J. Appl. Math. & Informatics Vol. **27**(2009), No. 1 - 2, pp. 97 - 103
Website: http://www.kcam.biz

INVOLUTION-PRESERVING MAPS WITHOUT THE LINEARITY ASSUMPTION AND ITS APPLICATION

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ABSTRACT. Suppose $\mathbf{F}$ is a field of characteristic not 2 and $\mathbf{F} \neq Z_3$. Let $M_n(\mathbf{F})$ be the linear space of all $n \times n$ matrices over $\mathbf{F}$, and let $\Gamma_n(\mathbf{F})$ be the subset of $M_n(\mathbf{F})$ consisting of all $n \times n$ involutory matrices. We denote by $\Phi_n(\mathbf{F})$ the set of all maps from $M_n(\mathbf{F})$ to itself satisfying $A - \lambda B \in \Gamma_n(\mathbf{F})$ if and only if $\phi(A) - \lambda\phi(B) \in \Gamma_n(\mathbf{F})$ for every $A, B \in M_n(\mathbf{F})$ and $\lambda \in \mathbf{F}$. It was showed that $\phi \in \Phi_n(\mathbf{F})$ if and only if there exist an invertible matrix $P \in M_n(\mathbf{F})$ and an involutory element ε such that either $\phi(A) = \varepsilon PAP^{-1}$ for every $A \in M_n(\mathbf{F})$ or $\phi(A) = \varepsilon PA^T P^{-1}$ for every $A \in M_n(\mathbf{F})$. As an application, the maps preserving inverses of matrces also are characterized.

AMS Mathematics Subject Classification : 15A04.
Key words and phrases : Involutory matrix, idempotent matrix, map.

1. Introduction

At present, some researchers are more interested in the study of the Preserver Problems without "linearity" assumption (see [1]-[8]). One of important techniques in the study of linear preserver problems is to reduce new linear preserver problems to the known ones (see [9]-[11]). However, the example using the reduce technique in the study of invariant preserver problem without the "linearity" assumption has not appeared in the literature. In this paper, after defining a sequence of sets (i.e. the following $N_n(k), k = 0, 1, \cdots, n^2$), we characterize involution-preserving maps without the linearity assumption by reducing it to the known idempotent preserver, furthermore, the maps preserving inverses of matrices are also characterized.

Suppose $\mathbf{F}$ is a field of characteristic not 2 and $\mathbf{F} \neq Z_3$. Let $M_n(\mathbf{F})$ be the linear space of all $n \times n$ matrices over $\mathbf{F}$, and let $\Omega_n(\mathbf{F})$ be a subset of $M_n(\mathbf{F})$. We denote by $\Phi(\Omega_n(\mathbf{F}))$ the set of all maps from $M_n(\mathbf{F})$ to itself satisfying $A - \lambda B \in \Omega_n(\mathbf{F})$ if and only if $\phi(A) - \lambda\phi(B) \in \Omega_n(\mathbf{F})$ for every $A, B \in M_n(\mathbf{F})$ and $\lambda \in \mathbf{F}$. Denote $\Gamma_n(\mathbf{F}) = \{A | A \in M_n(\mathbf{F}) \textit{ and } A^2 = I_n\}$,

Received October 17, 2007. Revised February 15, 2008. Accepted March 5, 2008.
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$P_n(\mathbf{F}) = \{A|A \in M_n(\mathbf{F})\ and\ A^2 = A\}$. In this paper, the general form of surjections in $\Phi(\Gamma_n(\mathbf{F}))$ are determined, and thereby, as an application, the the surjective maps preserving inverses of matrices are also characterized.

Denote by $[1, n]$ the set $\{1, 2, \cdots, n\}$. For any $i, j \in [1, n]$, E_{ij} expresses the matrix with 1 in the $(i, j)th$ position and 0 elsewhere. Let O_r and I_r be the $r \times r$ zero matrix and $r \times r$ identity matrix. Denote by " $\oplus$ " the usual direct sum of matrices. Now, we define a sequence of sets by

$$N_n(0) = \{O_n\}$$

$$N_n(k) = \{A|A \in M_n(\mathbf{F}), \exists\lambda \neq 0, B \in N_n(k-1), s.t.\ \lambda A + B \in P_n(F)\}, \forall k \in [1, n^2].$$

2. Main results

Lemma 1. *Suppose* $\mathbf{F}$ *is a field of characteristic not 2, then* $A \in P_n(\mathbf{F})$ *if and only if* $I_n - 2A \in \Gamma_n(\mathbf{F})$.

Proof. The proof is simple, so we omit it. □

Lemma 2. *Suppose* $\mathbf{F}$ *is a field of characteristic not 2, and* $\phi \in \Phi(\Gamma_n(\mathbf{F}))$ *is a surjective map, then* $\ker\phi = N_n(0)$.

Proof. Since $\phi \in \Phi(\Gamma_n(\mathbf{F}))$ is a surjective map, then $\ker\phi \neq \emptyset$, and thereby

$$\phi(I_n) - \lambda\phi(A) = \phi(I_n) \in \Gamma_n(\mathbf{F}), \forall\ A \in \ker\phi, \lambda \in \mathbf{F}$$

Hence $I_n - \lambda A \in \Gamma_n(\mathbf{F})$. Noting that $ch\mathbf{F} \neq 2$, we have $A = O_n$. i.e., $\ker\phi = N_n(0)$. □

Lemma 3. *Suppose* $\mathbf{F}$ *is a field of characteristic not 2, and* $\phi \in \Phi(\Gamma_n(\mathbf{F}))$ *is a surjective map, then* $\phi(I_n) \in \{-I_n, I_n\}$.

Proof. For any given $A \in \Gamma_n(\mathbf{F}) \setminus \{-I_n, I_n\}$, there exist an invertible matrix P and $k \in [1, n-1]$ such that $A = P(I_k \oplus -I_{n-k})P^{-1}$. Denote $B = PE_{k,k+1}P^{-1}$, obviously, $B \neq O_n$ and $A - \lambda B \in \Gamma_n(\mathbf{F})$ for any $\lambda \in \mathbf{F}$, and thereby $\phi(A) - \lambda\phi(B) \in \Gamma_n(\mathbf{F})$ for any $\lambda \in \mathbf{F}$.

If $\phi(A) \in \{-I_n, I_n\}$, then it is easily verified that $\phi(B) = O_n$ which contradicts with Lemma 2. Noting that ϕ is a surjective map, we complete the proof. □

Lemma 4. *Suppose* $\mathbf{F}$ *is an arbitrary field, then* $M_n(\mathbf{F}) = \bigcup_{k=0}^{n^2} N_n(k)$.

Proof. It is easy to see that for any given $k \in [0, n^2]$, $A \in N_n(k)$ if and only if $\mu A \in N_n(k), \forall\ \mu \in \mathbf{F} \setminus \{0\}$.

For any given $A \in M_n(\mathbf{F})$, there exist $A_k \in P_n(\mathbf{F}), a_k \in \mathbf{F}$ and $k \in [1, n^2]$ such that $A = \sum_{k=1}^{n^2} a_k A_k$. If $A = O_n$, then $A \in N_n(0)$. If $A \neq O_n$, without loss of generality, we assume $a_k \neq 0, \forall\ k \in [1, t]$ and $A = \sum_{k=1}^{t} a_k A_k$.

Now, we will prove $\sum_{k=1}^{t} a_k A_k \in N_n(t)$ by induction on t.

When $t = 1$, it follows from ${a_1}^{-1}(a_1 A_1) + O_n \in P_n(\mathbf{F})$ that $a_1 A_1 \in N_n(1)$.

Assume that $\sum_{k=1}^{s} a_k A_k \in N_n(s)$, then

$$ {a_{s+1}}^{-1} \sum_{k=1}^{s+1} a_k A_k - a_{s+1}^{-1} \sum_{k=1}^{s} a_k A_k = A_{s+1} \in P_n(\mathbf{F}), $$

and therefore $\sum_{k=1}^{s+1} a_k A_k \in N_n(s+1)$. Thus, $A \in \bigcup_{k=0}^{n^2} N_n(k)$. □

Lemma 5. *Suppose* $\mathbf{F}$ *is a field of characteristic not 2 and* $\mathbf{F} \neq Z_3$*, and* $\phi \in \Phi(\Gamma_n(\mathbf{F}))$ *satisfies* $\phi(I_n) = I_n$*, then*

$$ \phi(\mu I_n - 2A) = \mu I_n - 2\phi(A), \forall\ A \in M_n(\mathbf{F}), \mu \in \mathbf{F}. $$

Proof. Based on Lemma 4, we prove the conclusion by induction on $A \in N_n(k), k \in [0, n^2]$. It follows from $\mu I_n - (\mu - 1)I_n = I_n \in \Gamma_n(\mathbf{F})$, and $\mu I_n - (\mu + 1)I_n = -I_n \in \Gamma_n(\mathbf{F})$ that

$$ \phi(\mu I_n) - (\mu - 1)\phi(I_n) \in \Gamma_n(\mathbf{F}),\ \phi(\mu I_n) - (\mu + 1)\phi(I_n) \in \Gamma_n(\mathbf{F}) $$

Together with $\phi(I_n) = I_n$, we get $(\phi(\mu I_n) - \mu I_n) \pm I_n \in \Gamma_n(\mathbf{F})$, and thereby $\phi(\mu I_n) = \mu I_n$. i.e., the conclusion holds for $A = O_n \in N_n(0)$.

Assume the conclusion holds for any $B \in N_n(k-1)$. Now, we prove the conclusion holds for any $A \in N_n(k)$.

It follows from the definition of $N_n(k)$ that there exist $B \in N_n(k-1)$ and $\lambda \in \mathbf{F}\backslash\{0\}$ such that $\lambda A + B \in P_n(\mathbf{F})$. By Lemma 1, we obtain $(I_n - 2B) - 2\lambda A \in \Gamma_n(\mathbf{F})$. This, together with $\phi \in \Phi(\Gamma_n(\mathbf{F}))$, implies $\phi(I_n - 2B) - 2\lambda\phi(A) \in \Gamma_n(\mathbf{F})$. For $B \in N_n(k-1)$, using the inductive hypothesis, we have $\phi(I_n - 2B) = I_n - 2\phi(B)$. Hence, $I_n - 2\phi(B) - 2\lambda\phi(A) \in \Gamma_n(\mathbf{F})$. By Lemma 1, we derive $\lambda\phi(A) + \phi(B) \in P_n(\mathbf{F})$, so there exists an invertible matrix $P \in M_n(\mathbf{F})$ such that

$$ \lambda\phi(A) + \phi(B) = P(I_p \oplus O_{n-p})P^{-1}. \quad (1) $$

Let $\sigma \in \mathbf{F} \setminus \{-\lambda, 0, \lambda\}$ and $\varepsilon \in \{-1, 1\}$. We first prove

(i) $\phi(\varepsilon I_n - 2\sigma A) = \varepsilon I_n - 2\sigma\phi(A)$ for any $A \in N_n(k)$ holds.

It follows from $(\varepsilon I_n - 2\sigma A) + 2\sigma A = \varepsilon I_n \in \Gamma_n(\mathbf{F})$ and $\big((1 - \lambda\sigma^{-1}\varepsilon)I_n - 2B\big) + \lambda\sigma^{-1}(\varepsilon I_n - 2\sigma A) = I_n - 2(\lambda A + B) \in \Gamma_n(\mathbf{F})$ that

$$ \phi(\varepsilon I_n - 2\sigma A) + 2\sigma\phi(A) \in \Gamma_n(\mathbf{F}) \quad (2) $$

and

$$ \phi((1 - \lambda\sigma^{-1}\varepsilon)I_n - 2B) + \lambda\sigma^{-1}\phi(\varepsilon I_n - 2\sigma A) \in \Gamma_n(\mathbf{F}). \quad (3) $$

Since $B \in N_n(k-1)$, by the inductive hypothesis , we get

$$ \phi((1 - \lambda\sigma^{-1}\varepsilon)I_n - 2B) = (1 - \lambda\sigma^{-1}\varepsilon)I_n - 2\phi(B). \quad (4) $$

Substituting (1) into (2), we derive

$$(\phi(\varepsilon I_n - 2\sigma A) + 2\sigma(\lambda^{-1}P(I_p \oplus O_{n-p})P^{-1} - \lambda^{-1}\phi(B)))^2 = I_n. \quad (5)$$

Substituting (4) into (3), we obtain

$$((1 - \lambda\sigma^{-1}\varepsilon)I_n - 2\phi(B) + \lambda\sigma^{-1}\phi(\varepsilon I_n - 2\sigma A))^2 = I_n. \quad (6)$$

Denote $\xi = \lambda\sigma^{-1}$. It follows from $\sigma \in \mathbf{F} \setminus \{-\lambda, 0, \lambda\}$ that $\xi \in \mathbf{F} \setminus \{-1, 0, 1\}$. Denote

$$\phi(\varepsilon I_n - 2\sigma A) - 2\sigma\lambda^{-1}\phi(B) = PXP^{-1}. \quad (7)$$

Substituting $\xi = \lambda\sigma^{-1}$ and (7) into (5) and (6), we have

$$(X + 2\xi^{-1}(I_p \oplus O_{n-p}))^2 = I_n \quad (8)$$

and

$$(X + (\xi^{-1} - \varepsilon)I_n)^2 = \xi^{-2}I_n. \quad (9)$$

Combining (8) and (9), we obtain

$$(\xi^{-1} - \varepsilon)(X - \varepsilon I_n) = \xi^{-1}(X(I_p \oplus O_{n-p}) + (I_p \oplus O_{n-p})X) + 2\xi^{-2}(I_p \oplus O_{n-p}).$$

Noting that $\xi \in F \setminus \{-1, 0, 1\}$. We get $X = (\varepsilon - 2\xi^{-1})I_p \oplus \varepsilon I_{n-p}$. Hence

$$\begin{aligned}\phi(\varepsilon I_n - 2\sigma A) &= PXP^{-1} + 2\lambda^{-1}\sigma\phi(B)\\ &= \varepsilon I_n - 2\xi^{-1}P(I_p \oplus O_{n-p})P^{-1} + 2\lambda^{-1}\sigma\phi(B)\\ &= \varepsilon I_n - 2\lambda^{-1}\sigma(\lambda\phi(A) + \phi(B)) + 2\lambda^{-1}\sigma\phi(B)\\ &= \varepsilon I_n - 2\sigma\phi(A).\end{aligned} \quad (10)$$

Now, we prove

(ii) $\phi(\gamma I_n - 2\sigma A) = \gamma I_n - 2\sigma\phi(A)$, where $\gamma \in F \setminus \{-1, 1\}, \sigma \in F \setminus \{-\lambda, 0, \lambda\}$. It follows from $\gamma \in F \setminus \{-1, 1\}$ and $\sigma \in F \setminus \{-\lambda, 0, \lambda\}$ that there exists $\delta \in \{-1, 1\}$ such that $(\gamma - \delta)^{-1}\sigma \neq \pm\lambda$. In fact, if $(\gamma - 1)^{-1}\sigma = \pm\lambda$, then $(\gamma - 1)\lambda = \pm\sigma$, and thereby $(\gamma + 1)\lambda = 2\lambda \pm \sigma \notin \{-\sigma, \sigma\}$, (otherwise, it controdicts with $\sigma \in F \setminus \{-\lambda, 0, \lambda\}$ and $\lambda \neq 0$). Therefore, $(\gamma + 1)^{-1}\sigma \neq \pm\lambda$. Similarly, it follows from $(\gamma + 1)^{-1}\sigma = \pm\lambda$ that $(\gamma - 1)^{-1}\sigma \neq \pm\lambda$. Besides, it follows from $(\gamma I_n - 2\sigma A) - (\gamma - \delta)(I_n - 2\sigma(\gamma - \delta)^{-1}A) = \delta I_n \in \Gamma_n(F)$ and $((1 - \lambda\sigma^{-1}\gamma)I_n - 2B) + \lambda\sigma^{-1}(\gamma I_n - 2\sigma A) = I_n - 2(\lambda A + B) \in \Gamma_n(F)$ that

$$\phi(\gamma I_n - 2\sigma A) - (\gamma - \delta)\phi(I_n - 2\sigma(\gamma - \delta)^{-1}A) \in \Gamma_n(F) \quad (11)$$

and

$$\phi((1 - \lambda\sigma^{-1}\gamma)I_n - 2B) + \lambda\sigma^{-1}\phi(\gamma I_n - 2\sigma A) \in \Gamma_n(F). \quad (12)$$

Because of $(\gamma - \delta)^{-1}\sigma \neq \pm\lambda$, using (10) we have

$$\phi(I_n - 2\sigma(\gamma - \delta)^{-1}A) = I_n - 2\sigma(\gamma - \delta)^{-1}\phi(A). \quad (13)$$

Because of $B \in N_n(k - 1)$, by the inductive hypothesis, we get

$$\phi((1 - \sigma^{-1}\lambda\gamma)I_n - 2B) = (1 - \sigma^{-1}\lambda\gamma)I_n - 2\phi(B). \quad (14)$$

Substituting (1) and (13) into (11), we obtain

$$(\phi(\gamma I_n - 2\sigma A) - (\gamma - \delta)I_n + 2\sigma(\lambda^{-1}P(I_p \oplus O)P^{-1} - \lambda^{-1}\phi(B)))^2 = I_n. \tag{15}$$

Substituting (14) into (12), we derive

$$(\lambda\sigma^{-1}\phi(\gamma I_n - 2\sigma A) + (1 - \lambda\sigma^{-1}\gamma)I_n - 2\phi(B))^2 = I_n. \tag{16}$$

Let

$$\phi(\gamma I - 2\sigma A) - 2\sigma\lambda^{-1}\phi(B) - (\gamma - \delta)I = PYP^{-1}. \tag{17}$$

Substituting (17) into (15) and (16), we derive

$$(Y + 2\xi^{-1}(I_p \oplus O_{n-p}))^2 = I_n, \tag{18}$$

$$(Y + (\xi^{-1} - \delta)I_n)^2 = \xi^{-2}I_n. \tag{19}$$

Combining (18) and (19), we get

$$(\xi^{-1} - \delta)(Y - \delta I_n) = \xi^{-1}(Y(I_p \oplus O_{n-p}) + (I_p \oplus O_{n-p})Y) + 2\xi^{-2}(I_p \oplus O_{n-p}).$$

Noting that $Y = (\delta - 2\xi^{-1})I_p \oplus \delta I_{n-p}$, we have, for any $\sigma \in F \setminus \{-\lambda, 0, \lambda\}$

$$\phi(\gamma I_n - 2\sigma A) = PYP^{-1} + 2\lambda^{-1}\sigma\phi(B) + (\gamma - \delta)I_n = \gamma I_n - 2\sigma\phi(A). \tag{20}$$

Finally, we prove the conclusion in Lemma 5 Since $\mathbf{F} \neq Z_3$, then $|\mathbf{F} \setminus \{-\lambda, 0, \lambda\}| \geq 2$. For any $\mu \in \mathbf{F}, \sigma \in \mathbf{F} \setminus \{-\lambda, 0, \lambda\}$, it follows from

$$((1 - \mu\sigma)I_n + 2\sigma A) + \sigma(\mu I_n - 2A) = I_n \in \Gamma_n(F)$$

that

$$\phi((1 - \mu\sigma)I_n + 2\sigma A) + \sigma\phi(\mu I_n - 2A) \in \Gamma_n(F). \tag{21}$$

In views of $\sigma \in F \setminus \{-\lambda, 0, \lambda\}$, (10) and (20), we have

$$\phi((1 - \mu\sigma)I_n + 2\sigma A) = (1 - \mu\sigma)I_n + 2\sigma\phi(A). \tag{22}$$

Substituting (22) into (21), we get

$$I_n + \sigma(\phi(\mu I_n - 2A) - \mu I_n + 2\phi(A)) \in \Gamma_n(F), \forall\ \sigma \in F \setminus \{-\lambda, 0, \lambda\}.$$

Hence $\sigma(\phi(\mu I_n - 2A) - \mu I_n + 2\phi(A)) = O_n$. From $\sigma \neq 0$, we know that $\phi(\mu I_n - 2A) = \mu I_n - 2\phi(A)$ for any $A \in N_n(k)$. We complete the proof. □

Theorem 1. *Suppose* $\mathbf{F}$ *is a field of characteristic not 2 and* $\mathbf{F} \neq Z_3$, *and* $\phi \in \Phi(\Gamma_n(F))$ *is a surjective map. Then there exist an invertible matrix* $P \in M_n(F)$ *and an involutory element* $\varepsilon \in F$ *such that either* $\phi(A) = \varepsilon PAP^{-1}$ *for any* $A \in M \in (F)$ *or* $\phi(A) = \varepsilon PA^TP^{-1}$ *for any* $A \in M_n(F)$.

Proof. By Lemma 3 we have $\phi(I_n) \in \{-I_n, I_n\}$.

Case 1. $\phi(I_n) = I_n$. It follows from Lemma 5 that

$$\begin{aligned} A - \lambda B \in P_n(\mathbf{F}) &\Leftrightarrow (I - 2A) + 2\lambda B \in \Gamma_n(\mathbf{F}) \\ &\Leftrightarrow \phi(I - 2A) + 2\lambda\phi(B) = I - 2\phi(A) + 2\lambda\phi(B) \in \Gamma_n(\mathbf{F}) \\ &\Leftrightarrow \phi(A) - \lambda\phi(B) \in P_n(\mathbf{F}) \quad \forall\, A, B \in M_n(\mathbf{F}), \lambda \in \mathbf{F}. \end{aligned}$$

So $\phi \in \Phi(P_n(\mathbf{F}))$. By [3, theorem 1] , there exists an invertible matrix $P \in M_n(\mathbf{F})$ such that

$$\phi(A) = PAP^{-1},\ \forall\, A \in M_n(\mathbf{F}) \text{ or } \phi(A) = PA^TP^{-1},\ \forall\, A \in M_n(\mathbf{F}).$$

Case 2. $\phi(I_n) = -I_n$. Let $\psi(X) = -\phi(X)$, $\forall\, X \in M_n(\mathbf{F})$. Then $\psi \in \Phi(\Gamma_n(\mathbf{F}))$, $\psi(I_n) = I_n$, and ψ is a surjective map. Hence there exists an invertible matrix $P \in M_n(\mathbf{F})$ such that

$$\phi(A) = -PAP^{-1},\ \forall\, A \in M_n(\mathbf{F}) \text{ or } \phi(A) = -PA^TP^{-1},\ \forall\, A \in M_n(\mathbf{F}).$$

□

4. Application

Theorem 2. *Suppose* $\mathbf{F}$ *is a field of characteritic not 2 and* $\mathbf{F} \neq Z_3$. *A surjective map* $\phi : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *satisfies* $A - \lambda B = C^{-1}$ *if and only if* $\phi(A) - \lambda\phi(B) = \phi(C)^{-1}$, *where* $A, B, C \in M_n(\mathbf{F}), \lambda \in \mathbf{F}$ *and* C *is an invertible matrix. Then there exists an invertible matrix* $P \in M_n(\mathbf{F})$ *and an involutory element* $\varepsilon \in \mathbf{F}$ *such that either* $\phi(A) = \varepsilon PAP^{-1}$, $\forall\, A \in M_n(\mathbf{F})$ *or* $\phi(A) = \varepsilon PA^TP^{-1}$, $\forall\, A \in M_n(\mathbf{F})$.

Proof. Obviously, for any invertible matrix $A - \lambda B \in M_n(\mathbf{F})$, we have $\phi((A - \lambda B)^{-1}) = (\phi(A) - \lambda\phi(B))^{-1}$ and $\phi(A^{-1}) = (\phi(A))^{-1}$ while $\lambda = 0$. Hence,

$$\begin{aligned} &A - \lambda B \in \Gamma_n(\mathbf{F}) \Rightarrow (A - \lambda B)^{-1} = A - \lambda B \\ &\Rightarrow \phi(A) - \lambda\phi(B) = (\phi(A - \lambda B)^{-1})^{-1} = \phi(A - \lambda B) \\ &= \phi((A - \lambda B)^{-1}) = (\phi(A) - \lambda\phi(B))^{-1} \\ &\Rightarrow \phi(A) - \lambda\phi(B) \in \Gamma_n(\mathbf{F}). \end{aligned}$$

On the contrary, if $\phi(A) - \lambda\phi(B) \in \Gamma_n(\mathbf{F})$, then

$$\phi(A) - \lambda\phi(B) = (\phi(A) - \lambda\phi(B))^{-1} = \phi((A - \lambda B)^{-1}) = (\phi(A - \lambda B))^{-1}$$

This means $A - \lambda B = (A - \lambda B)^{-1}$, and thereby $A - \lambda B \in \Gamma_n(\mathbf{F})$. So $\phi \in \Phi(\Gamma_n(\mathbf{F}))$. Thus, by Theorem 1, we complete the proof. □

5. Remark

The method in this paper is also available for the upper triangular matrix space. The followings are some open problems related to this paper.

Can the surjection assumption be omitted?

When ch$\mathbf{F}$=2, characterizing the forms of elements in $\Phi(\Gamma_n(\mathbf{F}))$ is an open problem as well.

6. Acknowledgements

The author would like to thank the referees for their valuable comments which have improve this paper significantly.

This work was supported in part by the Chinese Natural Science Foundation under Grant No.10671026, the Natural Science Foundation of Heilongjiang Province under Grant No. A01-07 and the Fund of Heilongjiang Education Committee for Overseas Scholars under Grant No.1054HQ004.

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