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STRONG CONVERGENCE THEOREMS BY VISCOSITY APPROXIMATION METHODS FOR ACCRETIVE MAPPINGS AND NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper we present an iterative scheme for finding a common element of the set of zero points of accretive mappings and the set of fixed points of nonexpansive mappings in Banach spaces. By using viscosity approximation methods and under suitable conditions, some strong convergence theorems for approximating to this common elements are proved. The results presented in the paper improve and extend the corresponding results of Kim and Xu [Nonlinear Anal. TMA **61** (2005), 51-60], Xu [J. Math. Anal. Appl., **314** (2006), 631-643] and some others.

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1. Introduction and preliminaries

Throughout this paper, we always assume that E is a real Banach space, C is a nonempty closed convex subset of E and $S : C \to C$ is a mapping. We denote by $F(S) = \{x \in C : Sx = x\}$ the set of fixed points of mapping S. In the sequel, we use $\to$ to stands for the strong convergence and $\rightharpoonup$ to stands for the weak convergence.

Recall that $S : C \to C$ is *nonexpansive*, if

$$||Sx - Sy|| \leq ||x - y||, \ \ \forall x, y \in C.$$

Recall that a (possibly multivalued) mapping A with domain $D(A)$ and range $R(A)$ in E is said to be *accretive*, if for any $x_i \in D(A)$ and $y_i \in Ax_i \ \ (i = 1, 2)$, there exists a $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, \ j(x_2 - x_1)\rangle \geq 0,$$

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where $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$J(x) = \{x^* \in 2^{E^*} : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \quad x \in E.$$

A mapping $A : E \to E$ is said to be *$m-$accretive*, if $R(I + rA) = E, \ \forall r > 0$. Throughout this paper we always assume that $A : E \to E$ is $m-$accretive and has a *zero point* (i.e., the inclusion $0 \in A(z)$ is solvable). The set of zero points of A is denoted by

$$A^{-1}(0) = \{z \in D(A) : 0 \in A(z)\}.$$

For each $r > 0$, denote by J_r the *resolvent of A*, i.e.,

$$J_r = (I + rA)^{-1}. \tag{1.1}$$

It is well-known that if A is a $m-$accretive, then $J_r : E \to E$ is nonexpansive and

$$F(J_r) = A^{-1}(0), \quad \forall r > 0. \tag{1.2}$$

For each $r > 0$ we also denote by A_r the *Yosida approximation of A*, i.e., $A_r := \frac{1}{r}(I - J_r)$.

Recently, Dominguez et al [4], Kim and Xu [7] and Xu [9] introduced and studied the following iterative sequence $\{x_n\}$:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geq 0 \tag{1.3}$$

and proved some strong convergence theorems for the sequence (1.3) in the framework of uniformly smooth Banach spaces and reflexive Banach space with a weak continuous duality mapping, respectively, where $u \in C$ is a given point.

Inspired and motivated by the works given in [4, 5, 7, 8, 9], the purpose of this paper is to introduce the following composite iteration schemes:

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \\ \quad y_n = \beta_n x_n + (1 - \beta_n) S J_r(x_n) \end{cases} n \geq 0; \tag{1.4}$$

for finding a common element of the set of zero points of accretive mapping A and the set of fixed points of nonexpansive mapping S in Banach spaces, where $\{\alpha_n\}$ is a sequence in (0, 1), $\{\beta_n\}$ is a sequence in [0, 1], f is a contractive mapping, r is any given positive number, x_0 is a given point in E and $J_r = (I + rA)^{-1}$ is the resolvent of A. By using viscosity approximation methods and under suitable conditions, some strong convergence theorems to this common elements are proved. The results presented in the paper improve and extend the corresponding results of Dominguez et al [4], Kim and Xu [7], Xu [9] and [5, 8].

In order to prove our main results we need the following definitions and conclusions:

Definition 1.1 (Barbu [1]). The norm $||\cdot||$ of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*), if the limit

$$\lim_{t\to 0}\frac{||x+ty||-||x||}{t} \tag{1.5}$$

exists for each x, y in the unit sphere $U = \{x \in E : ||x|| = 1\}$.

It is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*, if the limit in (1.5) is attached uniformly for $x, y \in U$.

Definition 1.2. Following Browder [2], we say that a Banach space E has *a weakly continuous normalized duality mapping* $J : E \to E^*$, if J is single-valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $\{J(x_n)\}$ converges weak*ly to $J(x)$).

Lemma 1.1. *A Banach space E is uniformly smooth if and only if the normalized duality mapping J is single-valued and norm-to-norm uniformly continuous on any bounded subset of E.*

Lemma 1.2 [10]. *Let $\{a_n\}$ be a nonnegative real sequence such that:*

$$a_{n+1} \le (1-\lambda_n)a_n + \delta_n, \quad \forall n \ge n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in (0, 1) with $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{\delta_n\}$ is a sequence in R such that

$$\limsup_{\to\infty}\frac{\delta_n}{\lambda_n} \le 0 \ \text{or} \sum_{n=1}^{\infty}|\delta_n| < \infty,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.3 [3]. *Let E be a real Banach space, $J : E \to 2^{E^*}$ be the normalized duality mapping, then for any $x, y \in E$, the following conclusion holds:*

$$||x+y||^2 \le ||x||^2 + 2\langle y,\ j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Recall that if E is a real Banach space, C is a nonempty closed convex subset of E, $T : C \to C$ is a nonexpansive mapping with $F(T) \ne \emptyset$ and $f : C \to C$ is a contractive mapping. For any given $t \in (0,1)$, let z_t be the unique fixed point of the contraction $z \mapsto tf(z) + (1-t)Tz$ on C, i.e.,

$$z_t = tf(z_t) + (1-t)Tz_t. \tag{1.6}$$

Concerning the convergence of sequence $\{z_t\}$, Xu [11] proved the following result.

Lemma 1.4 (Xu [11]). *Let E be a uniformly smooth Banach space, C be a nonempty closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \ne \emptyset$ and $f \in \Pi_C$ (where Π_C is the collection of all contractions on*

C). Then the sequence $\{z_t\}$ defined by (1.6) converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \to F(T)$ by

$$Q(f) := \lim_{t\to 0} z_t, \quad f \in \Pi_C, \tag{1.7}$$

then $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f),\ J(p-Q(f))\rangle \geq 0, \quad f \in \Pi_C,\ p \in F(T). \tag{1.8}$$

In particular, if $f = u \in C$ is a constant, then the mapping Q defined by (1.7) is reduced to the sunny nonexpansive retraction of Reich from C onto $F(T)$:

$$\langle Q(u)-u,\ J(p-Q(u))\rangle \geq 0, \quad u \in C,\ p \in F(T). \tag{1.9}$$

Lemma 1.5 (Xu [10]). *Let E be a reflexive Banach space with a weakly continuous normalized duality mapping $J : E \to E^*$, C be a nonempty closed convex subset of E and $T : C \to C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0,1)$. Let $x_t \in C$ be the unique solution in C to the equation:*

$$x_t = tu + (1-t)Tx_t. \tag{1.10}$$

Then T has a fixed point if only if x_t remains bounded as $t \to 0+$, and in the case, $\{x_t\}$ converges as $t \to 0+$ strongly to $z \in F(T)$. If we define a mapping $Q : C \to F(T)$ by

$$Q(u) := \lim_{t\to 0+} x_t = z, \quad u \in C, \tag{1.11}$$

then Q is the sunny nonexpansive retraction from C onto $F(T)$, i.e., $Q(u)$ satisfies (1.9).

2. Main results

Theorem 2.1. *Let E be a real uniformly smooth Banach space, $r > 0$ be any given number, $A : E \to E$ be an $m-$accretive mapping, $S : E \to E$ be a nonexpansive mapping such that $F(S)\bigcap F(J_r) = F(S \circ J_r) \neq \emptyset$. Let $f : E \to E$ be a contractive mapping with a contractive constant $\alpha \in (0,1)$. Let $r > 0$ be any given positive number, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1]$ be two sequences satisfying the following conditions:*

(i) $\alpha_n \to 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \in [0,a)$, *for some* $a \in (0,1)$.

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then the sequence $\{x_n\}$ defined by (1.4) converges strongly to some common element $z \in F(S) \cap A^{-1}(0)$ which is a solution of the following variational inequality

$$\langle (f-I)z,\ J(z-y)\rangle \geq 0, \quad \forall y \in F(S)\bigcap A^{-1}(0).$$

Proof. We divide the proof of Theorem 2.1 into five steps:

(I) First prove that the sequences $\{x_n\}$ and $\{y_n\}$ defined by (1.4) are bounded and

$$||x_{n+1} - y_n|| = \alpha_n ||f(x_n) - y_n|| \to 0 \ \ (as\ n \to \infty). \tag{2.1}$$

In fact, for any given $p \in F(S) \bigcap A^{-1}(0)$, from (1.2) we know that

$$p = S(p) = SJ_k(p), \quad \forall k > 0. \tag{2.2}$$

From (1.4)

$$\begin{aligned} ||y_n - p|| &\le \beta_n ||x_n - p|| + (1-\beta_n)||SJ_r x_n - p|| \\ &\le \beta_n ||x_n - p|| + (1-\beta_n)||SJ_r x_n - SJ_r p|| \\ &\le ||x_n - p||. \end{aligned} \tag{2.3}$$

By using (1.4) again we have

$$\begin{aligned} ||x_{n+1} - p|| &\le \alpha_n ||f(x_n) - p|| + (1-\alpha_n)||y_n - p|| \\ &\le \alpha_n ||f(x_n) - f(p)|| + \alpha_n ||f(p) - p|| + (1-\alpha_n)||x_n - p|| \\ &\le \alpha_n \alpha ||x_n - p|| + \alpha_n ||f(p) - p|| + (1-\alpha_n)||x_n - p|| \\ &= (1 - \alpha_n(1-\alpha))||x_n - p|| + \frac{\alpha_n(1-\alpha)||f(p) - p||}{1-\alpha} \\ &\le \max\{||x_n - p||, \ \frac{||f(p) - p||}{1-\alpha}\} \\ &\le \cdots \\ &\le \max\{||x_0 - p||, \ \frac{||f(p) - p||}{1-\alpha}\}, \quad \forall n \ge 0. \end{aligned}$$

This implies that $\{x_n\}$ is a bounded sequence in E, and so $\{y_n\}$, $\{f(x_n)\}$ and $\{SJ_r x_n\}$ all are bounded sequences in E. By the assumption that $\{\alpha_n\} \to 0$, this implies that the conclusion (2.1) is true.

Now we denote

$$M = \sup_{n \ge 0}\{||x_n|| + ||SJ_r x_n|| + ||f(x_n)|| + ||y_n||\} < \infty \tag{2.4}$$

(II) Next prove that

$$||y_n - y_{n-1}|| \to 0 \ \text{ and } \ ||x_{n+1} - x_n|| \to 0 \ (\text{as } n \to \infty). \tag{2.5}$$

In fact, it follows from (1.4) and (2.2) that

$$\begin{aligned} y_n - y_{n-1} = &(1-\beta_n)(SJ_r x_n - SJ_r x_{n-1}) + \beta_n(x_n - x_{n-1}) \\ &+ (x_{n-1} - SJ_r x_{n-1})(\beta_n - \beta_{n-1}). \end{aligned}$$

This implies that

$$\begin{aligned} ||y_n - y_{n-1}|| &\le (1-\beta_n)||SJ_r x_n - SJ_r x_{n-1}|| + \beta_n ||x_n - x_{n-1}|| \\ &\quad + |\beta_n - \beta_{n-1}| ||x_{n-1} - SJ_r x_{n-1}|| \\ &\le (1-\beta_n)||x_n - x_{n-1}|| + \beta_n ||x_n - x_{n-1}|| \\ &\quad + |\beta_n - \beta_{n-1}| M \\ &\le ||x_n - x_{n-1}|| + |\beta_n - \beta_{n-1}| M. \end{aligned} \tag{2.6}$$

On the other hand, from (1.4) we have

$$\begin{aligned}||x_{n+1}-x_n|| &= ||\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) \\ &\quad - \alpha_{n-1} f(x_{n-1}) + (1-\alpha_n)y_n - (1-\alpha_n)y_{n-1} \\ &\quad + (1-\alpha_n)y_{n-1} - (1-\alpha_{n-1})y_{n-1}|| \\ &\leq \alpha_n\alpha||x_n - x_{n-1}|| + |\alpha_n - \alpha_{n-1}|||f(x_{n-1}|| \\ &\quad + (1-\alpha_n)||y_n - y_{n-1}|| + |\alpha_n - \alpha_{n-1}|||y_{n-1}|| \\ &\leq \alpha_n\alpha||x_n - x_{n-1}|| + 2M|\alpha_n - \alpha_{n-1}| + (1-\alpha_n)||y_n - y_{n-1}||\end{aligned} \tag{2.7}$$

Substituting (2.6) into (2.7) and simplifying we have

$$\begin{aligned}||x_{n+1}-x_n|| &\leq (1-\alpha_n(1-\alpha))||x_n - x_{n-1}|| \\ &\quad + \{2|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\}M\end{aligned} \tag{2.8}$$

By virtue of Lemma 1.2, we know that $||x_{n+1} - x_n|| \to 0$ *(as* $n \to \infty$*)*. Hence it follows from (2.6) that $||y_n - y_{n-1}|| \to 0$ *(as* $n \to \infty$*)*.

(*III*) *Next we prove that*

$$||SJ_r x_n - x_n|| \to 0 \ (as\ n \to \infty). \tag{2.9}$$

In fact, it follows from (1.4) that

$$\begin{aligned}||SJ_r x_n - x_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - y_n|| + ||y_n - SJ_r x_n|| \\ &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - y_n|| + \beta_n||x_n - SJ_r x_n||,\end{aligned}$$

i.e.,

$$(1-\beta_n)||SJ_r x_n - x_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - y_n||.$$

Since $\beta_n \in [0, a)$, $a \in (0,1)$, from (2.1) and (2.5) we have

$$||SJ_r x_n - x_n|| \to 0 \ \ (as\ n \to \infty).$$

Since E is a uniformly smooth real Banach space, $SJ_r : E \to E$ is a non-expansive mapping with $F(SJ_r) = F(S)\bigcap F(J_r) = F(S)\bigcap A^{-1}(0) \neq \emptyset$ and $f : E \to E$ is a contraction, by Lemma 1.4, the sequence $\{z_t\}$ defined by

$$z_t = z_t = tf(z_t) + (1-t)SJ_r z_t. \tag{2.10}$$

converges strongly to a point $z \in F(S \circ J_r) = F(S)\bigcap F(J_r)$. If we define $Q : \Pi_E \to F(S \circ F(S))$ by

$$Q(f) := \lim_{t\to 0} z_t, \ \ f \in \Pi_E$$

then $Q(f) = z$ solves the variational inequality:

$$\langle (I-f)z,\ J(y-z)\rangle \geq 0, \ \ \forall y \in F(S)\bigcap A^{-1}(0).$$

(*IV*) *Next we prove that*

$$\limsup_{n\to\infty}\langle z - f(z),\ J(z - x_n)\rangle \leq 0. \tag{2.11}$$

Indeed, it follows from (2.10) and Lemma 1.3 that for any $n \geq 0$ and $t > 0$,

$$\begin{aligned}
||z_t - x_n||^2 &= ||(1-t)(SJ_r z_t - x_n) + t(f(z_t) - x_n)||^2 \\
&\leq (1-t)^2||SJ_r z_t - x_n||^2 + 2t\langle f(z(t)) - x_n,\ J(z_t - x_n)\rangle \\
&\leq (1-t)^2\{||SJ_r z_t - SJ_r x_n|| + ||SJ_r x_n - x_n||\}^2 \\
&\quad + 2t\langle f(z(t)) - z_t + z_t - x_n,\ J(z_t - x_n)\rangle \\
&\leq (1-t)^2\{||z_t - x_n|| + ||SJ_r x_n - x_n||\}^2 \\
&\quad + 2t\langle f(z(t)) - z_t,\ J(z_t - x_n)\rangle + 2t||z_t - x_n||^2 \\
&\leq (1-t)^2\{||z_t - x_n||^2 + 2||z_t - x_n||||SJ_r x_n - x_n|| + ||SJ_r x_n - x_n||^2\} \\
&\quad + 2t\langle f(z(t)) - z_t,\ J(z_t - x_n)\rangle + 2t||z_t - x_n||^2 \\
&= (1-t)^2\{||z_t - x_n||^2 + \sigma_n(t)\} \\
&\quad + 2t\langle f(z(t)) - z_t,\ J(z_t - x_n)\rangle + 2t||z_t - x_n||^2,
\end{aligned}$$

Simplifying it we have

$$\begin{aligned}
\langle z_t - f(z(t)),\ J(z_t - x_n)\rangle &\leq \frac{t}{2}||z_t - x_n||^2 + \frac{1}{2t}\sigma_n(t) \\
&\leq \frac{t}{2}M_1 + \frac{1}{2t}\sigma_n(t)
\end{aligned} \tag{2.12}$$

where $M_1 = \sup_{t>0,\, n\geq 0} ||z_t - x_n||^2$ and

$$\begin{aligned}
\sigma_n(t) &= 2||z_t - x_n|| \cdot ||SJ_r x_n - x_n|| + ||SJ_r x_n - x_n||^2 \\
&\leq 2M_1||SJ_r x_n - x_n|| + ||SJ_r x_n - x_n||^2, \ \ \forall n \geq 0 \ \ \textit{and}\, t > 0.
\end{aligned} \tag{2.13}$$

Therefore by (2.9) we know that

$$\lim_{n\to\infty} \sigma_n(t) = 0 \ \ \textit{uniformly in}\ t \in (0,1).$$

Letting $n \to \infty$ and taking the lim sup in (2.12), we have

$$\limsup_{n\to\infty}\langle z_t - f(z_t),\ J(z_t - x_n)\rangle \leq \frac{t}{2}M, \ \ \forall t \in (0,1). \tag{2.14}$$

Taking the lim sup as $t \to 0$ in (2.14) and noting the fact that the two limits are interchangeable due to the fact the normalized duality mapping J is norm-to-norm uniformly continuous on bounded subsets of E, the conclusion (2.11) is obtained.

(V) *Finally we prove that* $\{x_n\}$ *converges strongly to* z.

Indeed, it follows from Lemma 1.3 and (2.3) that

$$\begin{aligned}||x_{n+1} - z||^2 &= ||(1-\alpha_n)(y_n - z) + \alpha_n(f(x_n) - z)||^2 \\ &\le (1-\alpha_n)^2||y_n - z||^2 + 2\alpha_n\langle f(x_n) - z,\ J(x_{n+1} - z)\rangle \\ &\le (1-\alpha_n)^2||x_n - z||^2 \\ &\quad + 2\alpha_n\langle f(x_n) - f(z) + f(z) - z,\ J(x_{n+1} - z)\rangle \\ &\le (1-\alpha_n)^2||x_n - z||^2 + 2\alpha_n\{\alpha||x_n - z||\cdot||x_{n+1} - z|| \\ &\quad + 2\alpha_n\langle f(z) - z,\ J(x_{n+1} - z)\rangle \\ &\le (1-\alpha_n)^2||x_n - z||^2 + \alpha_n\alpha\{||x_n - z||^2 + ||x_{n+1} - z||^2\} \\ &\quad + 2\alpha_n\langle f(z) - z,\ J(x_{n+1} - z)\rangle\end{aligned}$$

Simplifying it we have

$$\begin{aligned}||x_{n+1} - z||^2 \le \frac{1}{1-\alpha\alpha_n}&\{(1-\alpha_n(2-\alpha))||x_n - z||^2 + \alpha_n^2 M \\ &+ 2\alpha_n\langle z - f(z),\ J(z - x_{n+1})\rangle\}.\end{aligned} \tag{2.15}$$

Since $\alpha_n \to 0$, there exists a positive integer n_0 such that

$$1 - \alpha\alpha_n > \frac{1}{2}\ \ \forall n \ge n_0. \tag{2.16}$$

Again since

$$\begin{aligned}\frac{1}{1-\alpha\alpha_n}(1-\alpha_n(2-\alpha)) &= 1 - \frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n} \\ &\le (1 - 2\alpha_n(1-\alpha)),\ \ \forall n \ge n_0.\end{aligned} \tag{2.17}$$

Using (2.16) and (2.17), (2.15) can be written as follows:

$$\begin{aligned}||x_{n+1} - z||^2 &\le (1 - 2\alpha_n(1-\alpha))||x_n - z||^2 + 2\alpha_n^2 M \\ &\quad + 4\alpha_n\langle z - f(z),\ J(z - x_{n+1})\rangle,\ \ \forall n \ge n_0.\end{aligned} \tag{2.18}$$

Taking $a_n = ||x_n - z||^2$, $\lambda_n = 4\alpha_n(1-\alpha)$, $\delta_n = 2\alpha_n^2 M + 4\alpha_n\langle z - f(z),\ J(z - x_{n+1})\rangle$, by Lemma 1.2 we know that the sequence $x_n \to z$ as $n \to \infty$. This completes the proof. □

If E is a reflexive Banach space, then we have the following result.

Theorem 2.2. *Let E be a real reflexive Banach space with a weakly continuous normalized duality mapping $J : E \to E^*$. Let $A : E \to E$ be an $m-$accretive mapping such that $A^{-1}(0) \ne \emptyset$ and $\overline{D(A)}$ is convex. Let $r > 0$ be a given positive number, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1]$ be two sequences satisfying the following conditions:*

(i) $\alpha_n \to 0;\ \ \sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\beta_n \in [0, a)$, *for some* $a \in (0,1)$.

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty\ \ \sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then for any given point u and $x_0 \in E$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \\ \quad y_n = \beta_n x_n + (1-\beta_n) J_r(x_n) \end{cases} n \geq 0; \tag{2.19}$$

converges strongly to some common element $z \in A^{-1}(0)$ which is a solution of the following variational inequality

$$\langle (u - z,\ J(z - y)\rangle \geq 0,\ \ \forall y \in A^{-1}(0).$$

Proof. We only include the differences. By the same methods as given in the proof of Theorem 2.1, we can prove that $\{x_n\}$ and $\{y_n\}$ both are bounded and

$$||x_{n+1} - x_n|| \to 0 \ \ and \ |||x_n - J_r x_n|| \to 0 \ \ (as\ n \to \infty).$$

Next we prove that

$$\limsup_{n\to\infty} \langle u - Q(u),\ J(x_n - Q(u))\rangle \leq 0, \tag{2.20}$$

where $Q: E \to F(T)$ is a sunny nonexpansive retraction defined by

$$Q(u) := \lim_{t\to 0+} z_t = z,\ \ u \in C,$$

and z_t is the unique solution of the equation:

$$z_t = tu + (1-t) J_r z_t,\ \ t \in (0,1)$$

Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - Q(u),\ J(x_n - Q(u))\rangle = \lim_{n_k\to\infty} \langle u - Q(u),\ J(x_{n_k} - Q(u))\rangle. \tag{2.21}$$

Since E is reflexive we may assume that $x_{n_k} \rightharpoonup x^*$. Moreover, since $||x_n - J_r x_n|| \to 0$. this implies that $J_r x_{n_k} \rightharpoonup x^*$. By the definition of the resolvent J_r of $m-$accretive mapping A,

$$AJ_r = \frac{1}{r}(I - J_r).$$

This implies that

$$[J_r x_{n_k}, A(J_r x_{n_k})] = [J_r x_{n_k}, \frac{1}{r}(I - J_r)(x_{n_k})] \in Graph(A). \tag{2.22}$$

Taking the limit as $k \to \infty$ in (2.22), we know that $[x^*, 0] \in Graph(A)$, i.e., $x^* \in A^{-1}(0)$. By (2.21), Lemma 1.5 we have

$$\limsup_{n\to\infty} \langle u - Q(u),\ J(x_n - Q(u))\rangle = \langle u - Q(u),\ J(x^* - Q(u))\rangle \leq 0.$$

The conclusion (2.22) is proved.

By the same way as given in the proof of Theorem 2.1 we can prove that $x_n \to z$.

This completes the proof of Theorem 2.2. □

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