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PROXIMAL AUGMENTED LAGRANGIAN AND APPROXIMATE OPTIMAL SOLUTIONS IN NONLINEAR PROGRAMMING

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ABSTRACT. In this paper, we introduce some approximate optimal solutions and an augmented Lagrangian function in nonlinear programming, establish dual function and dual problem based on the augmented Lagrangian function , discuss the relationship between the approximate optimal solutions of augmented Lagrangian problem and that of primal problem, obtain approximate KKT necessary optimality condition of the augmented Lagrangian problem, prove that the approximate stationary points of augmented Lagrangian problem converge to that of the original problem. Our results improve and generalize some known results.

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1. Introduction

It is well known that dual method and penalty function method are popular methods in solving nonlinear optimization problems. Many constrained optimization problems can be formulated as an unconstrained optimization problem by dual method and penalty function method. Recently a general class of nonconvex constrained optimization problem has been reformulated as unconstrained optimization problem via augmented Lagrangian[1].

In [1] Rockafellar and Wets introduced an augmented Lagrangian for minimizing an extended real-valued function. Based on the augmented Lagrangian, a strong duality result without any convexity requirement in the primal problem was obtained under mild conditions. A necessary and sufficient condition for the exact penalization based on the augment Lagrangian function was given[1]. Chen

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et al.[2] , Huang and Yang[3] used augmented Lagrangian functions to construct the set-valued dual functions and corresponding dual problems, obtained weak and strong duality results of multiobjective optimization problem. And many literatures are devoted to investigate augmented Lagrangian problems. Necessary and sufficient optimality conditions, duality theory, saddle point theory as well as exact penalization results between the original constrained optimization problems and its unconstrained augmented Lagrangian problems have been established under mild conditions (see,e.g.,[4,6,7,8,9,15]). It is worth noting that most of these results are established on the basis of assumption that the set of optimal solutions of the primal constrained optimization problems is not empty.

However many mathematical programming problems do not have an optimal solution, moreover sometimes we do not need to find an exact optimal solution due to the fact that it is often very hard to find an exact optimal solution even if it does exists. As a mater of fact, many numerical methods only yield approximate optimal solutions thus we have to resort to approximate solution of nonlinear programming([10,11,12,13,14]). In [10] Liu used exact penalty function to transform a multiobjective programming problem with inequality constraints into an unconstrained problem and derived the Kuhn-Tucker conditions for ϵ-Pareto optimality of primal problem. In [14] Huang and Yang investigated relationship between approximate optimal values of nonlinear Lagrangian problem and that of primal problem. As we known, Ekeland's variational principle and penalty function methods are effective tools to study approximate solutions of constrained optimization problems and the augmented Lagrangian functions have some similar properties of penalty functions. Thus it is possible to apply them in the study of approximate solutions of constrained optimization problems.

In this paper, based on the results in [10] and [14], we investigate the possibility of obtaining the various versions of approximate solutions to a constrained optimization problem by solving an unconstrained program formulated by using an augmented Lagrangian function. As an application, a KKT type optimality condition is obtained for a kind of approximate solution to the augmented Lagrangian problem and we prove that the approximate stationary points of generalized augmented Lagrangian problem converge to that of the original problems.

The paper is organized as follows. In section 2, we present some concepts, basic assumptions and preliminary results. In section 3, we deal with the relationship between approximate solutions of augmented Lagrangian problem and that of the original problem. In section 4, we obtain an approximate KKT type optimality condition of augmented Lagrangian problem and prove that the approximate stationary points converge to that of the original problem.

2. Preliminaries

In this section, we present some definitions and Ekeland's variational principle. Consider the following constrained optimization problem:

$$\begin{array}{lll}(\mathrm{P}) & \inf & f(x)\\ & \text{s.t.} & x \in X,\\ & & g_j(x) \le 0 \quad j = 1...m,\end{array}$$

where $X \subset R^n$ is a nonempty and closed set, $f : X \to R$, $g_j : X \to R$, f and g_j are continuously differentiable functions. Let $S = \{x \in X, g_j(x) \le 0, j = 1, ..., m\}$, it is clear that S is the set of feasible solutions. For any $\epsilon > 0$, we denote by S_ϵ the set of ϵ feasible solution, i.e.,

$$S_\epsilon = \{x \in X : g_j(x) \le \epsilon, j = 1, ..., m\}$$

and by M_P the optimal value of problem (P).

Let $u \in I\!R$. We define a function $F : R^n \times R \to R$

$$F(x,u) = \begin{cases} f(x), & \text{if } g_j(x) \le u;\\ +\infty, & \text{else.}\end{cases}$$

So we have a perturbed problem

$$\begin{array}{lll}(P^*) & \inf & F(x,u)\\ & \text{s.t.} & x \in R^n\end{array}$$

Define the optimal value function by $p(u) = \inf_{x \in R^n} F(x,u)$, obviously $p(0)$ is the optimal value of problem (P).

Definition 2.1 [1]. A function $\sigma : I\!R^m \to I\!R_+ \cup \{+\infty\}$ is said to be an augmented function if it is proper, l.s.c., convex with the unique minimum value 0 at $0 \in R^m$

Define the dualizing parameterization function:

$$\bar{f}_p(x,u) = f(x) + \delta_{R^m_-}(G(x)+u) + \delta_X(x), \quad x \in R^n, u \in R^m, \tag{2.1}$$

where $G(x) = \{g_1(x), ..., g_m(x)\}$, δ_D is the indicator function of the set D, i.e.,

$$\delta_D(z) = \begin{cases} 0, & \text{if } z \in D;\\ +\infty, & \text{else.}\end{cases}$$

So a class of augmented Lagrangian of (P) with dualizing parameterization function $\bar{f}_p(x,u)$ defined by (2.1) can be expressed as

$$l_p(x,y,r) = \inf\{\bar{f}_p(x,u) - \langle y,u\rangle + r\sigma(u) : u \in R^m\} \tag{2.2}$$

When $\sigma(u) = \frac{1}{2} \parallel u \parallel^2$, the above abstract augmented Lagrangian can be formulated as the following proximal augmented Lagrangian (Example 11.57 in [1]).

$$l_p(x,y,r) = f(x) + \sum_{j=1}^{m} \begin{cases} y_j g_j(x) + \frac{r}{2} g_j^2(x), & \text{if } g_j(x) \ge -y_j/r;\\ \frac{-y_j^2}{2r} & \text{if } g_j(x) < -y_j/r.\end{cases} \tag{2.3}$$

In this paper, we will focus on the problems about proximal augmented Lagrangian.

The proximal augmented Lagrangian dual function corresponding l_p is defined as

$$\psi_p(y,r) = \inf\{l_p(x,y,r); x \in I\!R^n\} \quad y \in I\!R^m, r \geq 0 \tag{2.4}$$

The proximal augmented Lagrangian dual problem is defined as

$$\sup\ \psi_P(y,r) \quad \text{subject to } (y,r) \in I\!R^m \times (0,+\infty) \tag{2.5}$$

The following various definitions of approximate solutions are taken from Loridan[11].

Definition 2.2. Let $\epsilon > 0$, the point $x^* \in S$ is said to be an ϵ solution of (P) if

$$f(x^*) \leq f(x) + \epsilon \quad \forall x \in S$$

Definition 2.3. Let $\epsilon > 0$, the point $x^* \in S$ is said to be an ϵ-quasi solution of (P) if

$$f(x^*) \leq f(x) + \epsilon \parallel x - x^* \parallel \quad \forall x \in S$$

Definition 2.4. Let $\epsilon > 0$, the point $x^* \in S$ is said to be a regular ϵ -solution of (P) if it is both an ϵ solution and an ϵ-quasi solution of (P).

Definition 2.5. Let $\epsilon > 0$, the point $x^* \in S_\epsilon$ is said to be an almost ϵ- solution of (P) if

$$f(x^*) \leq f(x) + \epsilon \quad \forall x \in S$$

Definition 2.6. The point $x^* \in S$ is said to be an almost regular ϵ- solution of (P) if it is both an almost ϵ- solution and a regular ϵ- solution of (P).

Proposition 2.7 [13]. (Ekeland's variational principle) *Let $f : R^n \to R$ be proper lower semicontinous function which is bounded below. Then for any $\epsilon > 0$, there exists $x^* \in S$ such that*

$$(i) f(x^*) \leq f(x) + \epsilon, \quad \forall x \in S$$

$$(ii) f(x^*) < f(x) + \epsilon \parallel x - x^* \parallel, \quad \forall x \in S \backslash \{x^*\}$$

3. Approximate solutions of proximal augmented Lagrangian

In this section, we will discuss the relationship between approximate solutions of proximal augmented Lagrangian dual problem (Q) and that of primal problem (P).

The following Lemma is about the strong duality results between augmented Lagrangian dual problem and primal problem.

Lemma 3.1[1]. *Assume* f *is proper and that its dualizing parameterization function $\bar{f}_p(x,u)$ is proper, l.s.c., and level-bounded in* x *locally uniform in* u. *Suppose that there exists $(\bar{y},\bar{r}) \in I\!R^m \times (0,+\infty)$, such that*

$$\inf\{l(x,\bar{y},\bar{r}) : x \in I\!R^n\} > -\infty$$

Then zero duality gap holds:

$$p(0) = \sup_{(y,r)\in IR^m\times(0,+\infty)} \psi(y,r)$$

Proof. Please see it in [1. Theorem 11.59]. □

It is clear that when the abstract augmented Lagrangian dual problem is formulated as proximal augmented Lagrangian dual problem, we can derive that

$$\inf_{x\in X} f(x)$$

$$= \sup_{(y,r)\in IR^m\times(0,+\infty)} \inf_{x\in X} \left\{ f(x) + \sum_{j=1}^{m} \begin{cases} y_j g_j(x) + \frac{r}{2} g_j^2(x), & \text{if } g_j(x) \geq -y_j/r; \\ \frac{-y_j^2}{2r} & \text{if } g_j(x) < -y_j/r. \end{cases} \right\} \tag{3.1}$$

When $g_j(x) \geq -y_j/r$, the above expression (3.1) can be formulated as following:

$$\inf_{x\in S} f(x) = \sup_{(y,r)\in IR^m\times(0,+\infty)} \inf_{x\in S}\{f(x) + \sum_{j=1}^{m}[y_j g_j(x) + \frac{r}{2} g_j^2(x)]\} \tag{3.2}$$

and the right side of (3.2) can be formulated as following

$$\sup_{(y,r)\in IR^m\times(0,+\infty)} \inf_{x\in S} \left\{ f(x) + \sum_{j=1}^{m}[y_j g_j(x) + \frac{r}{2} g_j^2(x)] \right\}$$

$$= \inf_{x\in S} f(x) + \sup_{(y,r)\in IR^m\times(0,+\infty)} \sum_{j=1}^{m}(-\frac{y_j^2}{2r}) \tag{3.3}$$

Combining (3.2) with (3.3), we can derive that

$$\sup_{(y,r)\in IR^m\times(0,+\infty)} \sum_{j=1}^{m}(\frac{y_j^2}{2r}) = 0 \tag{3.4}$$

Theorem 3.2. *Assume* f *satisfies all the conditions mentioned in Lemma 3.1, then for any $\epsilon > 0$, there exists $y \in IR^m$ and $r \in IR$ such that every almost-ϵ solution of (Q) is an almost -ϵ solution (P).*

Proof. From the assumption, we know that there exist $x_\epsilon \in S_\epsilon$ such that

$$l_p(x_\epsilon, y, r) \leq l_p(x, y, r) + \epsilon \quad \forall x \in S \tag{3.5}$$

We consider following three cases: (a) $-\frac{y_j}{r} \leq g_j(x) \leq 0$; (b) $0 < g_j(x) \leq \epsilon$; (c) $g_j(x) < -\frac{y_j}{r}$.

(1) we consider the case (a): from the definition of proximal augmented Lagrangian, we have that

$$l_p(x_\epsilon, y, r) = f(x_\epsilon) + \sum_{j=1}^{m}\{y_j g_j(x_\epsilon) + \frac{r}{2} g_j^2(x_\epsilon)\}$$

$$l_p(x,y,r) = f(x) + \sum_{j=1}^{m}\{y_j g_j(x) + \frac{r}{2}g_j^2(x)\}.$$

For $x_\epsilon \in S_\epsilon$ is a every almost-ϵ solution of (Q), thus we can see that

$$f(x_\epsilon) + \sum_{j=1}^{m}\{y_j g_j(x_\epsilon) + \frac{r}{2}g_j^2(x_\epsilon)\} \leq f(x) + \sum_{j=1}^{m}\{y_j g_j(x) + \frac{r}{2}g_j^2(x)\} + \epsilon. \quad (3.6)$$

It is clear that $y_j g_j(x) + \frac{r}{2}g_j^2(x) \leq 0$, so we have that

$$f(x) + \sum_{j=1}^{m}\{y_j g_j(x) + \frac{r}{2}g_j^2(x)\} + \epsilon \leq f(x) + \epsilon$$

and

$$f(x_\epsilon) + \sum_{j=1}^{m}\{y_j g_j(x_\epsilon) + \frac{r}{2}g_j^2(x_\epsilon)\} \leq f(x) + \epsilon,$$

$$f(x_\epsilon) + \sum_{j=1}^{m}\inf_{x\in S_1}\{y_j g_j(x) + \frac{r}{2}g_j^2(x)\} \leq f(x) + \epsilon$$

where $S_1 = \{x \in S_\epsilon : -\frac{y_j}{r} \leq g_j(x) \leq 0\}$

$$f(x_\epsilon) - \sum_{j=1}^{m}(\frac{y_j^2}{2r}) \leq f(x) + \epsilon,$$

$$f(x_\epsilon) \leq f(x) + \sum_{j=1}^{m}(\frac{y_j^2}{2r}) + \epsilon \quad (3.7)$$

but from (3.4), we know that $\sum_{j=1}^{m}(\frac{y_j^2}{2r}) \to 0$. So we can derive that $f(x_\epsilon) \leq f(x) + \epsilon$.

(2) we consider the case (b): from the definition of proximal augmented Lagrangian, we have that

$$l_p(x_\epsilon,y,r) = f(x_\epsilon) + \sum_{i=1}^{m}\{y_j g_j(x_\epsilon) + \frac{r}{2}g_j^2(x_\epsilon)\},$$

$$l_p(x,y,r) = f(x) + \sum_{i=1}^{m}\{y_j g_j(x) + \frac{r}{2}g_j^2(x)\}.$$

For $0 < g_j(x) \leq \epsilon$ and $x_\epsilon \in S_\epsilon$ is a every almost-ϵ solution of (Q), thus we can see that

$$f(x_\epsilon) \leq f(x) + \epsilon$$

(3) we consider the case (c): from the definition of proximal augmented Lagrangian, we have that

$$l_p(x_\epsilon,y,r) = f(x_\epsilon) - \sum_{i=1}^{m}(\frac{y_j^2}{2r}),$$

$$l_p(x,y,r) = f(x) - \sum_{i=1}^{m}(\frac{y_j^2}{2r}).$$

For $x_\epsilon \in S_\epsilon$ is a every almost-ϵ solution of (Q), thus we can see that

$$f(x_\epsilon) \leq f(x) + \epsilon$$

From all of the three cases, we can draw a conclusion that every almost-ϵ solution of (Q) is an almost -ϵ solution (P).

Theorem 3.3. *Assume* f *satisfies all the conditions mentioned in Lemma 3.1, then for any $\epsilon > 0$, there exists $y \in IR^m$ and $r \in IR$ such that every regular almost-ϵ solution of (Q) is an regular almost -ϵ solution (P).*

Proof. From the definition of regular almost-ϵ solution , we need to prove following conclusion there exists $x_\epsilon \in S_\epsilon$, such that

$$f(x_\epsilon) \leq f(x) + \epsilon \quad \forall x \in S \tag{3.8}$$

$$f(x_\epsilon) \leq f(x) + \epsilon \parallel x - x^* \parallel \quad \forall x \in S \tag{3.9}$$

However from Theorem (3.2), we have proved (3.8), and it is easy to check that the proof of (3.9) is similar to that of (3.8), so we omit it. □

4. Approximate optimality conditions

In this section, we will discuss some approximate optimality conditions of constrained optimization problem, obtain necessary condition for a approximate solution of proximal augmented Lagrangian problem, prove that the first-order approximate necessary optimality condition converges to that of the original problem.

Let $\bar{x} \in S$. We denote

$$J_1(\bar{x}) = \{j : g_j(\bar{x}) = 0, j = 1, ..., m\}.$$

We say that the linear independence constrained constraint qualification (LICQ) for (P) holds at $\bar{x}$, if $\{\nabla g_j(\bar{x}) : j \in J_1(\bar{x})\}$ is linearly independent.

Suppose that $\bar{x} \in R^n$ is a local optimal solution to (P) and the (LICQ) for (P) holds at $\bar{x}$. Then the first-order necessary optimality condition is that there exists $\mu_j \geq 0$, $j \in J_1(\bar{x})$ such that

$$\nabla f(\bar{x}) + \sum_{j \in J_1(\bar{x})} \mu_j \nabla g_j(\bar{x}) = 0.$$

Proposition 4.1. *Suppose $\bar{x}_\epsilon \in R^n$ is a regular ϵ-solution for (P) and the (LICQ) for (P) holds at $\bar{x}_\epsilon \in R^n$. Then first-order approximate necessary condition is that there exists real number $\mu_j(\epsilon) \geq 0, j = 1, ..., m$, such that*

$$| \nabla f(\bar{x}_\epsilon) + \sum_{j \in J(\epsilon)} \mu_j(\epsilon) \nabla g_j(\bar{x}_\epsilon) |= \epsilon$$

where $J(\epsilon) = \{j : g_j(\bar{x}_\epsilon) = 0\}$

Proof. From the definition of regular ϵ-solution, we have that there exists $\bar{x}_\epsilon \in S$ such that

$$f(\bar{x}_\epsilon) \leq f(x) + \epsilon \parallel x - \bar{x}_\epsilon \parallel \quad \forall x \in S \tag{4.1}$$

We conclude that $\bar{x}_\epsilon$ is a local optimal solution of the following constrained optimization problem

$$(P^*) \qquad \inf \quad \{f(x) + \epsilon \parallel x - \bar{x}_\epsilon \parallel\}$$
$$\text{s.t.} \quad x \in S.$$

For the objective function $\{f(x) + \epsilon \parallel x - \bar{x}_\epsilon \parallel\}$ is only locally Lipschitz. Thus we apply the corollary of Proposition 2.4.3 in [16] and obtain the KKT necessary condition of (P^*)

$$\nabla f(\bar{x}_\epsilon) + \xi\epsilon + \sum_{j \in J(\epsilon)} \mu_j(\epsilon)\nabla g_j(\bar{x}_\epsilon) = 0 \quad \xi \in [-1, 1]$$

It follows that

$$| \nabla f(\bar{x}_\epsilon) + \sum_{j \in J(\epsilon)} \mu_j(\epsilon)\nabla g_j(\bar{x}_\epsilon) | \leq \epsilon \tag{4.2}$$

Suppose that x_ϵ^k is a local regular approximate optimal solution of (Q). Denote

$$J_1^{+k} = \{j : g_j(x^k) > -\frac{y_j^k}{r_k}, \quad j = 1, ..., m\},$$

$$J_1^k = \{j : g_j(x^k) = -\frac{y_j^k}{r_k}, \quad j = 1, ..., m\}$$

The following first-order approximate necessary condition of problem (Q) can be straightforwardly derived.

Lemma 4.2. *Let x_ϵ^k be a local regular ϵ- solution of problem (Q). Then we have that*

$$\nabla_x l_p(x_\epsilon^k, y_k, r_k) = \left| \nabla f(x_\epsilon^k) + \sum_{j \in J_1^{+k} \bigcup J_1^k} (y_k + g_j(x_\epsilon^k)\nabla g_j(x_\epsilon^k)) \right| \leq \epsilon \tag{4.3}$$

Theorem 4.3 (Convergence analysis). *Suppose $\{y_k\} \in R^m$ is bounded, $0 < r_k \to +\infty$, $x_\epsilon^k \in R^n$ be generated by some methods for solving the following problem (Q_k)*

$$\inf\{l_p(x, y^k, r_k); x \in I\!R^n\}, \quad y^k \in I\!R^m, r_k \geq 0.$$

We assume that each x_ϵ^k satisfies the first-order necessary optimaility condition stated in Lemma (4.2) and $x_\epsilon^k \to \bar{x} \in S$. Furthermore suppose that the (LICQ) for (P) hold at $\bar{x}$. Then the approximate first-order necessary condition for (P) holds at $\bar{x}$.

Proof. Since $x_\epsilon^k \to \bar{x} \in S$, we can see that $J_1^{+k} \bigcup J_1^k \subset J_1(\bar{x})$, when k is sufficiently large. In the following, we assume that k is sufficiently large. Let

$$u_j^k = y_j^k + r_k g_j(x_\epsilon^k), j \in J_1(\bar{x}) \bigcap J_1^{+k}; \quad u_j^k = 0, j \in J_1(\bar{x}) \backslash J_1^{+k}.$$

Then we have that

$$u_j^k \geq 0, \quad j \in J_1(\bar{x}) \tag{4.4}$$

and (4.3) can be formulated as

$$\mid \nabla f(x_\epsilon^k) + \sum_{j \in J_1(\bar{x})} u_j^k \nabla g_j(x_\epsilon^k)) \mid \leq \epsilon \tag{4.5}$$

Now we prove by contradiction that the sequence $\{\sum_{j \in J_1(\bar{x})} u_j^k\}$ is bounded as $k \to +\infty$. Otherwise assume that without loss of generality that

$$\sum_{j \in J_1(\bar{x})} u_j^k \to +\infty$$

and

$$\frac{u_j^k}{\sum_{j \in J_1(\bar{x})} u_j^k} \to u_j^*, \quad j \in J_1(\bar{x})$$

From (4.3) it is easy to check that $u_j^* \geq 0, j \in J_1(\bar{x})$. Dividing (4.4) by $\sum_{j \in J_1(\bar{x})} u_j^k$ and let it to the limit, we can see that

$$\sum_{j \in J_1(\bar{x})} u_j^* \nabla g_j(\bar{x}) = 0 \tag{4.6}$$

This contradicts the (LICQ) of (P) holds at $\bar{x}$, since $\sum_{j \in J_1(\bar{x})} u_j^* = 1$.

Hence$\{\sum_{j \in J_1(\bar{x})} u_j^k\}$ is bounded, so without loss of generality, we assume that

$$u_j^k \to u_j, \quad j \in J_1(\bar{x}) \tag{4.7}$$

and it is clear that

$$u_j \geq 0, \quad j \in J_1(\bar{x})$$

Thus taking limit in (4.5) as $k \to +\infty$ and applying (4.7), we obtain the approximate first-order necessary condition of (P). □

5. Conclusions

As we know Lagrangian method is a powerful tool to transform the constrained optimization problem into an unstrained optimization problem. However it will cause dual gap between primal problem and dual one without some convexity requirements. In [4, 7, 15], Huang and Yang introduced an augmented Lagrangian and studied various properties of augmented Lagrangian problem based on an assumption that the set of exact optimal solutions of the primal constrained optimization problem is not empty. But many mathematical programming problems do not have an optimal solution, moreover sometimes we do not need to find an exact optimal solution due to the fact that it is often very hard to find an exact optimal solution even if it does exists. And as a mater of fact many numerical methods only yield approximate optimal solutions. So in this paper, we consider the ϵ-quasi optimal solution and the augmented Lagrangian in nonlinear programming without the requirement that the set of optimal solutions of the primal constrained optimization problems is not empty, establish dual function and dual problem based on the generalized augmented Lagrangian, obtain approximate KKT necessary optimality condition of the augmented Lagrangian dual problem, prove that the approximate stationary points of augmented Lagrangian problem converge to that of the original problem. Our results generalized Huang and Yang's corresponding results in [4, 7, 15] into approximate case which is more suitable for numerical test.

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