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OPTIMALITY CRITERIA AND DUALITY FOR MULTIOBJECTIVE VARIATIONAL PROBLEMS INVOLVING HIGHER ORDER DERIVATIVES

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ABSTRACT. A multiobjective variational problem involving higher order derivatives is considered and Fritz-John and Karush-Kuhn-Tucker type optimality conditions for this problem are derived. As an application of Karush-Kuhn-Tucker optimality conditions, Wolfe type dual to this variational problem is constructed and various duality results are validated under generalized invexity. Some special cases are mentioned and it is also pointed out that our results can be considered as a dynamic generalization of the already existing results in nonlinear programming.

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1. Introduction

Calculus of variation is a powerful technique for the solution of various problems appearing in dynamics of rigid bodies, optimization of orbits, theory of variations and many other fields. The subject whose importance is fast growing in science and engineering primarily concern with finding optimal of a definite integral involving a certain function subject to fixed point boundary conditions. In [3] Courant and Hilbert, quoting an earlier work of Friedrichs [9], gave a dual relationship for a simple type of unconstrained variational problem. Subsequently, Hanson [11] pointed out that some of the duality results of mathematical programming have analogues in variational calculus. Exploring this relationship between mathematical programming and the classical calculus of variations, Mond and Hanson [13] formulated a constrained variational problem as a mathematical programming problem and using Valentine's [15] optimality conditions for the same, presented its Wolfe type dual variational problem for

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validating various duality results under convexity. Later Bector, Chandra and Husain [2] studied Mond-Weir type duality for the problem of [13] for weakening its convexity requirement. In [5] Chandra, Craven and Husain studied optimality and duality for a class of non-differentiable variational problem with non-differentiable term in the integrand of the objective functional while in [4] they derived optimality conditions and duality results for a constrained variational problem having terms with arbitrary norms in the objective as well as constrained functions.

Recently Husain and Jabeen [10] studied a wider class of variational problem in which the arc function is twice differentiable by extending the notion of invexity given in [14]. They obtained Fritz John as well as Karush- Kuhn-Tucker necessary optimality conditions as an application of Karush-Kuhn-Tucker optimality conditions studied various duality results for Wolfe and Mond and Weir type models.

In single objective programming we must settle on a single objective such as minimizing cost or maximizing profit. However, generally any real world problems can be identified with multiple conflicting criteria e.g., the problems of oil refinery scheduling, production planning, portfolio selection and many others can be modelled as multiobjective programming problems.

Duality results are very useful in the development of numerical algorithms for solving certain classes of optimization problems. Duality for multiobjective variational problem has been studied by a number of authors, notably Bector and Husain [1], Chen [7] and many others cited in these references. Applications of duality theory are prominent in physics, economics, management sciences, etc.

Since mathematical programming and classical calculus of variations have undergone independent development, it is felt that mutual adaptation of ideas and techniques may prove useful. Motivated with this idea in this exposition, we propose to study optimality criteria and duality for a wider class of multiobjective variational problems involving higher order derivative. These results not only generalize the results of Husain and Jabeen [10] and Bector and Husain [1] but also present a dynamic generalization of some of the results in multiobjective nonlinear programming already existing.

2. Invexity and generalized invexity

Invexity was introduced for functions in variational problems by Mond, Chandra and Husain [14] while Mond and Smart [12] defined invexity for functionals instead of functions. Here we introduce extended forms of definitions of invexity and various generalized invexity for functional in variational problems involving higher order derivatives.

Consider the real interval $I = [a, b]$, and the continuously differentiable function $\phi : I \times R^n \times R^n \times R^n \to R$, where x is twice differentiable with its first and second order derivatives $\dot{x}$ and $\ddot{x}$ respectively. If $x = (x^1, x^2, \ldots, x^n)^T$, the gradient vectors of f with respect to x, $\dot{x}$ and $\ddot{x}$ respectively denoted by

$$\phi_x = \left[\frac{\partial\phi}{\partial x^1}, \dots, \frac{\partial\phi}{\partial x^n}\right]^T, \quad \phi_{\dot{x}} = \left[\frac{\partial\phi}{\partial \dot{x}^1}, \dots, \frac{\partial\phi}{\partial \dot{x}^n}\right]^T, \quad \phi_{\ddot{x}} = \left[\frac{\partial\phi}{\partial \ddot{x}^1}, \dots, \frac{\partial\phi}{\partial \ddot{x}^n}\right]^T.$$

Definition 1 (Invexity). If there exists vector function $\eta(t, \dot{u}, \ddot{u}, x, \dot{x}, \ddot{x}) \in R^n$ with $\eta = 0$ and $x(t) = u(t)$, $t \in I$ and $D\eta = 0$ for $\dot{x}(t) = \dot{u}(t)$, $t \in I$ such that for a scalar function $\phi(t, x, \dot{x}, \ddot{x})$, the functional $\Phi(x, \dot{x}, \ddot{x}) = \int_I \phi(t, x, \dot{x}, \ddot{x})dt$ satisfies

$$\Phi(x, \dot{u}, \ddot{u}) - \Phi(x, \dot{x}, \ddot{x})$$
$$\geqq \int_I \{\eta^T \phi_x(t, x, \dot{x}, \ddot{x}) + (D\eta)^T \phi_{\dot{x}}(t, x, \dot{x}, \ddot{x}) + (D^2\eta)^T \phi_{\ddot{x}}(t, x, \dot{x}, \ddot{x})\}dt,$$

Φ is said to be invex in $x, \dot{x}$ and $\ddot{x}$ on I with respect to η.

Definition 2 (Pseudoinvexity). Φ is said to be pseudoinvex in $x, \dot{x} and \ddot{x}$ with respect to η if

$$\int_I \{\eta^T \phi_x(t, x, \dot{x}, \ddot{x}) + (D\eta)^T \phi_{\dot{x}}(t, x, \dot{x}, \ddot{x}) + (D^2\eta)^T \phi_{\ddot{x}}(t, x, \dot{x}, \ddot{x})\}dt \geqq 0$$

implies $\Phi(x, \dot{u}, \ddot{u}) \geqq \Phi(x, \dot{x}, \ddot{x})$.

Definition 3 (Quasi-invex). The functional Φ is said to quasi-invex in x, $\dot{x}$ and $\ddot{x}$ with respect to η if $\Phi(x, \dot{u}, \ddot{u}) \leqq \Phi(x, \dot{x}, \ddot{x})$ implies

$$\int_I \{\eta^T \phi_x(t, x, \dot{x}, \ddot{x}) + (D\eta)^T \phi_{\dot{x}}(t, x, \dot{x}, \ddot{x}) + (D^2\eta)^T \phi_{\ddot{x}}(t, x, \dot{x}, \ddot{x})\}dt \leqq 0.$$

3. Variational problem and optimality conditions

Before stating our variational problem and deriving its necessary optimality condition, we mention the following conventions for vectors x and y in n-dimensional Euclidian space R^n to be used throughout the analysis of this research.

$x < y, \quad \Leftrightarrow \quad x_i < y_i, \quad i = 1, 2, \dots, n.$
$x \leqq y, \quad \Leftrightarrow \quad x_i \leqq y_i, \quad i = 1, 2, \dots, n.$
$x \leq y, \quad \Leftrightarrow \quad x_i \leq y_i, \quad i = 1, 2, \dots, n, \quad \text{but } x \neq y$
$x \nleq y,$ is the negation of $x \leq y$

For $x, y \in R$, $x \leq y$ and $x < y$ have the usual meaning.
We present the following variational problem:

(VPE) $\quad$ Minimize$\left(\int_I f^1(t, x, \dot{x}, \ddot{x})dt, \dots, \int_I f^p(t, x, \dot{x}, \ddot{x})dt\right)$

Subject to

$$x(a) = 0 = x(b) \tag{1}$$
$$\dot{x}(a) = 0 = \dot{x}(b) \tag{2}$$
$$g(t, x, \dot{x}, \ddot{x}) \leqq 0, \quad t \in I \tag{3}$$
$$h(t, x, \dot{x}, \ddot{x}) = 0, \quad t \in I, \tag{4}$$

where

(1) $f^i : I \times R^n \times R^n \times R^n \to R$, $i = 1, 2, \ldots, p$, $g : I \times R^n \times R^n \times R^n \to R^m$ and $h : I \times R^n \times R^n \times R^n \to R^k$ are continuously differentiable function, and
(2) X designates the space of piecewise functions $x : I \to R^n$ possessing derivatives $\dot{x}$ and $\ddot{x}$ with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2x\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)ds,$$

where α is given boundary value; thus $D \equiv \frac{d}{dt}$ except at discontinuities.

In the results to follow, we use $C(I, R^m)$ to denote the space of continuous functions $\phi : I \to R^k$ with the uniform norm $\|\phi\| = \sup |\phi|_{t \in I}$; the partial derivatives of g and h are $m \times n$ and $k \times n$ matrices respectively; superscript T denotes matrix transpose.

We require the following definition of efficient solution for our further analysis.

Definition 4 (Efficient Solution). A feasible solution $\bar{x}$ is efficient for (VPE) if there exist no other feasible x for (VPE) such that for some $i \in P = \{1, 2, \ldots, p\}$,

$$\int_I f^i(t, x, \dot{x}, \ddot{x})dt < \int_I f^i(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}})dt$$

and

$$\int_I f^j(t, x, \dot{x}, \ddot{x})dt \leqq \int_I f^j(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}})dt \quad \text{for all } \; j \in P, \; j \neq i.$$

In relation to (VPE), we introduce the following set of problems $\bar{P}_r$ for each $r = 1, 2, \ldots, p$ in the spirit of [6], with a single objective,

$$(\bar{\mathrm{P}}_\mathrm{r}) \qquad \text{Minimize} \int_I f^r(t, x, \dot{x}, \ddot{x})dt$$

Subject to

$$x(a) = 0 = x(b), \;\; \dot{x}(a) = 0 = \dot{x}(b),$$
$$g(t, x, \dot{x}, \ddot{x}) \leqq 0, \quad t \in I,$$
$$h(t, x, \dot{x}, \ddot{x}) = 0, \quad t \in I,$$
$$\int_I f^i(t, x, \dot{x}, \ddot{x})dt \leqq \int_I f^i(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}})dt, \quad i = 1, 2, \ldots, p, \; i \neq r.$$

The following lemma can be proved on the lines of Chankong and Haimes [6].

Lemma 1. *x^* is an efficient solution of* (VPE) *if and only if $\bar{x}$ is an optimal solution of* $(\bar{\mathrm{P}}_\mathrm{r})$ *for each $r = 1, 2, \ldots, p$.*

Consider the following single objective variational problem considered in [10].

$$(\mathrm{P}_0) \qquad \textit{Minimize} \int_I \phi(t, x, \dot{x}, \ddot{x})dt$$

Subject to

$$x(a) = 0 = x(b), \;\; \dot{x}(a) = 0 = \dot{x}(b),$$
$$g(t, x, \dot{x}, \ddot{x}) \leqq 0, \;\; h(t, x, \dot{x}, \ddot{x}) = 0, \qquad t \in I,$$

where $\phi : I \times R^n \times R^n \times R^n \to R$.

The following proposition gives the Fritz-John type necessary optimality conditions obtained by Husain and Jabeen [10]. In this proposition, we have written the functions without arguments for brevity.

Proposition 1 ([10], (Fritz John Optimality Conditions)). *If $\bar{x}$ is an optimal solution of (P_0) and $h_x(x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot))$ maps X into the subspace of $C(I, R^k)$, then there exists Lagrange multiplier $\bar{\tau} \in R$, the piecewise smooth $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^k$, such that*

$$\begin{aligned}(\bar{\tau}\phi_x + \bar{y}(t)^T g_x + \bar{z}(t)^T h_x) - D(\bar{\tau}\phi_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}})&\\ +D^2(\bar{\tau}\phi_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}} + \bar{z}(t)^T h_{\ddot{x}}) = 0,& \quad t \in I,\\ \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0,& \quad t \in I,\\ (\bar{\tau}, \bar{y}(t)) \geqq 0,& \quad t \in I,\\ (\bar{\tau}, \bar{y}(t), \bar{z}(t)) \neq 0,& \quad t \in I.\end{aligned}$$

If $\bar{\tau} = 1$, then the above optimality conditions will reduce to the Karush-Kuhn-Tucker type optimality conditions and the solution $\bar{x}$ is referred to as a normal solution.

We now establish the following theorem that gives the necessary optimality conditions for (VPE).

Theorem 1 (Fritz-John Conditions). *Let $\bar{x}$ be an efficient solution of (VPE) and $h_x(x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot))$ maps X into the subspace of $C(I, R^k)$, then there exist $\bar{\lambda} \in R^k$ and the piecewise smooth $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^k$, such that*

$$\begin{aligned}(\bar{\lambda}^T f_x + \bar{y}(t)^T g_x + \bar{z}(t)^T h_x) - D(\bar{\lambda}^T f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}} + \bar{z}(t)^T h_{\dot{x}})&\\ +D^2(\bar{\lambda}^T f_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}} + \bar{z}(t)^T h_{\ddot{x}}) = 0,& \quad t \in I, \quad (5)\\ \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0,& \quad t \in I, \quad (6)\\ (\bar{\lambda}, \bar{y}(t)) \geqq 0,& \quad t \in I. \quad (7)\\ (\bar{\lambda}, \bar{y}(t), \bar{z}(t)) \neq 0,& \quad t \in I \quad (8)\end{aligned}$$

Proof. Since $\bar{x}$ is an efficient solution of (VPE) by Lemma 1, $\bar{x}$ is an optimal solution of $(\bar{P}_r)$, for each $r = 1, 2, \ldots, p$. From Proposition 1, it follows that, there exist scalars $\bar{\lambda}^{1r}, \bar{\lambda}^{2r}, \ldots, \bar{\lambda}^{pr}$ and piecewise smooth function $\bar{y} : I \to R^m$ and $\bar{z} : I \to R^k$, such that

$$\begin{aligned}&\bar{\lambda}^{rr} f_x^r + \sum_{\substack{i=1\\ i\neq r}}^{p} \bar{\lambda}^{ir} f_x^i + \sum_{j=1}^{m} \bar{y}^{jr}(t) g_x^j + \sum_{l=1}^{k} \bar{z}^{lr}(t) h_x^l\\ &\quad -D\left(\bar{\lambda}^{rr} f_{\dot{x}}^r + \sum_{\substack{i=1\\ i\neq r}}^{p} \bar{\lambda}^{ir} f_{\dot{x}}^i + \sum_{j=1}^{m} \bar{y}^{jr}(t) g_{\dot{x}}^j + \sum_{l=1}^{k} \bar{z}^{lr}(t) h_{\dot{x}}^l\right)\end{aligned}$$

$$+D^2\Big(\bar{\lambda}^{rr}f^r_{\ddot{x}}+\sum_{\substack{i=1\\ i\neq r}}^{p}\bar{\lambda}^{ir}f^i_{\ddot{x}}+\sum_{j=1}^{m}\bar{y}^{jr}(t)g^j_{\ddot{x}}+\sum_{l=1}^{k}\bar{z}^{lr}(t)h^l_{\ddot{x}}\Big)=0,\quad t\in I,$$

$$\bar{y}^T(t)g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})=0,\quad t\in I,$$
$$(\bar{\lambda}^{1r},\bar{\lambda}^{2r},\dots,\bar{\lambda}^{pr},\bar{y}^{1r}(t),\bar{y}^{2r}(t),\dots,\bar{y}^{mr}(t))\geqq 0,\quad t\in I,$$
$$(\bar{\lambda}^{1r},\bar{\lambda}^{2r},\dots,\bar{\lambda}^{pr},\bar{y}^{1r}(t),\bar{y}^{2r}(t),\dots,\bar{y}^{mr}(t),\bar{z}^{1r}(t),\bar{z}^{2r}(t),\dots,\bar{z}^{lr}(t))\neq 0,\quad t\in I.$$

Summing over r, we have

$$\sum_{r=1}^{p}\Big(\sum_{i=1}^{p}\bar{\lambda}^{ir}\Big)f^i_x+\sum_{r=1}^{p}\Big(\sum_{j=1}^{m}\bar{y}^{jr}(t)\Big)g^j_x+\sum_{r=1}^{p}\Big(\sum_{l=1}^{k}\bar{z}^{lr}(t)\Big)h^l_x$$
$$-D\Big(\sum_{r=1}^{p}\Big(\sum_{i=1}^{p}\bar{\lambda}^{ir}\Big)f^i_{\dot{x}}+\sum_{r=1}^{p}\Big(\sum_{j=1}^{m}\bar{y}^{jr}(t)\Big)g^j_{\dot{x}}+\sum_{r=1}^{p}\Big(\sum_{l=1}^{k}\bar{z}^{lr}(t)\Big)h^l_{\dot{x}}\Big)$$
$$+D^2\Big(\sum_{r=1}^{p}\Big(\sum_{i=1}^{p}\bar{\lambda}^{ir}\Big)f^i_{\ddot{x}}+\sum_{r=1}^{p}\Big(\sum_{j=1}^{m}\bar{y}^{jr}\Big)g^j_{\ddot{x}}+\sum_{r=1}^{p}\Big(\sum_{l=1}^{k}\bar{z}^{lr}(t)\Big)h^l_{\ddot{x}}\Big)=0,$$
$$\bar{y}^T(t)g(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})=0,\quad t\in I,$$
$$\Big(\sum_{r=1}^{p}\bar{\lambda}^{1r},\dots,\sum_{i=1}^{p}\bar{\lambda}^{pr};\sum_{i=1}^{p}\bar{y}^{1r}(t),\dots,\sum_{i=1}^{p}\bar{y}^{mr}(t)\Big)\geqq 0,\quad t\in I,$$
$$\Big(\sum_{r=1}^{p}\bar{\lambda}^{1r},\dots,\sum_{i=1}^{p}\bar{\lambda}^{pr};\sum_{i=1}^{p}\bar{y}^{1r}(t),\dots,\sum_{i=1}^{p}\bar{y}^{mr}(t);\sum_{r=1}^{p}\bar{z}^{1r}(t),\dots,\sum_{r=1}^{p}\bar{z}^{lr}(t)\Big)\neq 0.$$

Setting $\bar{\lambda}^i=\sum_{r=1}^{p}\bar{\lambda}^{ir}$, $\bar{y}^j(t)=\sum_{r=1}^{p}\bar{y}^{jr}(t)$, $t\in I$ and $\bar{z}^l(t)=\sum_{r=1}^{l}\bar{z}^{lr}(t)$, $t\in I$, we have

$$(\bar{\lambda}^Tf_x+\bar{y}(t)^Tg_x+\bar{z}(t)^Th_x)-D(\bar{\lambda}^Tf_{\dot{x}}+\bar{y}(t)^Tg_{\dot{x}}+\bar{z}(t)^Th_{\dot{x}})$$
$$+D^2(\bar{\lambda}^Tf_{\ddot{x}}+\bar{y}(t)^Tg_{\ddot{x}}+\bar{z}(t)^Th_{\ddot{x}})=0,\qquad t\in I,$$
$$\bar{y}(t)^Tg(t,\bar{x},\dot{\bar{x}},\ddot{\bar{x}})=0,\quad t\in I,$$
$$(\bar{\lambda},\bar{y}(t))\geqq 0,\quad t\in I,$$
$$(\bar{\lambda},\bar{y}(t),\bar{z}(t))\neq 0,\quad t\in I.$$

□

Theorem 2 (Karush-Kuhn-Tucker Conditions). *Let $\bar{x}$ be an efficient solution for (VPE) which is assumed to be normal for $(\bar{P}_r)$ for each $r=1,2,\dots,p$. Let the constraints of (P_r) satisfy Slater's Constraint Qualification [5] for each $r=1,2,\dots,p$. Then there exist $\bar{\lambda}^T\in R^k_+$, $\bar{y}:I\to R^m$ and $\bar{z}:I\to R^k$, such that the following relation hold for all $t\in I$,*

$$(\bar\lambda^T f_x + \bar y(t)^T g_x + \bar z(t)^T h_x) - D(\bar\lambda^T f_{\dot x} + \bar y(t)^T g_{\dot x} + \bar z(t)^T h_{\dot x})$$
$$+D^2(\bar\lambda^T f_{\ddot x} + \bar y(t)^T g_{\ddot x} + \bar z(t)^T h_{\ddot x}) = 0, \qquad t \in I, \tag{9}$$
$$\bar y(t)^T g(t, \bar x, \dot{\bar x}, \ddot{\bar x}) = 0, \qquad t \in I \tag{10}$$
$$\bar\lambda > 0, \quad y(t) \geqq 0, \qquad t \in I\,. \tag{11}$$

Proof. Since $\bar x$ is an efficient solution of (VPE) by Lemma 1, $\bar x$ is an optimal solution of $(\bar P_r)$, for each $r = 1, 2, \ldots, p$ then there exist scalars $\bar\lambda_{1r}, \bar\lambda_{2r}, \ldots, \bar\lambda_{pr}$ with $\bar\lambda_{rr} = 1$, $\bar y : I \to R^m$ and $\bar z : I \to R^k$, such that the following conditions are satisfied for all

$$f_x^r + \sum_{\substack{i=1\\ i\neq r}}^{k} \bar\lambda^{ir} f_x^r + \sum_{j=1}^{m} \bar y^{jr}(t) g_x^j + \sum_{l=1}^{k} \bar z^{lr}(t) h_x^l$$
$$-D\Big(f_{\dot x}^r + \sum_{\substack{i=1\\ i\neq r}}^{k} \bar\lambda^{ir} f_{\dot x}^r + \sum_{j=1}^{m} \bar y^{jr}(t) g_{\dot x}^j + \sum_{l=1}^{k} \bar z^{lr}(t) h_{\dot x}^l\Big)$$
$$+D^2\Big(f_{\ddot x}^r + \sum_{\substack{i=1\\ i\neq r}}^{k} \bar\lambda^{ir} f_{\ddot x}^r + \sum_{j=1}^{m} \bar y^{jr}(t) g_{\ddot x}^j + \sum_{l=1}^{k} \bar z^{lr}(t) h_{\ddot x}^l\Big) = 0, \quad t \in I,$$

$\bar y(t)^T g(t, \bar x, \dot{\bar x}, \ddot{\bar x}) = 0, \qquad t \in I,$
$\bar y(t) \geqq 0,$
$\bar\lambda_{ir} \geq 0, \quad i = 1, 2, \ldots, p,\ i \neq r.$

Summing over r and setting $\bar\lambda_i = \sum_{r=1}^{p} \bar\lambda^{ir}$ with $\bar\lambda^{ii} = 1$,

$$\bar y^j(t) = \sum_{r=1}^{p} \bar y^{jr}(t), \quad \bar z^l(t) = \sum_{r=1}^{p} \bar z^{lr}(t)$$

we have

$$(\bar\lambda^T f_x + \bar y(t)^T g_x + \bar z(t)^T h_x) - D(\bar\lambda^T f_{\dot x} + \bar y(t)^T g_{\dot x} + \bar z(t)^T h_{\dot x})$$
$$+D^2(\bar\lambda^T f_{\ddot x} + \bar y(t)^T g_{\ddot x} + \bar z(t)^T h_{\ddot x}) = 0, \quad t \in I,$$
$$\bar y(t)^T g(t, \bar x, \dot{\bar x}, \ddot{\bar x}) = 0, \quad t \in I,$$
$$\bar\lambda > 0, \quad \bar y(t) \geqq 0, \quad t \in I\,.$$

□

Remark. If $\lambda > 0$, then Theorem 1 reduces to Theorem 2 and then an efficient solution is called a normal solution as an analogy to the normality conditions which is equivalent to Slater's Constraint Qualification given in [5].

4. Wolfe type duality

In this section, we consider the following variational problem involving higher order derivatives, by suppressing the equality constraint in (VPE).

$$\text{(VP)} \quad \text{Minimize} \left(\int_I f^1(t,x,\dot{x},\ddot{x})dt, \ldots, \int_I f^p(t,x,\dot{x},\ddot{x})dt \right)$$

Subject to

$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b),$$
$$g(t,x,\dot{x},\ddot{x}) \leqq 0, \quad t \in I$$

We formulate the following Wolfe type dual to the problem (VP) and establish various duality results under invexity defined in the preceding section.

$$\text{(WD)} \quad \text{Maximize}\left(\int_i (f^1(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u}))dt \right.$$
$$\left. ,\ldots, \int_i (f^p(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u}))dt \right)$$

Subject to

$$u(a) = 0 = u(b), \quad \dot{u}(a) = 0 = \dot{u}(b),$$
$$(\lambda^T f_x + y(t)^T g_x) - D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}})$$
$$+D^2(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}) = 0, \qquad t \in I \tag{12}$$
$$y(t) \geqq 0, \qquad t \in I \tag{13}$$
$$\lambda > 0, \quad \lambda^T e = 1 \tag{14}$$
$$\text{where} \quad e = (1,1,\ldots,1)^T \quad \text{and} \quad \lambda \in R^k.$$

Theorem 3 (Weak Duality). *Let $x \in X$ be feasible for (VP) and (u,λ,y) be feasible for (WD), if $\int_I \lambda^T f dt$ is invex and $\int_I y(t)^T g dt$ is invex with respect to the same η. Then*

$$\int_1 f(t,x,\dot{x},\ddot{x})dt \leq \int_I \{f(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u})e\}dt$$

Proof.

$$\lambda^T \left(\int_I f^i(t,x,\dot{x},\ddot{x})dt - \int_I \{f(t,u,\dot{u},\ddot{u}) + y(t)^T g(t,u,\dot{u},\ddot{u})e\}dt \right)$$
$$= \int (\lambda^T f(t,x,\dot{x},\ddot{x}) - \lambda^T f(t,u,\dot{u},\ddot{u}) - (\lambda^T e) y(t)^T g(t,u,\dot{u},\ddot{u}))dt$$
$$= \int_I \lambda^T f(t,x,\dot{x},\ddot{x})dt - \int_I \lambda^T f(t,u,\dot{u},\ddot{u})dt - \int_I y(t)^T g(t,u,\dot{u},\ddot{u})dt,$$
$$\text{(by using } \lambda^T e = 1)$$

$$\begin{aligned}
\geqq \quad & \int_I \{\eta^T(\lambda^T f_x) + (D\eta)^T(\lambda^T f_{\dot{x}}) + (D^2\eta)^T(\lambda^T f_{\ddot{x}})\}dt \\
& - \int_I y(t)^T g(t, u, \dot{u}, \ddot{u})dt \qquad (15)
\end{aligned}$$

This is possible by invexity of $\int_I \lambda^T f dt$ Also from the feasibility of (VP) and (WD), we have

$$\begin{aligned}
& \int_I y(t)^T(t, x, \dot{x}, \ddot{x})dt - \int_I y(t)^T g(t, u, \dot{u}, \ddot{u})dt \\
& \quad \geqq \quad \int_I \{\eta^T(y^T g)_x + (D\eta)^T(y^T g)_{\dot{x}} + (D^2\eta)^T(y^T g)_{\ddot{x}}\}dt
\end{aligned}$$

(by definition of Invexity)

This implies

$$\begin{aligned}
-\int_I y(t)^T g(t, u, \dot{u}, \ddot{u})dt \quad \geqq \quad & \int_I \{\eta^T(y^T g)_x + (D\eta)^T(y^T g)_{\dot{x}} + (D^2\eta)^T(y^T g)_{\ddot{x}}\}dt \\
& - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})dt
\end{aligned}$$

Using this in (15), we have,

$$\begin{aligned}
& \lambda^T\Big(\int_I f(t, x, \dot{x}, \ddot{x})dt - \int_I \{f(t, u, \dot{u}, \ddot{u}) + y(t)^T g(t, u, \dot{u}, \ddot{u})e\}dt\Big) \\
& \quad \geqq \quad \int_I \eta^T\{\lambda^T f_x + (D\eta)^T(\lambda^T f_{\dot{x}}) + (D^2\eta)^T(\lambda^T f_{\ddot{x}})\}dt \\
& \qquad\quad + \int_I \{\eta^T(y^T g)_x + (D\eta)^T(y^T g)_{\dot{x}} + (D^2\eta)^T(y^T g)_{\ddot{x}}\}dt \\
& \qquad\quad - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})dt \\
& \quad = \quad \int_I \eta^T\{(\lambda^T f_x + y^T g_x) + (D\eta)^T(\lambda^T f_{\dot{x}} + y^T g_{\dot{x}}) \\
& \qquad\quad + (D\eta^2)^T(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}})\}dt - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})dt \\
& \quad = \quad \int_I \eta^T(\lambda^T f_x + y^T g_x)dt + \eta^T(\lambda^T f_{\dot{x}} + y^T g_{\dot{x}})|_{t=a}^{t=b} \\
& \qquad\quad - \int_I \eta^T D(\lambda^T f_{\dot{x}} + y^T g_{\dot{x}})dt + (D\eta)^T(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}})|_{t=a}^{t=b} \\
& \qquad\quad - \int_I (D\eta)^T D(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}})dt - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})dt
\end{aligned}$$

(by integration by parts)

Using the boundary conditions which give $D\eta = 0 = \eta$ at $t = a$, $t = b$

$$\begin{aligned}
&= \int_I \eta^T(\lambda^T f_x + y^T g_x)dt - \int_I \eta^T D(\lambda^T f_x + y^T g_x)dt + \eta^T D(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}})|_{t=a}^{t=b} \\
&\quad + \int_I \eta^T D^2(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}})dt - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})dt
\end{aligned}$$

(by integration by parts)

Using the boundary conditions which give $D\eta = 0 = \eta$ at $t = a$, $t = b$

$$\begin{aligned}
&= \int_I \eta^T\{(\lambda^T f_x + y^T g_x) - D(\lambda^T f_{\dot{x}} + y^T g_{\dot{x}}) + D^2(\lambda^T f_{\ddot{x}} + y^T g_{\ddot{x}})\}dt \\
&\quad - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})dt \\
&\geqq - \int_I y(t)^T g(t, x, \dot{x}, \ddot{x})dt, \qquad \text{(by equation (12))} \\
&\geqq 0 \qquad \text{(by (3) and (13))}
\end{aligned}$$

That is,

$$\int_I \lambda^T f(t, x, \dot{x}, \ddot{x})dt \geqq \int_I \{\lambda^T f(t, u, \dot{u}, \ddot{u}) + y(t)^T g(t, u, \dot{u}, \ddot{u})\}dt$$

or

$$\lambda^T(\int_I f(t, x, \dot{x}, \ddot{x}))dt \geqq \lambda^T(\int_I \{f(t, u, \dot{u}, \ddot{u}) + y(t)^T g(t, u, \dot{u}, \ddot{u})e\})dt$$

This yields,

$$\int_I f(t, x, \dot{x}, \ddot{x})dt \nleq \int_I \{f(t, u, \dot{u}, \ddot{u}) + y(t)^T g(t, u, \dot{u}, \ddot{u})e\}dt . \qquad \square$$

Theorem 4 (Strong Duality). *If $\bar{x}$ is efficient and normal solution of (VP), then there exist piecewise smooth $\bar{y} : I \to R^m$ such that $(\bar{x}, \bar{y})$ is feasible for (WD) and the corresponding objective values of the problems (VP) and (WD) are equal. If the hypotheses of Theorem 2 are satisfied, then $(\bar{x}, \bar{y})$ is an efficient solution of (WD).*

Proof. Since $\bar{x}$ is efficient and normal for (VP), by Theorem 2, it implies that there exist $\mu \in R^p$ and piecewise smooth $u : I \to R^m$ such that,

$$\begin{aligned}
(\mu^T f_x + u(t)^T g_x) - D(\mu^T f_{\dot{x}} + u(t)^T g_{\dot{x}}) + D^2(\mu^T f_{\ddot{x}} + u(t)^T g_{\ddot{x}}) &= 0, \qquad t \in I, \\
\bar{u}(t)^T g(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) &= 0, \qquad t \in I, \\
\mu > 0 \quad \bar{u}(t) &\geqq 0, \qquad t \in I.
\end{aligned}$$

Since $\mu > 0$, $\mu^T e \neq 0$.

$$\left(\left(\frac{\mu}{\mu^T e}\right)^T f_x + \frac{y(t)}{\mu^T e}^T g_x\right) - D\left(\left(\frac{\mu}{\mu^T e}\right)^T f_{\dot{x}} + \frac{y(t)}{\mu^T e}^T g_{\dot{x}}\right)$$
$$+D^2\left(\left(\frac{\mu}{\mu^T e}\right)^T f_{\ddot{x}} + \frac{y(t)}{\mu^T e}^T g_{\ddot{x}}\right) = 0, \qquad t \in I$$

$$\left(\frac{y(t)}{\mu^T e}\right)^T g(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0, \quad \frac{\mu}{\mu^T e} > 0, \frac{y(t)}{\mu^T e} \geqq 0, \qquad t \in I$$

Setting $\frac{\mu}{\mu^T e} = \bar{\lambda}_i$ and $\frac{y(t)}{\mu^T e} = \bar{y}(t)$ in the above relations, we have

$$(\bar{\lambda}^T f_x + \bar{y}(t)^T g_x) - D(\bar{\lambda}^T f_{\dot{x}} + \bar{y}(t)^T g_{\dot{x}})$$
$$+D^2(\bar{\lambda}^T f_{\ddot{x}} + \bar{y}(t)^T g_{\ddot{x}}) = 0, \qquad t \in I \tag{16}$$
$$\bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}, \ddot{\bar{x}}) = 0, \quad t \in I \tag{17}$$
$$\left.\begin{array}{r} \bar{\lambda} > 0 \;\; \bar{y}(t) \geqq 0 \\ \lambda^T e = 1 \end{array}\right\}, \quad t \in I \tag{18}$$

From (5) and (7), it follows that $(\bar{x}, \bar{\lambda}, \bar{y})$ is feasible for (WD). The equality of the objective of (VP) and (WD) is obvious in view of (17).

The efficiency of $(\bar{x}, \bar{\lambda}, \bar{y})$ for (WD) follows from Theorem 3.

As in [13], by employing chain rule in calculus, it can be easily seen that the expression $(\lambda^T f_x + y(t)^T g_x) - D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) + D^2(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}})$, may be regarded as a function θ of variables t, x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$, y, $\dot{y}$, $\ddot{y}$ and λ, where $\dddot{x} = D^3 x$and $\ddot{y} = D^2 y$. That is, we can write

$$\theta(t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \lambda)$$
$$= (\lambda^T f_x + y(t)^T g_x) - D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) + D^2(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}})$$

In order to prove converse duality between (VP) and (WD), the space $X$is now replaced by a smaller space X_2 of piecewise smooth thrice differentiable function $x : I \to R^n$with the norm $\|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty + \|D^3 x\|_\infty$. The problem (WD) may now be briefly written as,

$$\text{Minimize } -\left(\int_i (f^1(t, x, \dot{x}, \ddot{x}) + y(t)^T g(t, x, \dot{x}, \ddot{x}))dt\right.$$
$$\left., \ldots, \int_i (f^p(t, x, \dot{x}, \ddot{x}) + y(t)^T g(t, x, \dot{x}, \ddot{x}))dt\right)$$

Subject to

$$x(a) = 0 = x(b), \quad \dot{x}(a) = 0 = \dot{x}(b),$$
$$\theta(t, x, \dot{x}, \ddot{x}, \dddot{x}, y, \dot{y}, \ddot{y}, \lambda) = 0, \quad y(t) \geqq 0, \qquad t \in I$$

$\lambda > 0, \qquad \lambda^T e = 1$
where $e = (1, 1, \ldots, 1)^T \in R^p$ and $\lambda \in R^p$.

Consider $\theta(t, x(\cdot), \dot{x}(\cdot), \ddot{x}(\cdot), \dddot{x}(\cdot), y(\cdot), \dot{y}(\cdot), \ddot{y}(\cdot), \lambda) = 0$ as defining a mapping $\psi : X_2 \times Y \times R^p \to B$ where Y is a space of piecewise twice differentiable function and B is the Banach Space. In order to apply Theorem 1 to the problem (WD), the infinite dimensional inequality must be restricted. In the following theorem, we use $\psi'$to represent the Frèchèt derivative $[\psi_x(x, y, \lambda), \psi_y(x, y, \lambda), \psi_\lambda(x, y, \lambda)]$.

Theorem 5 (Converse Duality). *Let $(\bar{x}, \bar{\lambda}, \bar{y})$ be an efficient solution of (WD) and ψ' has a (weak*) closed range. Assume that*
(H_1) f and g are twice differentiable,
(H_2) the hypotheses of Theorem 3 hold, and
(H_3) $\sigma(t)^T(\sigma(t)^T\theta_x - D\sigma(t)^T\theta_{\dot{x}} + D^2\sigma(t)^T\theta_{\ddot{x}} - D^3\sigma(t)^T\theta_{\dddot{x}}) = 0$, $t \in I$
$\Rightarrow \sigma(t) = 0$, $t \in I$
Then $\bar{x}$ is an efficient solution of (VP).

Proof. Since $(\bar{x}, \bar{\lambda}, \bar{y})$ is an efficient solution of (WD) and ψ' has a closed range, then by Theorem 1, there exist $\alpha \in R^k$ and piecewise smooth $\beta : I \to R^n, \xi : I \to R^m$ and $\mu^T \in R^p$ such that

$$\begin{aligned}
&[-(\alpha f_x + (\alpha^T e)y(t)^T g_x) + \beta(t)^T\theta_x] - D[-(\alpha f_{\dot{x}} + (\alpha^T e)y(t)^T g_{\dot{x}}) + \beta(t)^T\theta_{\dot{x}}] \\
&\quad +[-(\alpha f_{\ddot{x}} + (\alpha^T e)y(t)^T g_{\ddot{x}}) + \beta(t)^T\theta_{\ddot{x}}] - D^3\beta(t)^T\theta_{\dddot{x}} = 0 \qquad (19)
\end{aligned}$$

$$-(\alpha^T e)g + \beta(t)^T\theta_y - D(\beta(t)^T\theta_{\dot{y}}) + D^2(\beta(t)^T\theta_{\ddot{y}}) - \xi(t) = 0 \qquad (20)$$

$$\beta(t)(f_x - Df_{\dot{x}} + D^2 f_{\ddot{x}}) + \mu^i + \gamma = 0 \qquad (21)$$

$$\mu^T\lambda = 0 \qquad (22)$$

$$\xi(t)^T\bar{y}(t) = 0 \qquad (23)$$

$$\gamma\left(\sum_{i=1}^{p}\lambda^i - 1\right) = 0 \qquad (24)$$

$$(\alpha, \mu, \gamma, \xi(t)) \geqq 0 \qquad (25)$$

$$(\alpha, \beta(t), \mu, \gamma, \xi(t)) \neq 0 \qquad (26)$$

Since $\lambda > 0$, (22) implies $\mu = 0$. Consequently (21) implies

$$\beta(t)^T(f_x^i - Df_{\dot{x}}^i + D^2 f_{\ddot{x}}^i) = -\gamma \qquad (27)$$

From the equality constraint of the dual problem (WD) together with (27), it follows

$$\begin{aligned}
y(t)^T g_x - Dy(t)^T g_{\dot{x}} + D^2 y(t)^T g_{\ddot{x}} &= -\sum_{i=1}^{p}\lambda^i(f_x^i - Df_{\dot{x}}^i + D^2 f_{\ddot{x}}^i) \\
(y(t)^T g_x - Dy(t)^T g_{\dot{x}} + D^2 y(t)^T g_{\ddot{x}})\beta(t) &= -\sum_{i=1}^{p}\lambda^i(f_x^i - Df_{\dot{x}}^i + D^2 f_{\ddot{x}}^i)\beta(t)
\end{aligned}$$

$$= \quad -\sum_{i=1}^{p} \lambda^i(-\gamma) = \gamma \tag{28}$$

Postmultiplying (19) by $\beta(t)$ and then using (27) and (28), we have

$$(\beta(t)^T \theta_x - D\beta(t)^T \theta_{\dot{x}} + D^2\beta(t)^T \theta_{\ddot{x}} - D^3\beta(t)^T \theta_{\dot{\ddot{x}}})\beta(t) = 0$$

This because of the hypothesis of (H3) yields

$$\beta(t) = 0, \qquad t \in I\,. \tag{29}$$

Therefore from (27), we have $\gamma = 0$.

Suppose $\alpha = 0$, then from (20), $\xi(t) = 0, t \in I$. Consequently we have $(\alpha, \beta(t), \mu, \gamma, \xi(t)) = 0$, $t \in I$. This is in contradiction to (26)

Hence $\alpha > 0$. The relation (20) in conjunction with (29) yields,

$$g(t, x, \dot{x}, \ddot{x}) = -\frac{\xi(t)}{\alpha^T e} \leqq 0 \tag{30}$$

This implies the feasibility of $\bar{x}$ for (VP). The relation (30) with (23) yields

$$y(t)^T g(t, x, \dot{x}, \ddot{x}) = 0, \quad t \in I\,. \tag{31}$$

This implies,

$$\int_i (f^i(t, u, \dot{u}, \ddot{u}) + y(t)^T g(t, u, \dot{u}, \ddot{u}))dt = \int_i f^i(t, x, \dot{x}, \ddot{x})dt$$

This along with an application of Theorem 3 accomplishes the efficiency of $\bar{x}$ for (VP). □

5. Natural boundary values

The duality results obtained in the preceding sections can easily be extended to the multiobjective variational problems with natural boundary values rather than fixed end points.

Primal (P_1) Minimize $\left(\int_I f^1(t, x, \dot{x}, \ddot{x})dt, \ldots, \int_I f^p(t, x, \dot{x}, \ddot{x})dt\right)$

Subject to

$$g(t, x, \dot{x}, \ddot{x}) \leqq 0, \qquad t \in I$$

Dual (D_1) Maximize $\int_I (f(t, u, \dot{u}, \ddot{u}) + y(t)^T g(t, u, \dot{u}, \ddot{u})e)dt$

Subject to

$$(\lambda^T f_x + y(t)^T g_x) - D(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}})$$
$$+D^2(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}) = 0, \quad t \in I$$

$$(\lambda^T f_{\dot{x}} + y(t)^T g_{\dot{x}}) = 0, \quad \text{at} \quad t = a \quad \text{and} \quad t = b$$
$$(\lambda^T f_{\ddot{x}} + y(t)^T g_{\ddot{x}}) = 0, \quad \text{at} \quad t = a \quad \text{and} \quad t = b$$
$$y \geqq 0, \quad \lambda > 0, \quad \lambda^T e = 1,$$

where $e = (1, 1, \ldots, 1)^T \in R^p$.

6. Nonlinear programming

If the problems (P_1) and (D_1) are independent of t, then they will reduce the following multiobjective nonlinear programming problems: studied in [8]

(NP): Minimize $f(x)$

Subject to

$$g(x) \leq 0.$$

(ND): Maximize $f(x) + y^T g(x) e$

Subject to

$$\lambda^T f_x + y^T g_x = 0, \quad y \geqq 0.$$

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