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AN ITERATIVE METHOD FOR EQUILIBRIUM PROBLEMS, VARIATIONAL INEQUALITY PROBLEMS AND FIXED POINT PROBLEMS

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ABSTRACT. In this paper, we introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of solutions of an equilibrium problem in a Hilbert space. We show that the iterative sequence converges strongly to a common element of the three sets. The results of this paper extend and improve the corresponding results announced by many others.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Recall that A mapping $S : C \to C$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. Recall that a mapping $A : C \to H$ is called monotone if for all $x, y \in C$,

$$\langle x - y, Ax - Ay\rangle \geq 0.$$

Recall that a mapping $A : C \to H$ is called inverse-strongly monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$$

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for all $x, y \in C$; see [4,8,14]. For such a case, A is called α-inverse-strongly monotone. If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $1/\alpha$-Lipschitz continuous.

The classical variational inequality problem is to find a $u \in C$ such that $\langle v-u, Au\rangle \geq 0$ for all $v \in C$. We denoted by $VI(A, C)$ the set of solutions of the variational inequality problem. The variational inequality has been extensively studied in the literature. See [2,3,12,13,15] and the references therein.

Let F be a bifunction of $C \times C$ into $\mathbf{R}$, where $\mathbf{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbf{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [5,6,12,13].

For finding an element of $F(S) \bigcap VI(C, A)$, Takahashi and Toyoda [16] introduced the following iterative scheme: $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n \geq 1 \tag{1.2}$$

and obtain a weak convergence theorem in a Hilbert space. Recently, Iiduka and Takahashi [7] proposed a new iterative scheme: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad n \geq 1 \tag{1.3}$$

and obtained a strong convergence theorem in a Hilbert space.

On the other hand, for finding an element of $EP(F) \cap F(S)$, Takahashi and Takahashi [13] introduced the following iterative scheme by the viscosity approximation method in a Hilbert space: $x_1 \in H$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy some appropriate conditions. Further, They proved $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)}f(z)$.

Very recently, Su et al. [12] introduced a new iterative scheme for finding a common element of $F(S) \cap VI(C, A) \cap EP(F)$ given as follows: $x_1 \in H$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(u_n - \lambda_n Au_n), \ n \geq 1. \end{cases}$$

In this paper, motivated and inspired by the above results, we introduce a new following iterative scheme: $x_1 \in H$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S P_C(u_n - \lambda_n A u_n), \ n \geq 1, \end{cases} \tag{1.5}$$

for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of a variational inequality for an α-inverse-strongly monotone mapping and the set of solutions of an equilibrium problem in a real Hilbert space. Furthermore, we show the iterative sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S)\cap VI(C,A)\cap EP(F)$, where $z = P_{F(S)\cap VI(C,A)\cap EP(F)}$ $f(z)$. The results of this paper extended and improved the results of Iiduka and Takahashi [7], Takahashi and Takahashi [13] and Su et al. [12].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. It is well known that for all $x, y \in H$ and $\lambda \in [0,1]$ there holds

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Let C be a nonempty closed convex subset of H. We denote by "$\rightarrow$" strong convergence and "$\rightharpoonup$" weak convergence. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \tag{2.1}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties:

$$P_C x \in C \text{ and } \langle x - P_C x, P_C x - y\rangle \geq 0, \text{ for all } y \in C.$$

In the context of the variational inequality problem, this implies that

$$u \in VI(C,A) \Leftrightarrow u = P_C(u - \lambda A u), \text{ for all } \lambda > 0. \tag{2.2}$$

A space X is said to satisfy Opial's condition [9] if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to point $x \in X$, we have

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \ \forall y \in X, \ y \neq x.$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y,g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$$

and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [10,11].

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbf{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

We need the following lemmas for the proof of our main results.

Lemma 2.1 [18].*Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.2 [17]. *Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbf{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.3 [1]. *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1) – (A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.4 [6]. *Assume that $F : C \times C \to \mathbf{R}$ satisfies (A1) – (A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : \ F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}$$

for all $x \in H$. Then, the following hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) *$EP(F)$ is closed and convex.*

3. Main results

In this section, we introduce an iterative method by the viscosity approximation method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of solutions of an equilibrium problem in a Hilbert space. We show that the iterative sequence converges strongly to a common element of the three sets.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1) – (A4) and S be a nonexpansive mapping of C into H. Let A be an α-inverse-strongly monotone mapping of C into H such that $F(S) \cap VI(C, A) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by (1.5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$, $\{\lambda_n\} \subset [0, 2\alpha]$ and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:*

(1) $\alpha_n + \beta_n + \gamma_n = 1$,
(2) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(4) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\alpha)$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,
(5) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap VI(C, A) \cap EP(F)$, where $z = P_{F(S)\cap VI(C,A)\cap EP(F)} f(z)$.

Proof. Let $Q = P_{F(S)\cap EP(F)\cap VI(C,A)}$. Then Qf is a contraction of H into C. In fact, there exists $a \in [0, 1)$ such that $\|f(x) - f(y)\| \leq a\|x - y\|$ for all $x, y \in H$. So, we have that

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq a\|x - y\|$$

for all $x, y \in H$. So, Qf is a contraction of H into C. Since H is complete, there exists a unique element of C, such that $z = Qf(z)$. Such a $z \in H$ is an element of C. For all $x, y \in C$ and $\lambda_n \in [0, 2\alpha]$,

$$\begin{aligned}
&\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 = \|(x - y) - \lambda_n(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda_n\langle x - y, Ax - Ay\rangle + \lambda_n^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2,
\end{aligned} \tag{3.1}$$

which implies that $I - \lambda_n A$ is nonexpansive. Let $v \in F(S) \cap EP(F) \cap VI(C, A)$. Then $v = P_C(v - \lambda Av)$, for all $\lambda > 0$. Put $w_n = P_C(u_n - \lambda_n Au_n)$, for all $n \geq 1$. We compute that

$$\begin{aligned}
\|w_n - v\| &= \|P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Av)\| \\
&\leq \|(u_n - \lambda_n Au_n) - (v - \lambda_n Av)\| \\
&\leq \|u_n - v\|.
\end{aligned} \tag{3.2}$$

From $u_n = T_{r_n} x_n$, we have

$$\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leq \|x_n - v\|, \text{ for all } n \geq 1. \tag{3.3}$$

Then we compute that

$$\begin{aligned}\|x_{n+1} - v\| &= \|\alpha_n(f(x_n) - v) + \beta_n(x_n - v) + \gamma_n(Sw_n - v)\| \\ &\leq \alpha_n\|f(x_n) - v\| + \beta_n\|x_n - v\| + \gamma_n\|w_n - v\| \\ &\leq \alpha_n a\|x_n - v\| + \alpha_n\|f(v) - v\| + (1 - \alpha_n)\|x_n - v\| \\ &\leq \max\{\|x_n - v\|, \frac{1}{1-a}\|f(v) - v\|\}.\end{aligned}$$

By induction, we get

$$\|x_n - v\| \leq \max\{\|x_1 - v\|, \frac{1}{1-a}\|f(v) - v\|\}, \ n \geq 1.$$

Therefore, $\{x_n\}$ is bounded. $\{u_n\}$, $\{Sw_n\}$ and $\{f(x_n)\}$ are also bounded. Next, we show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since $I - \lambda_n A$ is nonexpansive, we have

$$\begin{aligned}\|w_{n+1} - w_n\| &= \|P_C[(I - \lambda_{n+1}A)u_{n+1}] - P_C[(I - \lambda_n A)u_n]\| \\ &\leq \|(I - \lambda_{n+1}A)u_{n+1} - (I - \lambda_n A)u_n\| \\ &\leq \|(I - \lambda_{n+1}A)u_{n+1} - (I - \lambda_{n+1}A)u_n\| + |\lambda_n - \lambda_{n+1}|\|Au_n\| \\ &\leq \|u_{n+1} - u_n\| + |\lambda_n - \lambda_{n+1}|\|Au_n\|.\end{aligned} \tag{3.4}$$

On the other hand, from $u_n = T_{r_n} x_n$ and $u_{n+1} = T_{r_{n+1}} x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in C \tag{3.5}$$

and

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \quad \text{for all } y \in C \tag{3.6}$$

Putting $y = u_{n+1}$ in (3.5) and $y = u_n$ in (3.6), we have

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0$$

and $F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0$. So, from (A2) we have $\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0$ and hence

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbf{N}$. Then, we have

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\leq \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ &\leq \|u_{n+1} - u_n\|\{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1} - x_{n+1}\|\}\end{aligned}$$

and hence

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \|x_{n+1}-x_n\| + \frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\| \\ &\le \|x_{n+1}-x_n\| + \frac{1}{b}|r_{n+1}-r_n|L,\end{aligned}\tag{3.7}$$

where $L=\sup\{\|u_n-x_n\| : n\in\mathbf{N}\}$. From (3.4) we have

$$\|w_{n+1}-w_n\| \le \|x_{n+1}-x_n\| + \frac{1}{b}|r_{n+1}-r_n|L + |\lambda_n-\lambda_{n+1}|\|Au_n\|. \tag{3.8}$$

Define $x_{n+1}=(1-\beta_n)y_n+\beta_n x_n$, $n\ge 0$. Observe that from the definition of y_n, we obtain

$$\begin{aligned}y_{n+1}-y_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}Sw_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n)+\gamma_n Sw_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n)}{1-\beta_n} + \frac{\gamma_{n+1}}{1-\beta_{n+1}}Sw_{n+1} - \frac{\gamma_n}{1-\beta_n}Sw_n \\ &= \frac{\alpha_{n+1}f(x_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n)}{1-\beta_n} + (1-\frac{\alpha_{n+1}}{1-\beta_{n+1}})(Sw_{n+1}-Sw_n) \\ &\quad + (\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}})Sw_n.\end{aligned}\tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\begin{aligned}&\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n)\| + (1-\frac{\alpha_{n+1}}{1-\beta_{n+1}})\|w_{n+1}-w_n\| \\ &\quad + |\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}|\|Sw_n\| - \|x_{n+1}-x_n\| \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n)\| - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| \\ &\quad + \frac{1}{b}|r_{n+1}-r_n|L + |\lambda_n-\lambda_{n+1}|\|Au_n\| + |\frac{\alpha_n}{1-\beta_n} - \frac{\alpha_{n+1}}{1-\beta_{n+1}}|\|Sw_n\|.\end{aligned}$$

This together with the conditions (2)-(5) implies that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Hence using Lemma 2.1 , we have $\lim_{n\to\infty}\|y_n-x_n\|=0$. Consequently, we can obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|y_n-x_n\| = 0. \tag{3.10}$$

From (3.7) and $\sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty$, we have $\lim_{n\to\infty}\|u_{n+1}-u_n\|=0$. From (3.4) and $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$, we also have

$$\lim_{n\to\infty}\|w_{n+1}-w_n\| = 0. \tag{3.11}$$

Next we show $\|Sw_n - w_n\| \to 0$, as $n \to \infty$. First, we show $\|x_n - Sw_n\| \to 0$, as $n \to \infty$. t Notice that

$$\begin{aligned}\|x_n - Sw_n\| &\le \|x_n - Sw_{n-1}\| + \|Sw_{n-1} - Sw_n\| \\ &\le \alpha_{n-1}\|f(x_{n-1}) - Sw_{n-1}\| + \|w_n - w_{n-1}\|.\end{aligned}$$

From $\alpha_n \to 0$ and (3.11), we get

$$\lim_{n\to\infty} \|x_n - Sw_n\| = 0. \tag{3.12}$$

In sequence, we show $\|x_n - u_n\| \to 0$, as $n \to \infty$. For $v \in F(S) \cap FP(F) \cap VI(C, A)$, we have

$$\begin{aligned}\|u_n - v\|^2 &= \|T_{r_n}x_n - T_{r_n}v\|^2 \le \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle \\ &= \langle u_n - v, x_n - v\rangle \\ &= \frac{1}{2}(\|u_n - v\|^2 + \|x_n - v\|^2 - \|u_n - x_n\|^2)\end{aligned}$$

and hence $\|u_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - u_n\|^2$. Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n Sw_n - v\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|u_n - v\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n(\|x_n - v\|^2 - \|x_n - u_n\|^2) \\ &= \alpha_n\|f(x_n) - v\|^2 + (1 - \alpha_n)\|x_n - v\|^2 - \gamma_n\|x_n - u_n\|^2\end{aligned}$$

and hence

$$\begin{aligned}&\gamma_n\|x_n - u_n\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + \|x_n - x_{n+1}\|(\|x_n - v\| + \|x_{n+1} - v\|).\end{aligned}$$

So, from (3.10) and condition (1)-(3), we have

$$\|x_n - u_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.13}$$

Next we show $\|u_n - w_n\| \to 0$. For $v \in F(S) \cap FP(F) \cap VI(C, A)$, we compute that

$$\begin{aligned}&\|x_{n+1} - v\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|w_n - v\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|(I - \lambda_n A)u_n - (I - \lambda_n A)v\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n(\|u_n - v\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Av\|^2) \\ &\le \alpha_n\|f(x_n) - v\|^2 + (1 - \alpha_n)\|x_n - v\|^2 + a(b - 2\alpha)\|Au_n - Av\|^2\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& -a(b-2\alpha)\|Au_n - Av\|^2 \\
& \le \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\
& \le \alpha_n\|f(x_n) - v\|^2 + (\|x_n - v\| + \|x_{n+1} - v\|)\|x_{n+1} - x_n\|.
\end{aligned}$$

Since $\alpha_n \to 0$, $a, b \in (0, 2\alpha)$ and $\|x_{n+1} - x_n\| \to 0$, we obtain

$$\lim_{n\to\infty} \|Au_n - Av\| = 0. \tag{3.14}$$

From (2.1), we have

$$\begin{aligned}
\|w_n - v\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(v - \lambda_n Av)\|^2 \\
&\le \langle (u_n - \lambda_n Au_n) - (v - \lambda_n Av), w_n - v\rangle \\
&= \frac{1}{2}\{\|(u_n - \lambda_n Au_n) - (v - \lambda_n Av)\|^2 + \|w_n - v\|^2 \\
&\quad - \|(u_n - \lambda_n Au_n) - (v - \lambda_n Av) - (w_n - v)\|^2\} \\
&\le \frac{1}{2}\{\|u_n - v\|^2 + \|w_n - v\|^2 - \|(u_n - w_n) - \lambda_n(Au_n - Av)\|^2\} \\
&= \frac{1}{2}\{\|u_n - v\|^2 + \|w_n - v\|^2 - \|u_n - w_n\|^2 \\
&\quad + 2\lambda_n\langle u_n - w_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2\}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
& \|w_n - v\|^2 \\
& \le \|u_n - v\|^2 - \|u_n - w_n\|^2 + 2\lambda_n\langle u_n - w_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1} - v\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n Sw_n - v\|^2 \\
&\le \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|w_n - v\|^2 \\
&\le \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n(\|u_n - v\|^2 - \|u_n - w_n\|^2 \\
&\quad + 2\lambda_n\langle u_n - w_n, Au_n - Av\rangle - \lambda_n^2\|Au_n - Av\|^2) \\
&\le \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - \gamma_n\|u_n - w_n\|^2 \\
&\quad + 2\gamma_n\lambda_n\langle u_n - w_n, Au_n - Av\rangle - \lambda_n^2\gamma_n\|Au_n - Av\|^2.
\end{aligned}$$

From $\alpha_n \to 0$, (3.10) and (3.14), we obtain

$$\lim_{n\to\infty} \|u_n - w_n\| = 0. \tag{3.15}$$

From $\|Sw_n - w_n\| \le \|Sw_n - x_n\| + \|x_n - u_n\| + \|u_n - w_n\|$, we obtain

$$\lim_{n\to\infty} \|Sw_n - w_n\| = 0. \tag{3.16}$$

Next we show that

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \le 0, \tag{3.17}$$

where $z = P_{F(S)\cap VI(C,A)\cap EP(F)} f(z)$. To show it, choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z) - z, Sw_n - z\rangle = \lim_{i\to\infty} \langle f(z) - z, Sw_{n_i} - z\rangle.$$

As $\{w_{n_i}\}$ is bounded, we have that a subsequence $\{w_{n_{i_j}}\}$ of $\{w_{n_i}\}$ converges weakly to w. We may assume without loss of generality that $w_{n_i} \rightharpoonup w$. From (3.16), we obtain $Sw_{n_i} \rightharpoonup w$. Then we can obtain $w \in F(S)\cap VI(C,A)\cap EP(F)$.

In fact, let us first show that $w \in VI(C,A)$. Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v,u) \in G(T)$. Since $u - Av \in N_C v$ and $w_n \in C$, we have $\langle v - w_n, u - Av\rangle \geq 0$.

On the other hand, from $w_n = P_C(u_n - \lambda_n A u_n)$, we have $\langle v - w_n, w_n - (u_n - \lambda_n A u_n)\rangle \geq 0$ and hence $\langle v - w_n, \frac{w_n - u_n}{\lambda_n} + Au_n)\rangle \geq 0$. Therefore, we have

$$\begin{aligned}
&\langle v - w_{n_i}, u\rangle \geq \langle v - w_{n_i}, Av\rangle \\
&\geq \langle v - w_{n_i}, Av\rangle - \langle v - w_{n_i}, \frac{w_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i}\rangle \\
&= \langle v - w_{n_i}, Av - Au_{n_i} - \frac{w_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\
&= \langle v - w_{n_i}, Av - Aw_{n_i}\rangle + \langle v - w_{n_i}, Aw_{n_i} - Au_{n_i}\rangle - \langle v - w_{n_i}, \frac{w_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\
&\geq \langle v - w_{n_i}, Aw_{n_i} - Au_{n_i}\rangle - \langle v - w_{n_i}, \frac{w_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle,
\end{aligned}$$

which together with $\|w_n - u_n\| \to 0$ and A is *Lipschitz* continuous implies that $\langle v - w, u\rangle \geq 0$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in VI(C,A)$. Next, let us show that $w \in F(S)$. Assume $w \bar{\in} F(S)$. Since $w_{n_i} \rightharpoonup w$ and $Sw \neq w$, from Opial's condition, we have

$$\begin{aligned}
\liminf_{i\to\infty} \|w_{n_i} - w\| &< \liminf_{i\to\infty} \|w_{n_i} - Sw\| = \liminf_{i\to\infty} \|w_{n_i} - Sw_{n_i} + Sw_{n_i} - Sw\| \\
&= \liminf_{i\to\infty} \|Sw_{n_i} - Sw\| \leq \liminf_{i\to\infty} \|w_{n_i} - w\|.
\end{aligned}$$

This is a contradiction. Thus, we obtain $w \in F(S)$. Finally, we show that $w \in EP(F)$. By $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C.$$

From (A2), we also have $\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n)$ and hence $\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i})$. Since $\|u_n - w_n\| \to 0$ and $w_{n_i} \rightharpoonup w$, we have $u_{n_i} \rightharpoonup w$. It together with $\|x_n - u_n\| \to 0$ and (A4) implies that $0 \geq F(y, w)$ for all $y \in C$.

For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $F(y_t, w) \le 0$. So, from (A1) and (A4), we have

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, w) \le tF(y_t, y)$$

and hence $0 \le F(y_t, y)$. From (A3), we have $0 \le F(w, y)$ for all $y \in C$ and hence $w \in EP(F)$. Therefore $w \in F(S) \cap EP(F) \cap VI(C, A)$. Since $z = P_{F(S)\cap EP(F)\cap VI(C,A)} f(z)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle &= \limsup_{n\to\infty}\langle f(z) - z, Sw_n - z\rangle \\ &= \lim_{i\to\infty}\langle f(z) - z, Sw_{n_i} - z\rangle = \langle f(z) - z, w - z\rangle \le 0.\end{aligned}$$

For all $n \ge m$, we have

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n Sw_n - z, x_{n+1} - z\rangle \\ &= \alpha_n\langle f(x_n) - z, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\ &\quad + \gamma_n\langle Sw_n - z, x_{n+1} - z\rangle \\ &\le \frac{1}{2}\alpha_n a(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\ &\quad + \frac{1}{2}(1-\alpha_n)(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\ &\le \frac{1}{2}[1 - (1-a)\alpha_n]\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle,\end{aligned}$$

which implies that

$$\|x_{n+1} - z\|^2 \le [1 - (1-a)\alpha_n]\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle.$$

Finally by using condition (2), (3.17) and Lemma 2.1, we can obtain $x_n \to z = P_{F(S)\cap EP(F)\cap VI(C,A)} f(z)$, as $n \to \infty$. This completes the proof. □

4. Applications

From Theorem 3.1, we have the following results immediately.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into H such that $F(S) \cap VI(C, A) \neq \emptyset$. Let f be a contraction of H into itself. Suppose $x_1 \in H$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)SP_C[(I - \lambda_n A)P_C x_n],\ n \ge 1,$$

where $\{\alpha_n\} \subset [0,1)$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy the following conditions

(1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$,
(2) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\alpha)$ *and* $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z \in F(S) \cap VI(C, A)$, where $z = P_{F(S)\cap VI(C,A)}$ $f(z)$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$, $\beta_n = 1$ and $r_n = 1$ for all $n \in \mathbf{N}$ in Theorem 3.1. Then, we have $u_n = P_C x_n$. So, by Theorem 3.1, we can conclude the desired conclusion easily. $\square$

Remark 4.2. Putting $f(y) = x \in C$ for all $y \in H$ and S be a nonexpansive mapping of C in Corollary 4.1, we have $P_C x_n = x_n$. So, Theorem 3.1 in Iiduka and Takahashi's [7] can be directly obtained.

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