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EXISTENCE OF TRIPLE POSITIVE SOLUTIONS OF A KIND OF SECOND-ORDER FOUR-POINT BVP

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ABSTRACT. In this paper, we considered the following four-point boundary value problem

$$\begin{cases} x''(t) + h(t)f(t, x(t), x'(t)) = 0, & 0 < t < 1 \\ x'(0) = ax(\xi), & x'(1) = bx(\eta), \end{cases}$$

where $0 < \xi < \eta < 1$, $\delta = ab\xi - ab\eta + a - b > 0$, $0 < a < \frac{1}{\xi}$, $0 < b < \frac{1}{\eta}$. After the discussion of the Green function of the corresponding homogeneous system, we establish some criteria for the existence of positive solutions by using the generalized Leggett-William's fixed point theorem. The interesting point is the expression of the Green function, which is a difficulty for multi-point BVP.

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1. Introduction

As we all know, boundary value problems (BVPs) for second-order differential equations have been studied extensively. There are of many good results for the existence of solutions, multiple solutions, positive solutions that have been obtained, for details, see[1-3,8] and the references therein.

In the vast field of the research of differential boundary value problems, multi-point boundary value problems were being payed much attention. We refer the readers to [9-13] and the references therein. Recently, there are some authors studied a series four point boundary value problem, see [7,14,15]. In [7] and [15], the authors considered the second-order four-point boundary value problems at

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resonance

$$\begin{cases} x''(t) + q(t)f(t, x(t)) = 0, \quad 0 < t < 1, \\ x(0) = ax(\xi), \quad x(1) = bx(\eta), \end{cases} \tag{1.1}$$

and

$$\begin{cases} x''(t) + h(t)f(t, x(t)) = 0, \quad 0 < t < 1, \\ x'(0) = ax(\xi), \quad x'(1) = bx(\eta), \end{cases} \tag{1.2}$$

where the resonance condition in BVP (1.1) is $a\xi(1-b)+(1-a)(1-b\eta) = 0$, in BVP (1.2) is $b-a+ab(\eta-\xi) = 0$. The methods used in [10] and [11] are mainly the techniques of the upper and lower solutions and the coincidence degree theory.

In a late work [6], by using the generalized Leggett-William's fixed point theorem. Bai, Ge and Wang obtained the existence of triple positive solutions for four point boundary value problem

$$\begin{cases} x''(t) + q(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < 1, \\ x(0) = ax(\xi), \qquad x(1) = bx(\eta). \end{cases} \tag{1.3}$$

The emphasis is put on the nonlinear term involved with the first-order derivative and they supposed that $a\xi(1-b) + (1-a)(1-b\eta) > 0$ in BVP (1.3).

Motivated by the works mentioned above, in this paper, we aim to discuss the existence of three positive solutions for four-point boundary value problem:

$$\begin{cases} x''(t) + h(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < 1 \\ x'(0) = ax(\xi), \qquad x'(1) = bx(\eta), \end{cases} \tag{1.4}$$

where $0 < \xi < \eta < 1$, $\delta = ab\xi - ab\eta + a - b > 0$, $0 < a < \frac{1}{\xi}$, $0 < b < \frac{1}{\eta}$. Throughout, we will always assume the following conditions hold.

(A_1) $h : (0, 1) \to [0, \infty)$ is continuous and $h \not\equiv 0$ on any subinterval of $[0, 1]$.

(A_2) $f : [0, 1] \times (0, \infty) \times R \to [0, \infty)$ is continuous and $f \not\equiv 0$.

To the best knowledge of the authors, few work has been done to discuss second order differential equation with such nonresonance boundary value conditions. Our interest is in the discussion of the Green function of the corresponding homogeneous system, which has never been calculated up to now. So it need to overcome some difficulties to give its expression . Also, we give sufficient conditions to guarantee the Green function to be positive. By using the generalized Leggett-Williams' fixed point theorem gave in [5], some new results for the multiplicity of positive solutions of BVP (1.4) are obtained.

2. Some background material

For the convenience of the reader, we present some definitions and lemmas, which are important during the proof of our main results.

Definition 2.1. Let E be a Banach Space. $P \subset E$ is a nonempty convex closed set, P is said to be a cone provided that

(1) $au \in P$ for all $a \geq 0$, $u \in P$;
(2) $u \in P$, $-u \in P$ implies $u = 0$.

Definition 2.2. A map ψ is said to be a nonnegative concave continuous functional on P provided that $\psi : P \to [0, \infty)$ is continuous and

$$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y)$$

for all $x, y \in P$ and $0 \leq \lambda \leq 1$. Similarly, we say the map φ is a nonnegative continuous convex functional on P provided that $\varphi : P \to [0, \infty)$ is continuous and

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

for all x, $y \in P$ and $0 \leq \lambda \leq 1$.

Definition 2.3. Let $r > d > 0$, $L > 0$ be constants, ψ a nonnegative continuous concave functionals and α, β nonnegative continuous functionals on cone P. Define convex sets

$$P(\alpha, r;\ \beta, L) = \{y \in P \mid \alpha(y) < r,\ \beta(y) < L\},$$

$$\overline{P}(\alpha, r;\ \beta, L) = \{y \in P \mid \alpha(y) \leq r,\ \beta(y) \leq L\},$$

$$P(\alpha, r;\ \beta, L;\ \psi, d) = \{y \in P \mid \alpha(y) < r,\ \beta(y) < L,\ \psi > d\},$$

$$\overline{P}(\alpha, r;\ \beta, L;\ \psi, d) = \{y \in P \mid \alpha(y) \leq r,\ \beta(y) \leq L,\ \psi \geq d\}.$$

Next, we present a fixed point theorem established in [5], the following assumptions will be needed:

(H_1) There exists $K > 0$ such that $\|x\| \leq K \max\{\alpha(x), \beta(x)\}$ for all $x \in P$;
(H_2) $P(\alpha, r; \beta, L) \neq \emptyset$, for any $r > 0$, $L > 0$.

Lemma 2.1. [5] *Let E be a Banach space, $P \subset E$ is a cone and $r_2 \geq r > d > r_1 > 0$, $L_1, L_2 > 0$. Assume that α, β are nonnegative continuous convex functionals satisfying (H_1) and (H_2), ψ is a nonnegative concave functional on P such that $\psi(y) \leq \alpha(y)$ for all $y \in \overline{P}(\alpha, r_2; \beta, L_2)$ and $T : \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2)$ is a completely continuous operator. Suppose*

(S_1) $\{y \in \overline{P}(\alpha, r; \beta, L_2; \psi, d) | \psi(y) > d\} \neq \emptyset$, $\psi(Ty) > d$ for $y \in \overline{P}(\alpha, r; \beta, L_2; \psi, d)$,
(S_2) $\alpha(Ty) < r_1$, $\beta(Ty) < L_1$ for all $y \in \overline{P}(\alpha, r_1; \beta, L_1)$,
(S_3) $\psi(Ty) > d$ for all $y \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, d)$ with $\alpha(Ty) > r_2$.

Then T has at least three fixed points y_1, y_2, y_3 in $\overline{P}(\alpha, r_2; \beta, L_2)$ with

$$y_1 \in P(\alpha, r; \beta, L_1),\ \ y_2 \in \left\{\overline{P}(\alpha, r_2; \beta, L_2; \psi, d) | \psi(y) > d\right\}$$

and

$$y_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus \left(\overline{P}(\alpha, r_2; \beta, L_2; \psi, d) \cup \overline{P}(\alpha, r_1; \beta, L_1)\right)$$

Lemma 2.2. *The Green function for BVP*

$$\begin{cases} -x''(t)=0, & 0<t<1, \\ x'(0)=ax(\xi), & x'(1)=bx(\eta), \end{cases} \tag{2.1}$$

is

$$G(t,\ s)=\begin{cases} G_1(t,\ s), & 0\le s\le \min\{t,\xi\}\le 1, \\ G_2(t,\ s), & 0\le t\le s\le \xi, \\ G_3(t,\ s), & \xi\le s\le \min\{t,\eta\}\le 1, \\ G_4(t,\ s), & 0\le \max\{\xi,t\}\le s\le \eta, \\ G_5(t,\ s), & \eta\le s\le t\le 1, \\ G_6(t,\ s), & 0\le \max\{\eta,t\}\le s\le 1, \end{cases}$$

where

$$\begin{aligned} G_1(t,\ s)&=\frac{1}{\delta}(bt+1-b\eta), \\ G_2(t,\ s)&=\frac{1}{\delta}(bs+1-b\eta)+\frac{1}{\delta}(s-t)(ab\eta-ab\xi-a), \\ G_3(t,\ s)&=\frac{1}{\delta}(bt+1-b\eta)(as+1-a\xi), \\ G_4(t,\ s)&=\frac{1}{\delta}(bs+1-b\eta)(at+1-a\xi), \\ G_5(t,\ s)&=\frac{1}{\delta}(at+1-a\xi)+(s-t), \\ G_6(t,\ s)&=\frac{1}{\delta}(at+1-a\xi), \end{aligned}$$

and $\delta=ab\xi-ab\eta+a-b$.

Proof. Integrate the differential equation $-x''(t)=y(t)$ from 0 to t twice. Then we get

$$\begin{aligned} x(t)&=-\int_0^t\int_0^s y(\tau)\mathrm{d}\tau\mathrm{d}s+x'(0)t+x(0) \\ &=-\int_0^t(t-s)y(s)\mathrm{d}s+x'(0)t+x(0). \end{aligned} \tag{2.2}$$

So we have

$$\begin{aligned} x(\xi)&=-\int_0^\xi(\xi-s)y(s)\mathrm{d}s+x'(0)\xi+x(0), \\ x'(1)&=-\int_0^1 y(s)\mathrm{d}s+x'(0), \\ x(\eta)&=-\int_0^\eta(\eta-s)y(s)\mathrm{d}s+x'(0)\eta+x(0). \end{aligned}$$

Using the boundary value condition $x'(0) = ax(\xi),\ x'(1) = bx(\eta)$, we have

$$\begin{cases} (1-a\xi)x'(0) - ax(0) = -a\displaystyle\int_0^\xi (\xi - s)y(s)\mathrm{d}s, \\ (1-b\eta)x'(0) - bx(0) = -\displaystyle\int_0^\eta (\eta - s)y(s)\mathrm{d}s + \int_0^1 y(s)\mathrm{d}s. \end{cases}$$

Setting $\delta = ab\xi - ab\eta + a - b$, from the above equation we can get

$$\begin{aligned} x'(0) &= -\frac{1}{\delta}\left(ab\int_0^\eta (\eta - s)y(s)\mathrm{d}s - a\int_0^1 y(s)\mathrm{d}s - ab\int_0^\xi (\xi - s)y(s)\mathrm{d}s\right), \\ x(0) &= -\frac{1}{\delta}\left(b(1-a\xi)\int_0^\eta (\eta - s)y(s)\mathrm{d}s - (1-a\xi)\int_0^1 y(s)\mathrm{d}s\right) \\ &\quad + \frac{1}{\delta}\left(a(1-b\eta)\int_0^\xi (\xi - s)y(s)\mathrm{d}s\right). \end{aligned} \tag{2.3}$$

Substituting (2.3) to (2.2), we have

$$\begin{aligned} x(t) &= \int_0^t (s-t)y(s)\mathrm{d}s + \frac{1}{\delta}\int_0^\xi (s-\xi)(ab\eta - a - abt)y(s)\mathrm{d}s \\ &\quad + \frac{1}{\delta}\int_0^\eta (s-\eta)(abt - ab\xi + b)y(s)\mathrm{d}s + \frac{1}{\delta}\int_0^1 (at + 1 - a\xi)y(s)\mathrm{d}s \\ &= \int_0^1 G(t,s)y(s)\mathrm{d}s. \end{aligned} \tag{2.4}$$

From the definition of $G(t,s)$, we can get its properties as follows

(1) $G(t,s)$ is continuous on $[0,1]\times[0,1]$;

(2) For each $s \in [0,1]$, $G(t,s)$ is continuously differentiable on $[0,1]$ except $t = s$;

(3) $\dfrac{\partial G(t,s)}{\partial t}\Big|_{t=s^-} - \dfrac{\partial G(t,s)}{\partial t}\Big|_{t=s^+} = 1$;

(4) For each $s \in [0,1]$, $G(t,s)$ satisfies the corresponding homogeneous BVP(i.e. $y(t) \equiv 0$ in BVP(2.1)) on $[0,1]$ except $t = s$.

So, $G(t,s)$ is the Green function of BVP (2.1) on $[0,1]$. The proof is completed. □

Lemma 2.3. *If* $0 < \xi < \min\left\{\frac{\sqrt{a^2(1-b\eta)^2+4ab(1-b\eta)}}{2ab} - \frac{1-b\eta}{2b},\ \eta\right\}$, $0 < \eta < 1$, $0 < a < \frac{1}{\xi}$, $0 < b < \frac{1}{\eta}$ *and* $\delta > 0$, *then* $G(t,s) \geq 0$ *for* $0 \leq t, s \leq 1$. *Moreover, there exists a constant* $\gamma = \min\{\frac{b\xi+1-b\eta}{(b+1-b\eta)(a\eta+1-a\xi)}, \frac{b\xi+1-b\eta}{a+1-a\xi}\}$ *such that*

$$G(t,s) \geq \gamma \max_{\iota\in[0,1]} G(\iota, s), \quad \textit{for } t \in [\xi, 1], s \in [0,1],$$

Proof. Firstly, we show that $G(t,s) \geq 0$. It is clear that $G_1(t,s)$, $G_3(t,s)$, $G_4(t,s), G_6(t,s) \geq 0$, we only need to prove $G_2(t,s)$, $G_5(t,s) \geq 0$.

When $0 \le t \le s \le \xi$, we have

$$\begin{aligned}\delta G_2(t,s) &= (bs+1-b\eta)+(s-t)(ab\eta-ab\xi-a)\\ &\ge 1-b\eta-a\xi(1-b\eta+b\xi)\ (\text{for } 0\le t\le s\le\xi)\\ &=(1-a\xi)(1-b\eta)-ab\xi^2\\ &=-ab\xi^2-a(1-b\eta)\xi+(1-b\eta).\end{aligned} \tag{2.5}$$

It is clear that when $0<\xi<\frac{\sqrt{a^2(1-b\eta)^2+4ab(1-b\eta)}}{2ab}-\frac{1-b\eta}{2b}$, $(1-a\xi)(1-b\eta)-ab\xi^2>0$, $\delta\ G_2(t,s)>0$. And because $\delta>0$, $G_2(t,s)>0$.

When $\eta\le s\le t\le 1$, we have

$$\begin{aligned}G_5(t,s)&=\frac{1}{\delta}(at+1-a\xi)+(s-t)\\ &=\frac{1}{\delta}(a-\delta)t+\frac{1}{\delta}(1-a\xi)+s\\ &=\frac{1}{\delta}(ab\eta+b-ab\xi)t+\frac{1}{\delta}(1-a\xi)+s.\end{aligned}$$

Since $\delta>0$, $ab\eta+b-ab\xi>0$, thus $G_5(t,s)$ is increasing on t, then we know that $G_5(t,s)$ arrives its minimum at $t=s$. On the other hand, $G_5(t,s)\ge 0$ when $t=s$, thus $G_5(t,s)\ge 0$.

Secondly, we show that there exists a constant γ s.t. $G(t,s)\ge\gamma\max_{t\in[0,1]} G(t,s)$. By the continuity of $G(t,s)$, we have

$$\min_{t\in[\xi,1]}G(t,s)=\min_{t\in[\xi,1]}\{\min G_1(t,s),\ \min G_3(t,s),\ \min G_4(t,s),$$
$$\min G_5(t,s)\min G_6(t,s)\}.$$

So we have

$$\min_{t\in[\xi,1]}G(t,s)=\min\left\{\begin{array}{l}\frac{1}{\delta}(bs+1-b\eta)|_{s\in[0,\xi]},\ \frac{1}{\delta}(as+1-a\xi)|_{x\in[\eta,1]},\\ \frac{1}{\delta}(as+1-a\xi)(bs+1-b\eta)|_{s\in[\xi,\eta]},\ \frac{1}{\delta}\end{array}\right\}.$$

When $s\in[\xi,\eta]$,

$$as+1-a\xi\ge 1,\ bs+1-b\eta\ge b\xi+1-b\eta,$$

thus

$$(as+1-a\xi)(bs+1-b\eta)\ge b\xi+1-b\eta.$$

Noticing that $b\xi+1-b\eta<1$, we have

$$\min_{t\in[\xi,1]}G(t,s)=\frac{1}{\delta}(b\xi+1-b\eta).$$

On the other hand,

$$\max_{t\in[0,1]}G(t,s)=\max\left\{\begin{array}{l}\max G_1(t,s),\ \max G_2(t,s),\ \max G_3(t,s),\\ \max G_4(t,s),\ \max G_5(t,s),\ \max G_6(t,s)\end{array}\right\}.$$

So we have

$\max_{t\in[0,1]} G(t,s)$

$$= \max\left\{\begin{array}{l} \left\{\frac{1}{\delta}(b+1-b\eta),\ \frac{1}{\delta}(bs+1-b\eta)\right\}_{s\in[0,\xi]}, \\ \left\{\frac{1}{\delta}(b+1-b\eta)(as+1-a\xi),\ \frac{1}{\delta}(bs+1-b\eta)(as+1-a\xi)\right\}_{s\in[\xi,\eta]}, \\ \left\{\frac{1}{\delta}(a+1-a\xi)+(s-1),\ \frac{1}{\delta}(as+1-a\xi)\right\}_{s\in[\eta,1]} \end{array}\right\},$$

we can see that

$$\max_{t\in[0,1]} G(t,s) = \max\left\{\frac{1}{\delta}(b+1-b\eta)(a\eta+1-a\xi), \frac{1}{\delta}(a+1-a\xi)\right\}.$$

Consequently, we set

$$\gamma = \min\{\frac{b\xi+1-b\eta}{(b+1-b\eta)(a\eta+1-a\xi)}, \frac{b\xi+1-b\eta}{a+1-a\xi}\},$$

it holds that

$$\min_{t\in[\xi,1]} G(t,s) \geq \gamma \max_{\iota\in[0,1]} G(\iota,s), \text{ for } s\in[0,1].$$

The proof is complete. □

Remark 2.1. In lemma 2.3, if $\frac{\sqrt{a^2(1-b\eta)^2+4ab(1-b\eta)}}{2ab} - \frac{1-b\eta}{2b} < \eta$, ξ can be equal to $\frac{\sqrt{a^2(1-b\eta)^2+4ab(1-b\eta)}}{2ab} - \frac{1-b\eta}{2b}$, if $\xi = 0$ or $\eta = 1$, the problem (1.4) will be a three-point boundary value problem.From expression (2.5) we can see that $G(t,s)$ is not always positive, so we give conditions to guarantee that $G(t,s)$ is positive. But the condition is just sufficient.

Remark 2.2. It can be seen clearly that $0 < \gamma < 1$.

3. Multiplicity results of positive solutions

In this section, we will use lemma 2.1, 2.2 and 2.3 to acquire some new existence results of positive solutions for the second-order four-point boundary value problem

$$x''(t) + h(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < 1 \tag{3.1}$$

$$x'(0) = ax(\xi), \quad x'(1) = bx(\eta). \tag{3.2}$$

Let $X = C^1[0,1]$ be endowed with the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0,1]$, and the maximum norm $\|x\| = \max\{\max_{0\leq t\leq 1}|x(t)|,\ \max_{0\leq t\leq 1}|x'(t)|\}$.

Cone $P \subset X$ is defined as

$$P = \{x \in X \mid x(t) \geq 0,\ x \text{ is concave and nondecreasing on } [0,1]\}.$$

Set $T : P \to X$ by

$$(Tx)(t) = \int_0^1 G(t,s)h(s)f(s,x(s),x'(s))\mathrm{d}s,\ 0 \leq t \leq 1,$$

where $G(t,s)$ is as defined in lemma 2.2. Let

$$\alpha(x)=\max_{0\le t\le 1}x(t),\ \beta(x)=\max_{0\le t\le 1}|x'(t)|,\ \psi(x)=\min_{\xi\le t\le 1}x(t).$$

Then $\alpha,\ \beta : X\to[0,+\infty)$ are non-negative continuous convex functionals satisfying (H_1) and (H_2), and ψ is non-negative continuous concave functional on P, also it is clear that $\psi(x)\le\alpha(x)$ for all $x\in X$.

Lemma 3.4. *Let f, h be defined as above, and $G(t,s)\ge 0$. Then T is a completely continuous operator.*

Proof. To justify this, we first show that $T : P\to P$ is well defined.

Let $x\in P$, then $x(t)\ge 0$. There exists $\Gamma>0$ such that $\|x\|\le\Gamma$. It holds that

$$(Tx)(t)=\int_0^1 G(t,s)h(s)f(s,x(s),x'(s))\mathrm{ds}<+\infty. \tag{3.3}$$

Further, $G(t,s)\ge 0$, $f:[0,1]\times[0,\infty)\times R\to[0,\infty)$, $q:(0,1)\to[0,\infty)$, we have

$$(Tx)(t)=\int_0^1 G(t,s)h(s)f(s,x(s),x'(s))\mathrm{ds}\ge 0. \tag{3.4}$$

Then, it can be easily seen that $(Tx)(t)$ is nondecreasing on $t\in[0,1]$ for any $x\in P$. In fact, $(Tx)(\eta)\ge 0$, $(Tx)'(1)=(Tx)(\eta)$, so $(Tx)'(1)\ge 0$. Moreover, $(Tx)''(t)\le 0$, we have $(Tx)'(t)\ge 0$ on $t\in[0,1]$, then we know that $(Tx)(t)$ is nondecreasing on $t\in[0,1]$ for any $x\in P$, thus $T:P\to P$ is well defined.

T is completely continuous if and only if T is continuous in x and maps a bounded subset of P into a relatively compact set.

Let $x_n\to x$ as $n\to+\infty$ in P. For $t\in[0,1]$, we have

$$\begin{aligned}|(Tx_n)(t)-(Tx)(t)|&\le\int_0^1|G(t,s)||h(s)||f(s,x_n(s),x_n'(s))-f(s,x(s),x'(s))|\mathrm{ds}\\&\to 0\qquad\text{as } n\to+\infty,\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}|(Tx_n)'(t)-(Tx)'(t)|&\le\int_0^1|h(s)||f(s,x_n(s),x_n'(s))-f(s,x(s),x'(s)|\mathrm{ds}\\&\to 0\qquad\text{as } n\to+\infty,\end{aligned} \tag{3.6}$$

Let $\Omega\subset P$ be a bounded subset, so there exists $\overline{\Gamma}$, s.t. $\Omega\subset\{x\in P:\|x\|\le\overline{\Gamma}\}$, $\forall x\in\Omega$, we have

$$0\le\int_0^1 f(s,x(s),x'(s)\mathrm{ds}\le\max_{(s,u,v)\in[0,1]\times[0,r]\times[-r,r]}f(s,u,u')=:K,$$

so $T\Omega$ is uniformly bounded according to the properties of f and h.

For any $x \in \Omega$, for $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}|(Tx)(t_1) - (Tx)(t_2)| &\le \int_0^1 |G(t_1,s) - G(t_2,s)||h(s)||f(s,x(s),x'(s))|\mathrm{ds} \\ &\to 0 \qquad \text{as } t_1 \to t_2. \end{aligned} \tag{3.7}$$

And

$$\begin{aligned}|(Tx)'(t_1) - (Tx)'(t_2)| &\le \int_{t_1}^{t_2} |h(s)||f(s,x(s),x'(s))|\mathrm{ds} \\ &\to 0 \qquad \text{as } t_1 \to t_2. \end{aligned} \tag{3.8}$$

Therefore, combining (3.3)-(3.8), we can get $T : P \to P$ is completely continuous. □

Let

$$\begin{aligned} M &= \max_{0\le t\le 1}\int_0^1 G(t,s)h(s)\mathrm{ds}, \quad N = \int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right|_{t=0} h(s)\mathrm{ds}, \\ C &= \int_\xi^1 G(\xi,s)h(s)\mathrm{ds}. \end{aligned}$$

Theorem 3.5. *Suppose there exist constants $r_2 \ge \frac{d}{\gamma} > d > r_1 > 0, L_2 \ge L_1 > 0$ such that $\frac{d}{C} \le \min\{r_2/M,\ L_2/N\}$. If the following assumptions hold,*

(B_1) $f(t,u,v) < \min\{r_1/M, L_1/N\}$, for $(t,u,v) \in [0,1]\times[0,r_1]\times[-L_1,L_1]$;
(B_2) $f(t,u,v) > d/C$, for $(t,u,v) \in [\xi,1]\times[d,d/\gamma]\times[-L_2,L_2]$;
(B_3) $f(t,u,v) \le \min\{r_2/M,\ L_2/N\}$, for $(t,u,v) \in [0,1]\times[0,r_2]\times[-L_2,L_2]$,

then the problem (3.1), (3.2) has at least three positive solutions x_1, x_2 and x_3 with

$$\begin{aligned} &\max_{0\le t\le 1}|x_i'(t)| < L_2, && i = 1,\ 2,\ 3. \\ &\max_{0\le t\le 1} x_1(t) < r_1, && d \le \min_{\xi\le t\le 1} x_2(t) \le \max_{0\le t\le 1} x_2(t), \\ &\max_{0\le t\le 1} x_3(t) \le d/\gamma, && \min_{\xi\le t\le 1} x_3(t) \le d. \end{aligned}$$

Proof. X, P, T is well defined as above. Problem (3.1), (3.2) has a solution $x = x(t)$ if and only if x solves the operator equation $x(t) = Tx(t)$. We have showed that $T : P \to P$ is completely continuous, we will prove the result step by step.

Step 1, If $x \in \overline{P}(\alpha, r_2; \beta, L_2)$, then $\alpha(x) \le r_2$, $\beta(x) \le L_2$, and assumption (B_3) implies $f(t,x(t),x'(t)) \le \min\{r_2/M,\ L_2/N\}$. Consequently,

$$\begin{aligned}\alpha(Tx) &= \max_{t\in[0,1]}\left|\int_0^1 G(t,s)h(s)f(s,x(s),x'(s))ds\right| \\ &\le \frac{r_2}{M}\left|\max_{t\in[0,1]}\int_0^1 G(t,s)h(s)ds\right| = r_2. \end{aligned}$$

For $x \in P$, one has $Tx \in P$. Then $(Tx)(t)$ is concave and increasing on $t \in [0,1]$. It follows that

$$\max_{0\le t\le 1} |(Tx)'(t)| = (Tx)'(0).$$

Thus

$$\begin{aligned}\beta(Tx) &= \max_{0\le t\le 1} |(Tx)'(t)| = (Tx)'(0)\\ &= \int_0^1 \left|\frac{\partial G(t,s)}{\partial t}\right|_{t=0} h(s)f(s,x(s),x'(s))\mathrm{ds}\\ &\le \frac{L_2}{N}\cdot N = L_2\end{aligned}$$

Therefore $T : \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2)$. If $x \in \overline{P}(\alpha, r_1; \beta, L_1)$, then assumption ($\mathrm{B}_1$) yields $f(t, x(t), x'(t)) < \min\{r_1/M, L_1/N\}$, $0 \le t \le 1$. Similarly, we can obtain that $T : \overline{P}(\alpha, r_1; \beta, L_1) \to \overline{P}(\alpha, r_1; \beta, L_1)$. Hence, condition ($\mathrm{S}_2$) of lemma 2.1 is satisfied.

Step 2, in order to verify condition (S_1) in lemma 2.1, we choose $x(t) = d/\gamma$, $0 \le t \le 1$, then $x(t) = d/\gamma \in \overline{P}(\alpha, d/\gamma; \beta, L_2; \psi, d)$ and $\psi(x) = \psi(d/\gamma) > d$, consequently, $\{x \in \overline{P}(\alpha, d/\gamma; \beta, L_2;\ \psi, d) | \psi(y) > d\} \ne \emptyset$. Hence, if $x \in \overline{P}(\alpha, d/\gamma; \beta, L_2; \psi, d)$, then $d \le x(t) \le d/\gamma$ for $\xi \le t \le 1$. From assumption (B_2), we have $f(t, x(t), x'(t)) \ge d/C$ for $\xi \le t \le 1$. Also, we have

$$\begin{aligned}\psi(Tx) = (Tx)(\xi) &= \int_0^1 G(\xi,s)h(s)f(s,x(s),x'(s))\mathrm{ds}\\ &> \frac{d}{C}\cdot\int_\xi^1 G(\xi,s)h(s)\mathrm{ds} > d.\end{aligned}$$

Then

$$\psi(Tx) > d, \text{ for all } x \in \overline{P}(\alpha, d/\gamma; \beta, L_2, \psi, d).$$

This shows that condition (S_1) of lemma 2.1 is satisfied.

Step 3, we show finally that (S_3) of lemma 2.1 holds. Suppose that $x \in \overline{P}(\alpha, r_2; \beta, L_2, \psi, d)$ with $\alpha(Tx) > d/\gamma$. Then, by the definition of ψ and $Tx \in P$, applying Lemma 2.2, we have

$$\begin{aligned}\min_{\xi\le t\le 1} Tx(t) &= \min_{\xi\le t\le 1}\int_0^1 G(t,s)h(s)f(s,x(s),x'(s))\mathrm{ds}\\ &\ge \gamma\int_0^1 \max_{0\le t\le 1} G(t,s)h(s)f(s,x(s),x'(s))\mathrm{ds}\\ &\ge \gamma \max_{0\le t\le 1}(Tx).\end{aligned} \tag{3.9}$$

Then, we can get

$$\begin{aligned}\psi(Tx) &= \min_{\xi\le t\le 1} |(Tx)(t)| \ge \gamma \max_{0\le t\le 1} |(Tx)(t)|\\ &= \gamma\alpha(Tx) \ge \gamma\cdot d/\gamma = d.\end{aligned}$$

So, condition (S_3) of lemma 2.1 is also satisfied. Therefore, Lemma 2.1 yields that T has three fixed points, then the problem (3.1), (3.2) has three positive solutions $x_1,\ x_2,\ x_3$ in $\overline{P}(\alpha, r_2; \beta, L_2)$ with

$$x_1 \in P(\alpha, r_1; \beta, L_1), \quad x_2 \in \{\overline{P}(\alpha, r_2; \beta, L_2; \psi, d) \mid \psi(x) > d\}$$

and

$$x_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus (\overline{P}(\alpha, r_2; \beta, L_2; \psi, d) \cup \overline{P}(\alpha, r_1; \beta, L_1)) .$$

In addition, the three solutions are nondecreasing on $t \in [0, 1]$, for $x_1,\ x_2,\ x_3 \in P$, which completes our proof. □

4. Example

At the end of this paper, we give an example to illustrate our main results.

Example 4.6.

Consider the second-order boundary value problem of the differential equation

$$\begin{cases} x'' + f(t, x(t), x'(t)) = 0, \quad 0 \le t \le 1 \\ x'(0) = x(\frac{1}{4}), \quad x'(1) = \frac{1}{2} x(\frac{3}{4}), \end{cases}$$

where

$$f(t,u,v) = \begin{cases} \dfrac{t}{100} + \dfrac{4u^3}{11} + (\dfrac{v}{1000})^4, & 0 \le u \le \dfrac{7}{3}, \\ \dfrac{t}{100} + \dfrac{4}{11} \times (\dfrac{7}{3})^3 + (\dfrac{v}{1000})^4, & u > \dfrac{7}{3}. \end{cases} \tag{4.1}$$

Set $h(t) = 1$, and it can be easily seen that $h,\ f$ satisfy our requirements. Choose $L_1 = L_2 = 100$, $r_1 = \frac{2}{3}$, $r_2 = 200$, $d = 1$, and we can see from the above equation that $a = 1,\ b = \frac{1}{2},\ \xi = \frac{1}{4},\ \eta = \frac{3}{4}$. Then by direct calculations, we can obtain that

$$\delta = \frac{1}{4}, \quad \gamma = \frac{3}{7}, \quad M = \frac{181}{32}, \quad N = \frac{7}{2}, \quad C = \frac{11}{4}, \quad d/C = \frac{4}{11}.$$

So the nonlinear term f satisfies

$$f(t,u,v) < 0.01 + 0.10775 + 10^{-4} < 0.11786 < \min\left\{\frac{2/3}{181/32}, \frac{100}{7/2}\right\},$$
$$\text{for } (t,u,v) \in [0,1] \times [0,1/2] \times [-100,100];$$

$$f(t,u,v) > 4/11, \text{ for } (t,u,v) \in [1/4,1] \times [1,7/3] \times [-100,100];$$

$$f(t,u,v) \le 0.01 + 4.62 + 10^{-4} < \min\left\{200/\frac{181}{32}, 100/\frac{7}{2}\right\},$$
$$\text{for } (t,u,v) \in [0,1] \times [0,200] \times [-100,100].$$

Then the conditions in Theorem 3.2 are all satisfied. So BVP (3.1), (3.2) has at least three positive solutions $x_1,\ x_2,\ x_3$ such that

$$\max_{0\le t\le 1}|x_i'(t)|<100,\qquad i=1,\ 2,\ 3,$$

$$\max_{0\le t\le 1}x_1(t)<\frac{1}{2},\qquad 1\le\min_{\frac14\le t\le 1}x_2(t)\le\max_{0\le t\le 1}x_2(t)\le 200,$$

$$\max_{0\le t\le 1}x_3(t)\le\frac{7}{3},\qquad \min_{\frac14\le t\le 1}x_3(t)\le 1.$$

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