = 1, b \in R$, we have

Corollary 1. *Suppose that $\{\phi_n(x-n)\}_{n\in Z}$ is a frame for the subspace $V_{1,b}$, such that the lattice sampling sequence $\{\phi(n+b)\}_{n\in Z} \in l^2, b \in R$. Then, there exists a function $S \in V_{1,b}$ such that*

$$f(x) = \sum_{n\in Z} f(n+b)S(x-n),\ \forall f \in V_{a,b} \tag{40}$$

holds in the $L^2(R)$ sense if and only if the following condition:

$$\frac{1}{\hat{\phi}^*(\omega)}\chi_{E_\phi} \in L^2[0,2\pi] \tag{41}$$

holds. In this case

$$\hat{S}(\omega) = \begin{cases} \frac{\hat{\phi}(\omega)}{\hat{\phi}^*(\omega)} & \omega \in E_\phi \\ 0 & \omega \notin E_\phi \end{cases} \tag{42}$$

holds for a.e. $\omega \in R$.

If $a > 0, b = 0$, we have

Corollary 2. *Suppose that $\{\phi_n(x-an)\}_{n\in Z}$ is a frame for the subspace $V_{a,0}$, such that the lattice sampling sequence $\{\phi(an)\}_{n\in Z} \in l^2$. Then, there exists a function $S \in V_{a,0}$ such that*

$$f(x) = \sum_{n\in Z} f(an)S(x-an),\ \forall f \in V_{a,b} \tag{43}$$

holds in the $L^2(R)$ sense if and only if the following condition:

$$\frac{1}{\hat{\phi}^*(\omega)}\chi_{E_\phi} \in L^2[0,\frac{2\pi}{a}] \tag{44}$$

holds. In this case

$$\hat{S}(\omega) = \begin{cases} \frac{\hat{\phi}(\omega)}{\hat{\phi}^*(\omega)} & \omega \in E_\phi \\ 0 & \omega \notin E_\phi \end{cases} \tag{45}$$

holds for a.e. $\omega \in R$.

Remark. If the parameter $a = 1, b = 0$ in Theorem 1, then we get [6, Theorem 2].

At last, we give an example that satisfies our conditions but not those in previous paper.

Example. Let $\hat{\phi}(\omega) = \chi_{[-4\pi a,4\pi a)}(\omega), 0 < a < \frac{1}{2}$. Then $G_\phi(\omega) = \chi_{[-4\pi a,4\pi a)}(\omega)$ in $[-2\pi, 2\pi)$. Since $\hat{\phi}^*(\omega) = \sum_{k\in Z}\hat{\phi}(\omega + k\pi)$ in $L^2[0,\pi]$[5], we have

$$\hat{\phi}^*(\omega)\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]} = \chi_{[-\pi a,\pi a]}(\omega).$$

It is easy to verify that $\{\phi(x-2n)\}_{n\in Z}$ is a frame for $V_{2,3}$[7]. By [5], $\{\phi(x-2n)\}_{n\in Z}$ is not a Riesz basis for a subspace $V_{2,3}$ and [6, Theorem 2] cannot be applied to deal with the $\phi(t)$. However, $\frac{1}{\hat{\phi}^*(\omega)}\chi_{E_\phi} \in L^2[0,\pi]$ implies that our sampling theorem is available. The $\hat{S}(\omega)$ is given by

$$\hat{S}(\omega) = \begin{cases} \frac{\hat{\phi}(\omega)}{\hat{\phi}^*(\omega)} & \omega \in E_\phi \\ 0 & \omega \notin E_\phi \end{cases} = \chi_{[-\pi a,\pi a]}(\omega).$$

4. Conclusion

In this paper, the necessary and sufficient condition for lattice sampling theorem to hold for frame in subspaces of $L^2(R)$ is studied. This conclusion is useful in providing theoretical basis for constructing function S to satisfy the sampling theorem. We find a weaker sampling theorem than the integer sampling theorem such that (20) holds and generalize Zhao and Liu's result to lattice sampling sequence.

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