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AN EXTREMAL PROBLEM ON POTENTIALLY $K_{r,r} - ke$-GRAPHIC SEQUENCES

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ABSTRACT. For $1 \leq k \leq r$, let $\sigma(K_{r,r} - ke, n)$ be the smallest even integer such that every n-term graphic sequence $\pi = (d_1, d_2, \ldots, d_n)$ with term sum $\sigma(\pi) = d_1 + d_2 + \cdots + d_n \geq \sigma(K_{r,r} - ke, n)$ has a realization G containing $K_{r,r} - ke$ as a subgraph, where $K_{r,r} - ke$ is the graph obtained from the $r \times r$ complete bipartite graph $K_{r,r}$ by deleting k edges which form a matching. In this paper, we determine $\sigma(K_{r,r} - ke, n)$ for even r (≥ 4) and $n \geq 7r^2 + \frac{1}{2}r - 22$ and for odd r (≥ 5) and $n \geq 7r^2 + 9r - 26$.

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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers with $d_i \leq n-1$ for each i is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The set of all graphic non-increasing sequences in NS_n is denoted by GS_n. For a sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, denote $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$. For given a graph H, a sequence $\pi \in GS_n$ is *potentially H-graphic* if there exists a realization of π containing H as a subgraph. Gould et al. [3] considered the following variation of the classical Turán-type extremal problems: for given a graph H, determine the smallest even integer $\sigma(H, n)$ such that every sequence $\pi \in GS_n$ with $\sigma(\pi) \geq \sigma(H, n)$ is potentially H-graphic. If $H = K_{r+1}$, the complete graph on $r+1$ vertices, this problem was considered by Erdős et al. [2] where they showed that $\sigma(K_3, n) = 2n$ for $n \geq 6$ and conjectured that $\sigma(K_{r+1}, n) = (r-1)(2n-r) + 2$ for sufficiently large n. Gould et al. [3] and Li et al. [5] independently proved it for $r = 3$. Recently, Li et al. [6,7] proved that the conjecture is true for $r = 4$ and $n \geq 10$ and for $r \geq 5$ and

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$n \geq \binom{r}{2} + 3$. For $H = K_{r,s}$, the $r \times s$ complete bipartite graph, Gould et al. [3] determined $\sigma(K_{2,2}, n)$ for $n \geq 4$. Yin et al. [8] determined $\sigma(K_{3,3}, n)$ for $n \geq 6$ and $\sigma(K_{4,4}, n)$ for $n \geq 8$. Yin et al. [9] also determined $\sigma(K_{r,r}, n)$ for even r (≥ 4) and $n \geq 4r^2 - r - 6$ and for odd r (≥ 3) and $n \geq 4r^2 + 3r - 8$. Recently, Yin et al. [10,11] further determined $\sigma(K_{r,s}, n)$ for $s \geq r \geq 1$ and sufficiently large n. The purpose of this paper is to determine $\sigma(K_{r,r} - ke, n)$ for even r (≥ 4) and $n \geq 7r^2 + \frac{1}{2}r - 22$ (Theorem 6) and for odd r (≥ 5) and $n \geq 7r^2 + 9r - 26$ (Theorem 7), where $1 \leq k \leq r$ and $K_{r,r} - ke$ is the graph obtained from $K_{r,r}$ by deleting k edges which form a matching.

2. Preliminaries

In order to prove our main results, we need the following known theorems.

Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence. Denote $f(\pi) = max\{i \mid d_i \geq i\}$ and define an n-by-n matrix $\overline{A} = (a_{ij})$ as follows: if $d_i \geq i$, then

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq j \leq d_i + 1 \text{ and } j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

and if $d_i < i$, then

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leq j \leq d_i, \\ 0 & \text{otherwise.} \end{cases}$$

$f(\pi)$ and $\overline{A}$ are called the *trace* and the *left-most off-diagonal matrix* of π, respectively. The column sum vector of $\overline{A}$, denoted by $\overline{\pi} = (\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$, is called the *corrected conjugate vector* of π. Clearly, the row sum vector of $\overline{A}$ is π and $\sigma(\overline{\pi}) = \sigma(\pi)$.

Theorem 1. [1] *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence with even $\sigma(\pi)$. Then π is graphic if and only if $d_1 + d_2 + \cdots + d_i \leq \overline{d_1} + \overline{d_2} + \cdots + \overline{d_i}$ for each $i = 1, 2, \ldots, f(\pi)$.*

For a non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, let $d'_1 \geq d'_2 \geq \cdots \geq d'_{n-1}$ be the rearrangement of $d_1 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1}$. Then $\pi' = (d'_1, d'_2, \ldots, d'_{n-1})$ is called the *residual sequence* of π. It is easy to see that if π' is graphic then π is also graphic, since a realization G of π can be obtained from a realization G' of π' by adding a new vertex of degree d_n and joining it to the vertices whose degrees are reduced by one in going from π to π'. In fact, more is true:

Theorem 2. [4] *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ be a non-increasing sequence. Then π is graphic if and only if π' is graphic.*

Theorem 3. [7] *If $r \geq 5$, then $\sigma(K_{r+1}, n) \leq 2n(r-2)+8$ for $2r+2 \leq n \leq \binom{r}{2}+3$ and $\sigma(K_{r+1}, n) = (r-1)(2n-r) + 2$ for $n \geq \binom{r}{2} + 3$.*

Theorem 4. [8] *Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in GS_n$, where $d_{r+s} \geq r + s - 1$ and $d_n \geq r$. Then π is potentially $K_{r,s}$-graphic.*

Theorem 5. [8] *Let* $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{r+s}, d_{r+s+1}, \ldots, d_n) \in GS_n$, *where* $d_r \geq r+s-1$, $d_{r+s} \leq r+s-2$ *and* $d_n \geq r$. *If* $n \geq (r+s)(s-1)$, *then* π *is potentially* $K_{r,s}$*-graphic.*

In order to prove our main results, we also borrow an idea from [8,11]. Let

$$\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{2r}, d_{2r+1}, \ldots, d_n) \in NS_n,$$

where $d_1 \geq \cdots \geq d_{r-1} \geq r$, $d_r \geq r-1$, $d_{r+1} \geq \cdots \geq d_{2r} \geq r$ and $d_{2r+1} \geq \cdots \geq d_n \geq r$. Let

$$\pi_1' = \begin{cases} (d_2, \ldots, d_r, d_{r+1}-1, \ldots, d_{r+d_1}-1, d_{r+d_1+1}, \ldots, d_n) \\ \text{if } d_1 \leq n-r, \\ (d_2-1, \ldots, d_{d_1+r-n+1}-1, d_{d_1+r-n+2}, \ldots, d_r, d_{r+1}-1, \ldots, d_n-1) \\ \text{if } d_1 > n-r, \end{cases}$$

and $\pi_1'' = (d_2^{(1)}, \ldots, d_r^{(1)}, d_{r+1}^{(1)}, \ldots, d_{2r}^{(1)}, d_{2r+1}^{(1)}, \ldots, d_n^{(1)})$, where $d_2^{(1)} \geq \cdots \geq d_r^{(1)}$ is the rearrangement of the first $r-1$ terms in π_1', $d_{r+i}^{(1)} = d_{r+i}-1$ for $1 \leq i \leq r$ and $d_{2r+1}^{(1)} \geq \cdots \geq d_n^{(1)}$ is the rearrangement of the final $n-2r$ terms in π_1'.

For $\pi_1'' = (d_2^{(1)}, \ldots, d_r^{(1)}, d_{r+1}^{(1)}, \ldots, d_{2r}^{(1)}, d_{2r+1}^{(1)}, \ldots, d_n^{(1)})$, if $d_2^{(1)} \geq \cdots \geq d_{r-1}^{(1)} \geq r$ and $d_r^{(1)} \geq r-1$, we can similarly define π_2'' as follows: let

$$\pi_2' = \begin{cases} (d_3^{(1)}, \ldots, d_r^{(1)}, d_{r+1}^{(1)}-1, \ldots, d_{r+d_2^{(1)}}^{(1)}-1, d_{r+d_2^{(1)}+1}^{(1)}, \ldots, d_n^{(1)}) \\ \text{if } d_2^{(1)} \leq n-r, \\ (d_3^{(1)}-1, \ldots, d_{d_2^{(1)}+r-n+2}^{(1)}-1, d_{d_2^{(1)}+r-n+3}^{(1)}, \ldots, d_r^{(1)}, d_{r+1}^{(1)}-1, \ldots, \\ d_n^{(1)}-1) \ \text{if } d_2^{(1)} > n-r, \end{cases}$$

and $\pi_2'' = (d_3^{(2)}, \ldots, d_r^{(2)}, d_{r+1}^{(2)}, \ldots, d_{2r}^{(2)}, d_{2r+1}^{(2)}, \ldots, d_n^{(2)})$, where $d_3^{(2)} \geq \cdots \geq d_r^{(2)}$ is the rearrangement of the first $r-2$ terms in π_2', $d_{r+i}^{(2)} = d_{r+i}^{(1)}-1$ for $1 \leq i \leq r$ and $d_{2r+1}^{(2)} \geq \cdots \geq d_n^{(2)}$ is the rearrangement of the final $n-2r$ terms in π_2'. For $k = 3, 4, \ldots, r-1$ in turn, if $d_k^{(k-1)} \geq \cdots \geq d_{r-1}^{(k-1)} \geq r$ and $d_r^{(k-1)} \geq r-1$, the definitions of π_k' and π_k'' are similar.

For $\pi_{r-1}'' = (d_r^{(r-1)}, d_{r+1}^{(r-1)}, \ldots, d_{2r}^{(r-1)}, d_{2r+1}^{(r-1)}, \ldots, d_n^{(r-1)})$, if $d_r^{(r-1)} \geq r-1$, we define π_r'' as follow: let

$$\pi_r' = (d_{r+1}^{(r-1)}-1, \ldots, d_{d_r^{(r-1)}+r}^{(r-1)}-1, d_{d_r^{(r-1)}+r+1}^{(r-1)}, \ldots, d_n^{(r-1)}),$$

and $\pi_r'' = (d_{r+1}^{(r)}, \ldots, d_{2r}^{(r)}, d_{2r+1}^{(r)}, \ldots, d_n^{(r)})$, where $d_{r+1}^{(r)} \geq \cdots \geq d_{2r}^{(r)}$ is the rearrangement of the first r terms in π_r' and $d_{2r+1}^{(r)} \geq \cdots \geq d_n^{(r)}$ is the rearrangement of the final $n-2r$ terms in π_r'. By the definition of π_r'', the following Proposition 1 is obvious.

Proposition 1. *Let* $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{2r}, d_{2r+1}, \ldots, d_n) \in NS_n$, *where* $d_1 \geq \cdots \geq d_{r-1} \geq r$, $d_r \geq r-1$, $d_{r+1} \geq \cdots \geq d_{2r} \geq r$ *and* $d_{2r+1} \geq \cdots \geq d_n \geq r$. *Let* π_r'' *be defined as above. If* π_r'' *is graphic, then* π *is potentially* $K_{r,r}-e$*-graphic.*

For the defined sequence $\pi''_r = (d^{(r)}_{r+1}, \ldots, d^{(r)}_{2r}, d^{(r)}_{2r+1}, \ldots, d^{(r)}_n)$ in Proposition 1, if $d^{(r)}_{r+1} \geq \cdots \geq d^{(r)}_{2r} \geq 1$ and $d^{(r)}_{2r+1} \geq \cdots \geq d^{(r)}_n \geq 1$, we define

$$\pi'_{r+1} = (d^{(r)}_{r+2} - 1, \ldots, d^{(r)}_{r+d^{(r)}_{r+1}+1} - 1, d^{(r)}_{r+d^{(r)}_{r+1}+2}, \ldots, d^{(r)}_n),$$

and $\pi''_{r+1} = (d^{(r+1)}_{r+2}, \ldots, d^{(r+1)}_{2r}, d^{(r+1)}_{2r+1}, \ldots, d^{(r+1)}_n)$, where $d^{(r+1)}_{r+2} \geq \cdots \geq d^{(r+1)}_{2r}$ is the rearrangement of the first $r-1$ terms in π'_{r+1} and $d^{(r+1)}_{2r+1} \geq \cdots \geq d^{(r+1)}_n$ is the rearrangement of the final $n-2r$ terms in π'_{r+1}. For $k = 2, 3, \ldots, r$ in turn, if $d^{(r+k-1)}_{r+k} \geq \cdots \geq d^{(r+k-1)}_{2r} \geq 1$ and $d^{(r+k-1)}_{2r+1} \geq \cdots \geq d^{(r+k-1)}_n \geq 1$, the definitions of π'_{r+k} and π''_{r+k} are similar.

Proposition 2. [11] *Let $\pi''_r = (d^{(r)}_{r+1}, \ldots, d^{(r)}_{2r}, d^{(r)}_{2r+1}, \ldots, d^{(r)}_n)$ be a defined sequence as in Proposition 1, $1 \leq k \leq r$ and let π''_{r+k} be defined as above. If π''_{r+k} is graphic, then π''_r is also graphic.*

Lemma 1. [8] *Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, $m = \max\{d_1, d_2, \ldots, d_n\}$ and $\sigma(\pi)$ be even. Let $\pi^* = (d^*_1, d^*_2, \ldots, d^*_n)$ be the rearrangement sequence of π, where $m = d^*_1 \geq d^*_2 \geq \cdots \geq d^*_n$ is the rearrangement of $d_1, d_2, \ldots, d_n$. If there exists an integer n_1 $(\leq n)$ such that $d^*_{n_1} \geq h \geq 1$ and $n_1 \geq \frac{1}{h}\left\lceil\frac{(m+h+1)^2}{4}\right\rceil$, then π is graphic.*

We now prove the following:

Lemma 2. *Let $r \geq 4$ and $n \geq \frac{r^2}{2} + \frac{5}{2}r$. Let $\pi = (d_1, \ldots, d_r, d_{r+1}, \ldots, d_{2r}, d_{2r+1}, \ldots, d_n) \in GS_n$ with $d_r \leq 2r-2$ and $d_n \geq r$. If there exists an integer $t \in \{1, 2, \ldots, \lceil\frac{r}{2}\rceil - 1\}$ such that $d_{r+t} \geq 2r-2-t$ and $d_{2r} \geq r+t-1$, then π is potentially $K_{r,r} - e$-graphic.*

Proof. Rearrange the terms in π to get that

$$(p_1, \ldots, p_{r-t}, p_{r-t+1}, \ldots, p_r, p_{r+1}, \ldots, p_{r+t}, p_{r+t+1}, \ldots, p_n),$$

where $p_1 = d_1, \ldots, p_{r-t} = d_{r-t}$; $p_{r-t+1} = d_{r+1}, \ldots, p_r = d_{r+t}$; $p_{r+1} = d_{r-t+1}, \ldots, p_{r+t} = d_r$; $p_{r+t+1} = d_{r+t+1}, \ldots, p_n = d_n$. For convenience, the new sequence is still denoted by π. Clearly, $p_1 \geq \cdots \geq p_r \geq 2r-2-t, p_{r+1} \geq \cdots \geq p_{2r} \geq r+t-1$ and $2r-2 \geq p_{2r+1} \geq \cdots \geq p_n \geq r$. By Proposition 1, it is enough to prove that π''_r is graphic.

Since $\pi''_{r-t-2} = (p^{(r-t-2)}_{r-t-1}, \ldots, p^{(r-t-2)}_r, \ldots, p^{(r-t-2)}_{2r}, p^{(r-t-2)}_{2r+1}, \ldots, p^{(r-t-2)}_n)$ satisfies

(1) $2r-2 \geq p^{(r-t-2)}_{r-t-1} \geq \cdots \geq p^{(r-t-2)}_r \geq d_{r+t} - (r-t-2) \geq r$,

(2) $2r-2 \geq p^{(r-t-2)}_{2r+1} \geq \cdots \geq p^{(r-t-2)}_n \geq t+2$,

(3) $(p^{(r-t-2)}_{r-t-1} - r) + \cdots + (p^{(r-t-2)}_{r-1} - r) + (p^{(r-t-2)}_r - (r-1)) \leq (r-2)(t+1) + (r-1) \leq (r-2)\lceil\frac{r}{2}\rceil + r - 1 < \frac{r^2}{2} + \frac{r}{2} \leq n - 2r$,

we get that $\pi''_r = (p^{(r)}_{r+1}, \ldots, p^{(r)}_{r+t-1}, p^{(r)}_{r+t}, \ldots, p^{(r)}_{2r}, p^{(r)}_{2r+1}, \ldots, p^{(r)}_n)$ satisfies

(4) $n - r - 1 \geq p^{(r)}_{r+1} \geq \cdots \geq p^{(r)}_{r+t-1}$,

(5) $2r - 2 \geq p^{(r)}_{r+t} \geq \cdots \geq p^{(r)}_{2r} \geq t - 1$,

(6) $2r - 2 \geq p^{(r)}_{2r+1} \geq \cdots \geq p^{(r)}_{n} \geq t + 1$.

Hence $\pi''_{r+t-1} = (p^{(r+t-1)}_{r+t}, \ldots, p^{(r+t-1)}_{2r}, p^{(r+t-1)}_{2r+1}, \ldots, p^{(r+t-1)}_{n})$ satisfies

(7) $2r - 2 \geq p^{(r+t-1)}_{r+t} \geq \cdots \geq p^{(r+t-1)}_{2r} \geq 0$,

(8) $2r - 2 \geq p^{(r+t-1)}_{2r+1} \geq \cdots \geq p^{(r+t-1)}_{n} \geq 2$.

Thus, $\frac{1}{2}[\frac{(2r-2+2+1)^2}{4}] \leq \frac{r^2}{2} + \frac{r}{2} \leq n - 2r$. By Lemma 1, π''_{r+t-1} is graphic, and hence π''_r is also graphic by Proposition 2. □

3. Main results

For convenience, we first introduce the following notations. Let $r = 8k + t$, where $k \geq 0$ and $0 \leq t \leq 7$. If $t \in \{2, 3, 6, 7\}$, let $F_t = \{(8k + t, n) \mid k \geq 0 \text{ and } n \geq 16k + 2t\}$. If $t \in \{0, 1, 4, 5\}$, let $F'_t = \{(8k+t, n) \mid k \geq 0, n \geq 16k + 2t$ and n is odd$\}$ and $F''_t = \{(8k+t, n) \mid k \geq 0, n \geq 16k+2t$ and n is even$\}$. Denote $E_1 = F_2 \cup F'_0 \cup F''_4$, $E_2 = F_6 \cup F'_4 \cup F''_0$, $E_3 = F_7 \cup F'_1 \cup F''_5$ and $E_4 = F_3 \cup F'_5 \cup F''_1$.

3.1 $\sigma(K_{r,r} - ke, n)$ for even r and $n \geq 7r^2 + \frac{1}{2}r - 22$.

Lemma 3. *Let r be even, $r \geq 4$ and $n \geq 2r$. Then*

$$\sigma(K_{r,r} - re, n) \geq \begin{cases} (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1), & \text{if } (r, n) \in E_1, \\ (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 1 - (n - r + 1), & \text{if } (r, n) \in E_2. \end{cases}$$

Proof. Suppose $(r, n) \in E_1$. Consider $\pi = ((n-1)^{r-1}, 2r - 3, 2r - 4, \ldots, \frac{3}{2}r - 1, (\frac{3}{2}r - 2)^{n - \frac{3}{2}r + 2})$, where x^y stands for y consecutive terms, each equal to x. Then $\sigma(\pi) = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r - (n - r + 1)$ is even and $f(\pi) = \frac{3}{2}r - 2$. It follows from the left-most off-diagonal matrix $\overline{A}$ of π that $\overline{\pi} = (\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$ satisfies $\overline{d_1} = \overline{d_2} = \cdots = \overline{d_{\frac{3}{2}r-2}} = n - 1$. Clearly, $d_1 + d_2 + \cdots + d_i \leq \overline{d_1} + \overline{d_2} + \cdots + \overline{d_i}$ for each $i = 1, 2, \ldots, f(\pi)$. By Theorem 1, $\pi \in GS_n$. Let $\pi_1 = (r - 2, r - 3, \ldots, \frac{r}{2}, (\frac{r}{2} - 1)^{n - \frac{3}{2}r + 2})$. If π is potentially $K_{r,r} - re$-graphic, then there exist integers t and s, $t \geq s \geq 1$ and $t + s = r + 1$ such that π_1 is potentially $K_{s,t} - se$-graphic , where $K_{s,t} - se$ is the graph obtained from $K_{s,t}$ by deleting s edges which form a matching. Hence, there are at least s terms in π_1 which are greater than or equal to $r - s$, a contradiction. So π is not potentially $K_{r,r} - re$-graphic. Thus $\sigma(K_{r,r} - re, n) \geq \sigma(\pi) + 2 = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1)$.

Now assume $(r, n) \in E_2$. Consider $\pi = ((n-1)^{r-1}, 2r - 3, 2r - 4, \ldots, \frac{3}{2}r - 1, (\frac{3}{2}r - 2)^{n - \frac{3}{2}r + 1}, \frac{3}{2}r - 3)$. Then $\sigma(\pi) = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r - 1 - (n - r + 1)$ is even and $f(\pi) = \frac{3}{2}r - 2$. By the left-most off-diagonal matrix $\overline{A}$ of π that $\overline{\pi} = (\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$ satisfies $\overline{d_1} = \overline{d_2} = \cdots = \overline{d_{\frac{3}{2}r-3}} = n - 1, \overline{d_{\frac{3}{2}r-2}} = n - 2$. Clearly, $d_1 + d_2 + \cdots + d_i \leq \overline{d_1} + \overline{d_2} + \cdots + \overline{d_i}$ for each $i = 1, 2, \ldots, f(\pi)$. By Theorem 1, $\pi \in GS_n$. Similarly, we also can prove that π is not potentially $K_{r,r} - re$-graphic. Thus $\sigma(K_{r,r} - re, n) \geq \sigma(\pi) + 2 = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 1 - (n - r + 1)$. □

Lemma 4. *Let r be even, $r \geq 4$ and $n = r^2 + r - 2$. Then*

$$\sigma(K_{r,r} - e, n) \leq (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1) + (\frac{3}{2}r^3 - \frac{1}{8}r^2 - 5r).$$

Proof. By Theorem 3,

$$\begin{aligned}
&\sigma(K_{r,r} - e, n) \leq \sigma(K_{2r}, n) \leq (4r - 6)n + 8 \\
&= (\tfrac{5}{2}r - 2)n + (\tfrac{3}{2}r - 4)(r^2 + r - 2) + 8 \\
&= (\tfrac{5}{2}r - 2)n - \tfrac{11}{8}r^2 + \tfrac{5}{4}r + 2 - (n - r + 1) + (\tfrac{3}{2}r^3 - \tfrac{1}{8}r^2 - \tfrac{33}{4}r + 13) \\
&\leq (\tfrac{5}{2}r - 2)n - \tfrac{11}{8}r^2 + \tfrac{5}{4}r + 2 - (n - r + 1) + (\tfrac{3}{2}r^3 - \tfrac{1}{8}r^2 - 5r).
\end{aligned}$$

□

Lemma 5. *Let r be even, $r \geq 4$ and $n \geq r^2 + r - 2$. Let $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $d_n \geq r$. If $\sigma(\pi) \geq (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1)$, then π is potentially $K_{r,r} - e$-graphic.*

Proof. If $d_r \geq 2r - 1$, then by $n \geq (r + 2)(r - 1)$ and Theorems 4 and 5, π is potentially $K_{r,r}$-graphic, and hence π is potentially $K_{r,r} - e$-graphic.

Now assume $d_r \leq 2r - 2$. If $d_{r+t} \leq 2r - 3 - t$ for any $t \in \{1, 2, \ldots \frac{r}{2} - 1\}$, then $\sigma(\pi) \leq (n-1)(r-1) + (2r-2) + (2r-4) + \cdots + (\frac{3}{2}r - 1) + (\frac{3}{2}r - 2)(n - \frac{3}{2}r + 3) = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 1 - (n - r + 1) < (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1) \leq \sigma(\pi)$, a contradiction. Hence there exists an integer $t \in \{1, 2, \ldots, \frac{r}{2} - 1\}$ such that $d_{r+t} \geq 2r - 2 - t$. If $d_{2r} \leq \frac{3}{2}r - 3$, then $\sigma(\pi) \leq (n-1)(r-1) + (2r-2)r + (\frac{3}{2}r - 3)(n - 2r + 1) = (\frac{5}{2}r - 4)n - r^2 + \frac{9}{2}r - 2 < (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1) \leq \sigma(\pi)$, a contradiction. Hence $d_{2r} \geq \frac{3}{2}r - 2$. By Lemma 2, π is potentially $K_{r,r} - e$-graphic. □

Lemma 6. *Let r be even, $r \geq 4$ and $n = r^2 + r - 2 + t$, where $0 \leq t \leq 6r^2 - \frac{r}{2} - 20$. Then*

$$\sigma(K_{r,r} - e, n) \leq (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1) + \frac{r}{4}(6r^2 - \frac{r}{2} - 20) - \frac{1}{4}rt.$$

Proof. Use induction on t. It is known from Lemma 4 that the result holds for $t = 0$. Now assume that the result holds for $t - 1$, $0 \leq t - 1 \leq 6r^2 - \frac{r}{2} - 21$. Let $n = r^2 + r - 2 + t$, and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1) + \frac{r}{4}(6r^2 - \frac{r}{2} - 20) - \frac{1}{4}rt$. We only need to prove that π is potentially $K_{r,r} - e$-graphic. Obviously, $\sigma(\pi) \geq (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1)$. If $d_n \geq r$, then by Lemma 5, π is potentially $K_{r,r} - e$-graphic. If $d_n \leq r - 1$, then the residual sequence π' of π satisfies $\sigma(\pi') = \sigma(\pi) - 2d_n \geq (\frac{5}{2}r - 2)(n - 1) - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - ((n - 1) - r + 1) + \frac{r}{4}(6r^2 - \frac{r}{2} - 20) - \frac{1}{4}r(t - 1)$. By the induction hypothesis, π' is potentially $K_{r,r} - e$-graphic, and hence so is π. □

Lemma 7. *Let r be even, $r \geq 4$ and $n \geq 7r^2 + \frac{1}{2}r - 22$. Then*

$$\sigma(K_{r,r} - e, n) \leq (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1).$$

Proof. It is enough to prove that (*): if $\pi = (d_1, \ldots, d_n) \in GS_n$ and $\sigma(\pi) \geq (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1)$, then π is potentially $K_{r,r} - e$-graphic. Apply induction on n. By Lemma 6, (*) holds for $n = 7r^2 + \frac{1}{2}r - 22$. Now suppose that (*) holds for $n - 1 \geq 7r^2 + \frac{1}{2}r - 22$. We will prove that (*) holds for n. If $d_n \geq r$, then by Lemma 5, π is potentially $K_{r,r} - e$-graphic. If $d_n \leq r - 1$, then the residual sequence π' satisfies $\sigma(\pi') = \sigma(\pi) - 2d_n \geq (\frac{5}{2}r - 2)(n - 1) - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - ((n - 1) - r + 1)$. By the induction hypothesis, π' and π are potentially $K_{r,r} - e$-graphic. □

Theorem 6. *Let $r \geq 4$ be even, $1 \leq k \leq r$ and $n \geq 7r^2 + \frac{1}{2}r - 22$. Then*

$$\sigma(K_{r,r} - ke, n) = \begin{cases} (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1), & \text{if } (r, n) \in E_1, \\ (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 1 - (n - r + 1), & \text{if } (r, n) \in E_2. \end{cases}$$

Proof. Since $K_{r,r} - e$ contains $K_{r,r} - ke$ as a subgraph and $K_{r,r} - ke$ contains $K_{r,r} - re$ as a subgraph, it is well known that $\sigma(K_{r,r} - re, n) \leq \sigma(K_{r,r} - ke, n) \leq \sigma(K_{r,r} - e, n)$. By Lemmas 3 and 7, for $(r, n) \in E_1$,

$$\sigma(K_{r,r} - ke, n) = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1),$$

and for $(r, n) \in E_2$, $(\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 1 - (n - r + 1) \leq \sigma(K_{r,r} - ke, n) \leq (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 2 - (n - r + 1)$. Since $\sigma(K_{r,r} - ke, n)$ is even, we have $\sigma(K_{r,r} - ke, n) = (\frac{5}{2}r - 2)n - \frac{11}{8}r^2 + \frac{5}{4}r + 1 - (n - r + 1)$ for $(r, n) \in E_2$. □

3.2 $\sigma(K_{r,r} - ke, n)$ for odd r and $n \geq 7r^2 + 9r - 26$.

Lemma 8. *Let r be odd, $r \geq 5$ and $n \geq 2r$. Then*

$$\sigma(K_{r,r} - re, n) \geq \begin{cases} (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1), & \text{if } (r, n) \in E_3, \\ (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{7}{8} - (n - r + 1), & \text{if } (r, n) \in E_4. \end{cases}$$

Proof. Suppose that $(r, n) \in E_3$. Consider $\pi = ((n-1)^{r-1}, 2r - 3, 2r - 4, \ldots, \frac{3}{2}r - \frac{1}{2}, (\frac{3}{2}r - \frac{3}{2})^{\frac{r}{2}+\frac{3}{2}}, (\frac{3}{2}r - \frac{5}{2})^{n-2r+1})$. Then $\sigma(\pi) = (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r - \frac{1}{8} - (n - r + 1)$ is even and $f(\pi) = \frac{3}{2}r - \frac{3}{2}$. It follows from the left-most off-diagonal matrix $\overline{A}$ of π that $\overline{\pi} = (\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$ satisfies $\overline{d_1} = \overline{d_2} = \cdots = \overline{d_{\frac{3}{2}r-\frac{5}{2}}} = n - 1$, $\overline{d_{\frac{3}{2}r-\frac{3}{2}}} = 2r - 2$. Clearly, $d_1 + d_2 + \cdots + d_i \leq \overline{d_1} + \overline{d_2} + \cdots + \overline{d_i}$ for each $i = 1, 2, \ldots, f(\pi)$. By Theorem 1, $\pi \in GS_n$. Let $\pi_1 = (r - 2, r - 3, \ldots, \frac{r}{2} + \frac{1}{2}, (\frac{r}{2} - \frac{1}{2})^{\frac{r}{2}+\frac{3}{2}}, (\frac{r}{2} - \frac{3}{2})^{n-2r+1})$. If π is potentially $K_{r,r} - re$-graphic, then there exist integers t and s, $t \geq s \geq 1$ and $t + s = r + 1$ such that π_1 is potentially $K_{s,t} - se$-graphic. If $s < \frac{r}{2} + \frac{1}{2}$, then there are at least s terms in π_1 which are greater than or equal to $r - s$, which is impossible. If $s = t = \frac{r}{2} + \frac{1}{2}$, then there are at least $r + 1$ terms in π_1 which are greater than or equal to $\frac{r}{2} - \frac{1}{2}$, which is also impossible . Thus, π is not potentially $K_{r,r} - re$-graphic. Hence $\sigma(K_{r,r} - re, n) \geq \sigma(\pi) + 2 = (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1)$.

Now suppose $(r, n) \in E_4$. Consider $\pi = ((n-1)^{r-1}, 2r - 3, 2r - 4, \ldots, \frac{3}{2}r - \frac{1}{2}, (\frac{3}{2}r - \frac{3}{2})^{\frac{r}{2}+\frac{3}{2}}, (\frac{3}{2}r - \frac{5}{2})^{n-2r}, \frac{3}{2}r - \frac{7}{2})$. Then $\sigma(\pi) = (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r -$

$\frac{9}{8}-(n-r+1)$ is even and $f(\pi)=\frac{3}{2}r-\frac{3}{2}$. By the left-most off-diagonal matrix $\overline{A}$ of π, $\overline{\pi}=(\overline{d_1},\overline{d_2},\ldots,\overline{d_n})$ satisfies $\overline{d_1}=\overline{d_2}=\cdots=\overline{d_{\frac{3}{2}r-\frac{7}{2}}}=n-1$, $\overline{d_{\frac{3}{2}r-\frac{5}{2}}}=n-2, \overline{d_{\frac{3}{2}r-\frac{3}{2}}}=2r-2$. Clearly, $d_1+d_2+\cdots+d_i\leq\overline{d_1}+\overline{d_2}+\cdots+\overline{d_i}$ for each $i=1,2,\ldots,f(\pi)$. By Theorem 1, $\pi\in GS_n$. Similarly, we can prove that π is not potentially $K_{r,r}-re$-graphic. Thus, $\sigma(K_{r,r}-re,n)\geq\sigma(\pi)+2=(\frac{5}{2}r-\frac{5}{2})n-\frac{11}{8}r^2+\frac{5}{2}r+\frac{7}{8}-(n-r+1)$. □

Lemma 9. *Let r be odd, $r\geq5$ and $n=r^2+r-2$. Then*

$$\sigma(K_{r,r}-e)\leq(\frac{5}{2}r-\frac{5}{2})n-\frac{11}{8}r^2+\frac{5}{2}r+\frac{15}{8}-(n-r+1)+\frac{1}{4}(6r^3+2r^2-32r+24).$$

Proof. By Theorem 3,

$$\begin{aligned}&\sigma(K_{r,r}-e,n)\leq\sigma(K_{2r},n)\leq(4r-6)n+8\\&=(\tfrac{5}{2}r-\tfrac{5}{2})n+(\tfrac{3}{2}r-\tfrac{7}{2})(r^2+r-2)+8\\&=(\tfrac{5}{2}r-\tfrac{5}{2})n-\tfrac{11}{8}r^2+\tfrac{5}{2}r+\tfrac{15}{8}-(n-r+1)+(\tfrac{3}{2}r^3+\tfrac{3}{8}r^2-9r+\tfrac{97}{8})\\&=(\tfrac{5}{2}r-\tfrac{5}{2})n-\tfrac{11}{8}r^2+\tfrac{5}{2}r+\tfrac{15}{8}-(n-r+1)+\tfrac{1}{4}(6r^3+\tfrac{3}{2}r^2-36r+\tfrac{97}{2})\\&=(\tfrac{5}{2}r-\tfrac{5}{2})n-\tfrac{11}{8}r^2+\tfrac{5}{2}r+\tfrac{15}{8}-(n-r+1)+\tfrac{1}{4}(6r^3+(\tfrac{3}{2}r^2+\tfrac{9}{2})-36r+44)\\&\leq(\tfrac{5}{2}r-\tfrac{5}{2})n-\tfrac{11}{8}r^2+\tfrac{5}{2}r+\tfrac{15}{8}-(n-r+1)+\tfrac{1}{4}(6r^3+2r^2-32r+24).\end{aligned}$$ □

Lemma 10. *Let $r\geq5$ be odd, $n\geq r^2+r-2$ and $\pi=(d_1,d_2,\ldots,d_n)\in GS_n$ with $d_n\geq r$. If $\sigma(\pi)\geq(\frac{5}{2}r-\frac{5}{2})n-\frac{11}{8}r^2+\frac{5}{2}r+\frac{15}{8}-(n-r+1)$, then π is potentially $K_{r,r}-e$-graphic.*

Proof. If $d_r\geq2r-1$, then by $n\geq(r+2)(r-1)$ and Theorems 4 and 5, π is potentially $K_{r,r}$-graphic, and hence π is potentially $K_{r,r}-e$-graphic. Assume $d_r\leq2r-2$. If $d_{r+t}\leq2r-3-t$ for any $t\in\{1,2,\ldots,\frac{r}{2}-\frac{1}{2}\}$, then $\sigma(\pi)\leq(n-1)(r-1)+(2r-2)+(2r-4)+\cdots+(\frac{3}{2}r-\frac{3}{2})+(\frac{3}{2}r-\frac{5}{2})(n-\frac{3}{2}r+\frac{3}{2})<(\frac{5}{2}r-\frac{5}{2})n-\frac{11}{8}r^2+\frac{5}{2}r+\frac{15}{8}-(n-r+1)\leq\sigma(\pi)$, a contradiction. Hence there exists $t\in\{1,2,\ldots,\frac{r}{2}-\frac{1}{2}\}$ such that $d_{r+t}\geq2r-2-t$. There are two cases.

Case 1. There exists $t\in\{1,2,\ldots,\frac{r}{2}-\frac{3}{2}\}$ such that $d_{r+t}\geq2r-2-t$. If $d_{2r}\leq\frac{3}{2}r-\frac{7}{2}$, then $\sigma(\pi)\leq(n-1)(r-1)+(2r-2)r+(\frac{3}{2}r-\frac{7}{2})(n-2r+1)=(\frac{5}{2}r-\frac{9}{2})n-r^2+\frac{11}{2}r-\frac{5}{2}<(\frac{5}{2}r-\frac{5}{2})n-\frac{11}{8}r^2+\frac{5}{2}r+\frac{15}{8}-(n-r+1)\leq\sigma(\pi)$, a contradiction. Hence $d_{2r}\geq\frac{3}{2}r-\frac{5}{2}$. By Lemma 2, π is potentially $K_{r,r}-e$-graphic.

Case 2. $d_{r+t}\leq2r-3-t$ for any $t\in\{1,2,\ldots,\frac{r}{2}-\frac{3}{2}\}$ and $d_{\frac{3}{2}r-\frac{1}{2}}\geq\frac{3}{2}r-\frac{3}{2}$. If $d_{2r}\leq\frac{3}{2}r-\frac{5}{2}$, then $\sigma(\pi)\leq(n-1)(r-1)+(2r-2)+(2r-4)+\cdots+(\frac{3}{2}r-\frac{1}{2})+(\frac{3}{2}r-\frac{3}{2})(\frac{r}{2}+\frac{3}{2})+(\frac{3}{2}r-\frac{5}{2})(n-2r+1)=(\frac{5}{2}r-\frac{7}{2})n-\frac{11}{8}r^2+\frac{7}{2}r-\frac{1}{8}<(\frac{5}{2}r-\frac{5}{2})n-\frac{11}{8}r^2+\frac{5}{2}r+\frac{15}{8}-(n-r+1)\leq\sigma(\pi)$, a contradiction. Hence $d_{2r}\geq\frac{3}{2}r-\frac{3}{2}$. By Lemma 2, π is potentially $K_{r,r}-e$-graphic. □

Lemma 11. *Let r be odd, $r \geq 5$ and $n = r^2+r-2+t$, where $0 \leq t \leq 6r^2+8r-24$. Then*

$$\sigma(K_{r,r}-e,n) \leq (\frac{5}{2}r-\frac{5}{2})n-\frac{11}{8}r^2+\frac{5}{2}r+\frac{15}{8}-(n-r+1)+\frac{r-1}{4}(6r^2+8r-24)-\frac{r-1}{4}t.$$

Proof. Use induction on t. It follows from Lemma 9 that Lemma 11 holds for $t = 0$. Now suppose that Lemma 11 holds for $0 \leq t-1 \leq 6r^2+8r-25$. We will prove that Lemma 11 holds for t. Let $n = r^2+r-2+t$, and $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n-r+1) + \frac{r-1}{4}(6r^2+8r-24) - \frac{r-1}{4}t$. Clearly, $\sigma(\pi) \geq (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n-r+1)$. If $d_n \geq r$, then by Lemma 10, π is potentially $K_{r,r} - e$-graphic. If $d_n \leq r-1$, then the residual sequence π' satisfies $\sigma(\pi') = \sigma(\pi) - 2d_n \geq (\frac{5}{2}r - \frac{5}{2})(n-1) - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - ((n-1) - r + 1) + \frac{r-1}{4}(6r^2 + 8r - 24) - \frac{r-1}{4}(t-1)$. By the induction hypothesis, π' and π are potentially $K_{r,r} - e$-graphic. Thus, $\sigma(K_{r,r} - e, n) \leq (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n-r+1) + \frac{r-1}{4}(6r^2 + 8r - 24) - \frac{r-1}{4}t$. □

Lemma 12. *Let r be odd, $r \geq 5$ and $n \geq 7r^2 + 9r - 26$. Then*

$$\sigma(K_{r,r} - e, n) \leq (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n - r + 1).$$

Proof. It is enough to prove that $(*)$: if $\pi = (d_1, d_2, \ldots, d_n) \in GS_n$ with $\sigma(\pi) \geq (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n-r+1)$, then π is potentially $K_{r,r} - e$-graphic. Apply induction on n. By Lemma 11, $(*)$ holds for $n = 7r^2 + 9r - 26$. Now suppose that $(*)$ holds for $n - 1 \geq 7r^2 + 9r - 26$. We will prove that $(*)$ holds for n. If $d_n \geq r$, then by Lemma 10, π is potentially $K_{r,r} - e$-graphic. If $d_n \leq r-1$, then residual sequence π' satisfies $\sigma(\pi') = \sigma(\pi) - 2d_n \geq (\frac{5}{2}r - \frac{5}{2})(n-1) - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - ((n-1) - r + 1)$. By the induction hypothesis, π' and π are potentially $K_{r,r} - e$-graphic. □

Theorem 7. *Let $r \geq 5$ be odd, $1 \leq k \leq r$ and $n \geq 7r^2 + 9r - 26$. Then*

$$\sigma(K_{r,r}-ke,n) = \begin{cases} (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n-r+1), & \text{if } (r,n) \in E_3, \\ (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{7}{8} - (n-r+1), & \text{if } (r,n) \in E_4. \end{cases}$$

Proof. By Lemmas 8 and 12, $\sigma(K_{r,r} - ke, n) = (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n-r+1)$ for $(r,n) \in E_3$ and $(\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{7}{8} - (n-r+1) \leq \sigma(K_{r,r} - ke, n) \leq (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{15}{8} - (n-r+1)$ for $(n,r) \in E_4$. Since $\sigma(K_{r,r} - ke, n)$ is even, we have $\sigma(K_{r,r} - ke, n) = (\frac{5}{2}r - \frac{5}{2})n - \frac{11}{8}r^2 + \frac{5}{2}r + \frac{7}{8} - (n-r+1)$ for $(r,n) \in E_4$. □

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