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HIGH ACCURACY POINTS OF WAVELET APPROXIMATION

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ABSTRACT. The accuracy of wavelet approximation at resolution $h = 2^{-k}$ to a smooth function f is limited by $O(h^M)$, where M is the number of vanishing moments of the mother wavelet ψ; that is, the approximation order of wavelet approximation is $M - 1$. High accuracy points of wavelet approximation are of interest in some applications such as signal processing and numerical approximation. In this paper, we prove the scaling and translating properties of high accuracy points of wavelet approximation. To illustrate the results in this paper, we also present two examples of high accuracy points of wavelet approximation.

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1. Introduction

One of the main features of wavelets is the scaling and translating of the scaling function and/or the mother wavelet to construct a basis of a Hilbert space $\mathcal{H}$, usually $L^2(\mathbb{R})$ ([1],[2]). Any numerical method based on wavelets starts by projecting a function in a Hilbert space $\mathcal{H}$ onto some subspace V_k of $\mathcal{H}$. This projection produces a discretization error.

Some methods then decompose V_k into subspaces $W_{k-1}+W_{k-2}+\cdots+W_{k-j}+V_{k-j}$ by the fast wavelet transform, perform calculations in the decomposed space, and then reconstruct V_k by the inverse fast wavelet transform.

We are completely focused on estimating the error resulting from projecting a function from $\mathcal{H}$ onto a subspace V_k of $\mathcal{H}$, and finding methods to minimize this error. Wavelet approximation at high accuracy points minimizes this error. Moreover, approximation order at high accuracy points is greater than the global approximation order of wavelet approximation.

Motivated by error and approximation order of wavelet approximations, it is desirable to investigate properties, such as scaling and translating properties, of

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high accuracy points of wavelet approximation. By proving scaling and translating properties of high accuracy points of wavelet approximations, it suffices to work with high accuracy points of wavelet approximation at one level only.

The main objective of this paper is to prove the scaling and translating properties of high accuracy points of wavelet approximation. Also we find high accuracy points of wavelet approximation for some examples.

This paper is organized as follows. Preliminaries of the wavelet theory are introduced in Section 2. The main result is stated in Section 3. Finally, numerical examples are given in Section 4.

2. Preliminaries

A *multiresolution analysis* (MRA) of $L^2(\mathbb{R})$ ([3]) is a sequence of closed subspaces $\{V_k\}_{k\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ with the following properties:

$$\begin{array}{rlcl} (i) & V_k \subset V_{k+1}, & & \\ (ii) & \bigcup_{k\in\mathbb{Z}} V_k \text{ is dense in } L^2(\mathbb{R}) & \text{and} & \bigcap_{k\in\mathbb{Z}} V_k = \{0\}, \\ (iii) & f(x) \in V_k & \Longleftrightarrow & f(2x) \in V_{k+1}, \\ (iv) & f(x) \in V_0 & \Longleftrightarrow & f(x-j) \in V_0 \text{ for each } j \in \mathbb{Z}, \\ (v) & \multicolumn{3}{l}{\text{there exists a function } \phi(x) \in V_0, \text{ called the } \textit{scaling function,}} \\ & \multicolumn{3}{l}{\text{such that } \{\phi(x-j)\}_{j\in\mathbb{Z}} \text{ forms a Riesz basis of } V_0.} \end{array} \tag{1}$$

If the integer translates $\{\phi(x-j)\}_{j\in\mathbb{Z}}$ of a scaling function ϕ forms an orthogonal basis of V_0, then we call ϕ an *orthogonal scaling function.*

Let W_j denote the orthogonal complement of V_j in V_{j+1}, i.e.,

$$V_{j+1} = V_j \bigoplus W_j,$$

where $\oplus$ is a direct sum. The space $L^2(\mathbb{R})$ can then be represented as the direct sum

$$L^2(\mathbb{R}) = \overline{\bigoplus_{j\in\mathbb{Z}} W_j}.$$

A function $\psi(x)$ is called a *(mother) wavelet* if the integer translates $\Big\{\psi(x-j)\Big\}_{j\in\mathbb{Z}}$ form a Riesz basis for W_0. We assume that such a function $\psi(x)$ exists. We denote the scaled translates of ϕ by

$$\phi_j^k(x) = 2^{k/2}\phi(2^k x - j). \tag{2}$$

Then for fixed $k \in \mathbb{Z}$, the $\Big\{\phi_j^k\Big\}_{j\in\mathbb{Z}}$ form a Riesz basis of V_k.

Since $\phi(x) \in V_0 \subset V_1$ and $\psi(x) \in W_0 \subset V_1$, we must have *recursion relations*

$$\phi(x) = \sqrt{2}\sum_{j\in\mathbb{Z}} h_j\, \phi(2x-j), \tag{3}$$

$$\psi(x) = \sqrt{2} \sum_{j \in \mathbb{Z}} g_j \, \psi(2x - j). \tag{4}$$

for some sequences $\{h_j\}_{j\in\mathbb{Z}}$, $\{g_j\}_{j\in\mathbb{Z}}$ in ℓ^2. The recursion relation coefficients satisfy

$$\sum_{j \in \mathbb{Z}} h_j = \sum_{j \in \mathbb{Z}} (-1)^j g_j = \sqrt{2}. \tag{5}$$

Orthogonal wavelets are usually constructed from a scaling function ϕ in (1) such that $\left\{\phi(x-j)\right\}_{j\in\mathbb{Z}}$ form an orthonormal basis of V_0. Mallat has shown that if $\left\{\phi(x-j)\right\}_{j\in\mathbb{Z}}$ is an orthonormal basis of V_0, then the function

$$\psi(x) := \sum_{j \in \mathbb{Z}} (-1)^j h_{1-j} \phi(x - j),$$

is an orthogonal wavelet ([3].)

We assume that the basis functions are normalized, that is,

$$\int_{-\infty}^{\infty} \phi(x)\, dx = 1. \tag{6}$$

Let $\mathcal{P}_k$ be the orthogonal projection from $L^2(\mathbb{R})$ onto V_k; that is, $\mathcal{P}_k f$ is the best $L^2(\mathbb{R})$ approximation to $f \in L^2(\mathbb{R})$ from V_k. Since $\{\phi_j^k\}_{j\in\mathbb{Z}}$ is an orthonormal basis of V_k, we obtain the orthogonal projector $\mathcal{P}_k$ from $L^2(\mathbb{R})$ onto V_k as

$$\mathcal{P}_k f(x) = \sum_{j \in \mathbb{Z}} \langle f, \phi_j^k \rangle \phi_j^k(x). \tag{7}$$

We call $\mathcal{P}_k f$ the *wavelet approximation* or *scaling function approximation* to f at resolution $h = 2^{-k}$, and $\langle f, \phi_j^k \rangle$ the *wavelet coefficients.*

We define a function H by

$$H(\xi) = \frac{1}{\sqrt{2}} \sum_{j \in \mathbb{Z}} h_j e^{-ij\xi} \qquad \text{where } i = \sqrt{-1}$$

so that the Fourier transform $\hat{\phi}$ of the scaling function ϕ satisfies

$$\hat{\phi}(\xi) = \hat{\phi}\Big(\frac{\xi}{2}\Big) H\Big(\frac{\xi}{2}\Big). \tag{8}$$

$H(\xi)$ is a 2π-periodic function, given in the form of Fourier series.

The *discrete moments* of the scaling function ϕ and the wavelet ψ are defined by

$$m_p = \frac{1}{\sqrt{2}} \sum_{j \in \mathbb{Z}} j^p h_j, \tag{9}$$

$$n_p = \frac{1}{\sqrt{2}} \sum_{j \in \mathbb{Z}} j^p g_j, \tag{10}$$

where h_j and g_j satisfy (5) and p is any nonnegative integer.

The *continuous moments* of the scaling function ϕ and the wavelet ψ are defined by

$$\mathcal{M}_p = \int_{-\infty}^{\infty} x^p \phi(x)\,dx, \tag{11}$$

$$\mathcal{N}_p = \int_{-\infty}^{\infty} x^p \psi(x)\,dx, \tag{12}$$

for any nonnegative integer p. Since the scaling function is normalized as (6), $\mathcal{M}_0 = 1$.

A recursive formula to calculate the continuous moments of ϕ from the coefficients h_j (used in calculating discrete moments m_j) is

$$\begin{aligned} \mathcal{M}_0 &= 1, \\ \mathcal{M}_p &= \frac{1}{2^p - 1} \sum_{\ell=1}^{p} \binom{p}{\ell} m_\ell \mathcal{M}_{p-\ell}, \qquad \text{for } p \geq 1 \end{aligned} \tag{13}$$

(see ([5]). The continuous moments of ψ can be calculated by

$$\mathcal{N}_p = \frac{1}{2^p} \sum_{\ell=0}^{p} \binom{p}{\ell} n_\ell \mathcal{M}_{p-\ell}, \qquad \text{for } p \geq 0. \tag{14}$$

3. High accuracy points

Wavelet approximation at high accuracy points minimizes the error resulting from projecting a function from $\mathcal{H}$ onto a subspace V_k of $\mathcal{H}$. Moreover, approximation order at high accuracy points is greater than the global approximation order of wavelet approximations.

Motivated by error and approximation order of wavelet approximation, we prove the scaling and translating properties of high accuracy points of wavelet approximations in this section. By proving scaling and translating properties of high accuracy points of wavelet approximations, it suffices to work with high accuracy points of wavelet approximation at one level only.

Throughout the paper, M is a positive integer and ϕ is a scaling function with M vanishing moments for the corresponding wavelet ψ.

The following two Lemmas will be used in the proof of Theorem 1.

Lemma 1. ([4]) *The following are equivalent:*

- *the first M vanishing moments of the wavelet ψ are zero:*

$$\mathcal{N}_i = 0 \qquad \text{for } i = 0, 1, \ldots, M-1 \quad \text{and} \quad \mathcal{N}_M \neq 0.$$

- *The polynomials $1, x, \ldots, x^{M-1}$ are linear combinations of the translates $\phi(x - j)$.*

- *Smooth function can be approximated with error $\mathcal{O}(h^M)$ at resolution $h = 2^{-k}$:*

$$\|f - \mathcal{P}_k f\| = \mathcal{O}(h^M). \tag{15}$$

Lemma 2. *For $p = 0, 1, 2, \ldots, M-1$*

$$x^p = \sum_{j\in\mathbb{Z}} c_j^p\, \phi(x-j),$$

where

$$c_j^p := \langle x^p, \phi(x-j)\rangle = \begin{cases} \mathcal{M}_p, & j = 0 \\ \sum_{i=0}^{p} \binom{p}{i} j^i \mathcal{M}_{p-i}, & j \in \mathbb{Z}\setminus\{0\}. \end{cases} \tag{16}$$

Proof. It can be easily proved by using the binomial expansion and the definition of the continuous moments of the scaling function. □

We are in a position to prove the main result of this paper. We first prove the translating property of high accuracy points of wavelet approximation with resolution $2^{-k} = 1$ at level $k = 0$.

Theorem 1. *Let $f(x) = x^M$. If $\mathcal{P}_0 f(s) = f(s)$ for some $s \in [0, 1)$, that is, s is a high accuracy point of wavelet approximations on the unit interval $[0, 1)$, then for any integer i*

$$\mathcal{P}_0 f(s+i) = f(s+i),$$

that is, if

$$s^M = \sum_{j\in\mathbb{Z}} c_j^M\, \phi(x-j), \tag{17}$$

then for any integer i

$$(s+i)^M = \sum_{j\in\mathbb{Z}} c_j^M\, \phi(x-j), \tag{18}$$

where c_j^M is defined as (16).

Proof. It suffices to show that we have equality for $i = 1$. Note that

$$\sum_{j\in\mathbb{Z}} c_j^M\, \phi(s+1-j) = \sum_{j\in\mathbb{Z}} c_{j+1}^M\, \phi(s-j). \tag{19}$$

By the binomial expansion, we have

$$\begin{aligned} (s+1)^M &= \sum_{i=0}^{M} \binom{M}{i} s^i \\ &= \sum_{j\in\mathbb{Z}} \sum_{i=0}^{M} \binom{M}{i} c_j^i\, \phi(s-j) \end{aligned}$$

$$= \sum_{i=0}^{M} \binom{M}{i} \mathcal{M}_i \,\phi(s) + \sum_{j\neq 0} \sum_{i=0}^{M} \sum_{\ell=0}^{i} \binom{M}{i} \binom{i}{\ell} j^\ell \mathcal{M}_{i-\ell} \,\phi(s-j)$$

$$= c_1^M \phi(s) + \sum_{j\neq 0} \sum_{i=0}^{M} \sum_{\ell=0}^{i} \binom{M}{i} \binom{i}{\ell} j^\ell \mathcal{M}_{i-\ell} \,\phi(s-j).$$

It is enough to show that

$$\sum_{i=0}^{M} \sum_{\ell=0}^{i} \binom{M}{i} \binom{i}{\ell} j^\ell \mathcal{M}_{i-\ell} = c_{j+1}^M \tag{20}$$

for $j \neq 0$.

Let $n = M - i$. Then by expanding the double summation and then by collecting terms with $\mathcal{M}_i$ first, we have

$$\begin{aligned} \sum_{i=0}^{M} \sum_{\ell=0}^{i} \binom{M}{i} \binom{i}{\ell} j^\ell \mathcal{M}_{i-\ell} &= \sum_{k=0}^{M} \sum_{n=0}^{k} \binom{M}{M-n} \binom{M-n}{k-n} j^{k-n} \mathcal{M}_{M-k} \\ &= \sum_{k=0}^{M} \sum_{n=0}^{k} \binom{M}{k} \binom{k}{n} j^{k-n} \mathcal{M}_{M-k} \\ &= \begin{cases} \mathcal{M}_M, & j = -1 \\ \displaystyle\sum_{k=0}^{M} \binom{M}{k} (j+1)^k \mathcal{M}_{M-k}, & j \in \mathbb{Z} \setminus \{-1\} \end{cases} \\ &= c_{j+1}^M. \end{aligned}$$

This completes the proof. □

Corollary 2. *Let ϕ be a scaling function with M vanishing moments for the wavelet ψ. Let f be a polynomial of degree less than or equal to M. Then the translating property of high accuracy points in Theorem 1 also holds; that is, if*

$$f(s) = \sum_{j\in\mathbb{Z}} \langle f(x), \phi(x-j) \rangle \,\phi(s-j), \tag{21}$$

then

$$f(s+i) = \sum_{j\in\mathbb{Z}} \langle f(x), \phi(x-j) \rangle \,\phi(s+i-j) \tag{22}$$

for any integer i.

Proof. By Lemma 1 wavelet approximations are exact for polynomials up to degree $M-1$ for all real x. For the term x^M, it holds from Theorem 1. □

Now we prove the scaling property of high accuracy points of a wavelet approximation with resolution 2^{-k} at level k.

Theorem 3. *If s is a high accuracy point of wavelet approximation at level 0, that is,*

$$f(s) = \sum_{j\in\mathbb{Z}} \langle f(\cdot), \phi(\cdot - j)\rangle\, \phi(s-j) = \sum_{j\in\mathbb{Z}} \langle f(\cdot), \phi_j^0(\cdot)\rangle\, \phi_j^0(s), \tag{23}$$

then $2^{-k}s$ is a high accuracy point of wavelet approximation at any integer level k, that is,

$$f(2^k s) = \sum_{j\in\mathbb{Z}} \langle f(2^k\cdot), \phi_j^k(\cdot)\rangle\, \phi_j^k(s). \tag{24}$$

Proof. By substitution, we note that

$$\left\langle f(2^k\cdot), \phi_j^k(\cdot)\right\rangle = 2^{-k/2}\langle f(\cdot), \phi_j^0(\cdot)\rangle,$$

and

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \langle f(2^k\cdot), \phi_j^k(\cdot)\rangle\, \phi_j^k(s) &= \sum_{j\in\mathbb{Z}} 2^{-k/2}\langle f(\cdot), \phi_j^0(\cdot)\rangle\, 2^{k/2}\phi_j^0(2^k s) \\
&= \sum_{j\in\mathbb{Z}} \langle f(\cdot), \phi_j^0(\cdot)\rangle\, \phi_j^0(2^k s) \\
&= f(2^k s).
\end{aligned}$$

Hence, $2^{-k}s$ is a high accuracy point of wavelet approximation at level k. □

4. Examples

In this section, we give two examples to illustrate the general theory.

We base our examples on the Daubechies orthonormal scaling functions ϕ with $M = 2, 3$ vanishing moments for wavelets ψ given in [2]. We find that there exist two high accuracy points s on each interval of length 2^{-k}, where k is a level number.

Example 1. In this example we take the Daubechies orthonormal scaling function ϕ with 2 vanishing moments for wavelets ψ given in [2]. The recursion coefficients of ϕ are

$$h_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, \quad h_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad h_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad h_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}.$$

Some discrete moments of the scaling function ϕ are

$$m_0 = 1, \quad m_1 = \frac{3-\sqrt{3}}{2}, \quad m_2 = 3 - \frac{3\sqrt{3}}{2}, \quad m_3 = \frac{27-17\sqrt{3}}{4}.$$

Some continuous moments of the scaling function ϕ are

$$\mathcal{M}_0 = 1, \quad \mathcal{M}_1 = \frac{3-\sqrt{3}}{2}, \quad \mathcal{M}_2 = 3 - \frac{3\sqrt{3}}{2}, \quad \mathcal{M}_3 = \frac{27}{4} - \frac{107\sqrt{3}}{28}.$$

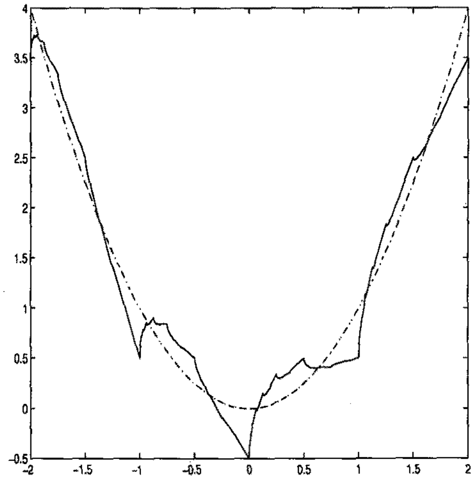

FIGURE 1. Wavelet approximation (solid) with Daubechies 2 vanishing moment for x^2 (dashdot) at level 0. The two high accuracy points on $[0,1)$ at level 0 are $s \approx 0.081055$ and 0.634766, the meeting points of x^2 and wavelet approximation on $[0,1)$.

The global accuracy of the wavelet approximation is of order 1, that is, the wavelet approximation is exact up to the linear functions.

If the level k is 0 and the integer translate j is 0, then it suffices to find the high accuracy points of the wavelet approximation on the unit interval $[0,1)$. We find that the two high accuracy points of the wavelet approximation on $[0,1)$ are $s \approx 0.081055$ and $s \approx 0.634866$.

If the level k and the integer translate j are any integers, then, by the Theorems 1 and 3, the high accuracy points on the interval $\left[j/2^k, (1+j)/2^k\right)$ are approximately $(0.081055+j)/2^k$ and $(0.634766+j)/2^k$.

Example 2. In this example we take the Daubechies orthonormal wavelets ϕ with 3 vanishing moments in [2]. The recursion coefficients are

$$h_0 = \frac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{16\sqrt{2}}, \qquad h_1 = \frac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{16\sqrt{2}},$$
$$h_2 = \frac{10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}}}{16\sqrt{2}}, \qquad h_3 = \frac{10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}}}{16\sqrt{2}},$$
$$h_4 = \frac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{16\sqrt{2}}, \qquad h_5 = \frac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16\sqrt{2}}.$$

Some discrete moments of the scaling function ϕ are

$$\begin{aligned}
m_0 &= 1,\\
m_1 &= \frac{1}{2}\left(5-\sqrt{5+2\sqrt{10}}\right) \approx 0.817401,\\
m_2 &= \frac{1}{2}\left(15+\sqrt{10}-\sqrt{5+2\sqrt{10}}\right) \approx 0.668145,\\
m_3 &= \frac{1}{4}\left(100+15\sqrt{10}-44\sqrt{5+2\sqrt{10}}\right) \approx -0.158633.
\end{aligned}$$

Some continuous moments of the scaling function ϕ are

$$\begin{aligned}
\mathcal{M}_0 &= 1,\\
\mathcal{M}_1 &= \frac{1}{2}\left(5-\sqrt{5+2\sqrt{10}}\right) \approx 0.817401,\\
\mathcal{M}_2 &= \frac{1}{2}\left(15+\sqrt{10}-\sqrt{5+2\sqrt{10}}\right) \approx 0.668145,\\
\mathcal{M}_3 &= 25+\frac{15}{4}\sqrt{10}-\left(\frac{71}{7}+\frac{3}{64}\sqrt{10}\right)\sqrt{5+2\sqrt{10}} \approx 0.445460.
\end{aligned}$$

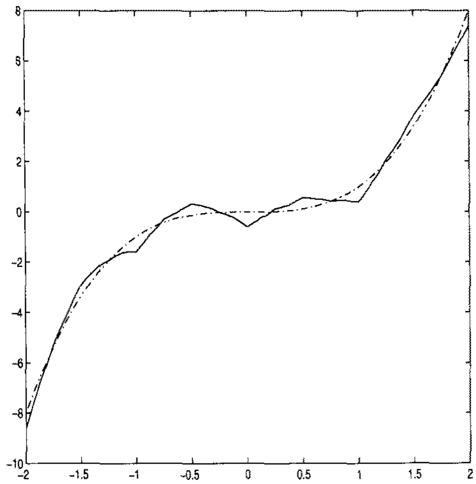

FIGURE 2. Wavelet approximation (solid) with Daubechies 3 vanishing moment for x^3 (dashdot) at level 0. The two high accuracy points on $[0,1)$ at level 0 are $s \approx 0.218750$ and 0.751465, the meeting points of x^3 and wavelet approximation on $[0,1)$.

The global accuracy of the wavelet approximation is of order 2, that is, the wavelet approximation is exact up to the quadratic functions.

If the level k is 0 and the integer translate j is 0, then it suffices to find the high accuracy points of the wavelet approximation on the unit interval $[0,1)$. We find that the two high accuracy points of the wavelet approximation on $[0, 1)$ are $s \approx 0.218750$ and $s \approx 0.751465$.

If the level k and the integer translate j are any integers, then, by the Theorems 1 and 3, the high accuracy points on the interval $\left[j/2^k, (1+j)/2^k\right)$ are approximately $(0.218750 + j)/2^k$ and $(0.751465 + j)/2^k$.

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