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ONE NEW TYPE OF INTERLEAVED ITERATIVE ALGORITHM FOR H-MATRICES

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ABSTRACT. In the theory and the applications of Numerical Linear Algebra, the class of H-matrices is very important. In recent years, many appeared works have proposed iterative criterion for H-matrices. In this paper, we provide a new type of interleaved iterative algorithm, which is always convergent in finite steps for H-matrices and needs fewer iterations than those proposed in the related works, and a corresponding algorithm for general matrix, which eliminates the redundant computations when the given matrix is not an H-matrix. Finally, several numerical examples are presented to show the effectiveness of the proposed algorithms.

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1. Introduction

H-matrices play a vital role in both the theory and the applications of Numerical Linear Algebra. Many appeared works have proposed various criterions(algorithms) that identify whether a matrix to be solved is an H-matrix or not[1-9]. At first, we give notations and definitions as follows:

In this paper, $C^{n\times n}(R^{n\times n})$ will be used to denote the set of all $n \times n$ complex(real) matrices. $N = \{1, 2, \ldots, n\}$. Let $A = (a_{ij}) \in C^{n\times n}$, and

$$R_i(A) = \sum_{j\neq i} |a_{ij}|\,,\ i \in N, \qquad N_0(A) = \{\, i \mid |a_{ii}| = R_i(A)\,, i \in N\},$$

$$N_1(A) = \{\, i \mid |a_{ii}| > R_i(A)\,, i \in N\}\,, N_2(A) = \{\, i \mid 0 < |a_{ii}| < R_i(A)\,, i \in N\}.$$

If $|a_{ii}| > R_i(A)\,, \forall i \in N$, then A is called a strictly diagonally dominant matrix. And if there exists a positive diagonal matrix D such that AD is strictly diagonally dominant, then A is called a generalized diagonally dominant matrix

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(GDDM), we denote this by $A \in \widetilde{D}$. It is well known that A is a GDDM if and only if A is a nonsingular H-matrix.

Matrix A is called a reducible matrix, if there exists a subset $K: \phi \neq K \subset N$, satisfies

$$a_{ij} = 0\,, \text{ for any } i \in K,\, j \in N \backslash K.$$

If A is not a reducible matrix, we call A is an irreducible matrix.

Definition 1. We define the comparison matrix of A, $\mu(A) = (\alpha_{ij})$, by

$$\alpha_{ij} = \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & i \neq j. \end{cases}$$

If the eigenvalues of $\mu(A)$ have positive real parts, we call $\mu(A)$ an M-matrix. We say that A is an H-matrix if and only if $\mu(A)$ is an M-matrix.

It is obvious that, as defined above, every H-matrix is nonsingular.

Definition 2. Let A be an irreducible matrix, if for all $i \in N$,

$$|a_{ii}| \geq R_i(A), \tag{1}$$

and there exists at least one strict inequality in (1), then A is called an irreducible diagonally dominant matrix.

Lemma 1[10]. *Let A is an irreducible matrix. If for all $i \in N$,*

$$|a_{ii}| \geq \mathrm{R}_i(A), \tag{2}$$

and there exists at least one strict inequality in (2), then A is an H-matrix.

We know that A is an H-matrix if $N_0(A) \bigcup N_2(A) = \emptyset$, and A is not an H-matrix if $N_1(A) = \emptyset$. So in this paper, set $N_0(A) \bigcup N_2(A) \neq \emptyset$, and $N_1(A) \neq \emptyset$.

Because it is difficult to find a proper D for an H-matrix such that AD is strictly diagonally dominant, an efficient iterative algorithm is required. Recently, Li *et al.* in [2] have proposed a non-parameter iterative method for generalized diagonally dominant matrices, and T. Kohno *et al.* in [3] gave an algorithmic procedure to eliminate redundant computations of iterations when A is not an H-matrix. Liu and He in [1] provide two improved algorithm by means of interleaved iteration, which need fewer iterations than that of Li *et al.*in [1] and T. Kohno *et al.* in [3], and other methods show in [4-9].

In this paper, we provide an interleaved iterative algorithms for H-matrices, which is always convergent in finite steps and needs fewer iterations than those in [1-3], and then give a corresponding algorithm for general matrix to eliminate redundant iterations when the given matrix is not an H-matrix. Finally, several numerical examples are presented to show the effectiveness of the proposed algorithms.

2. The algorithms

First, set $A = (a_{ij}) \in C^{n \times n}$, satisfying $a_{ii} \neq 0$, for all $i \in N$, we will use the notations as follows:

$$\gamma_i = \sum_{j\in N_0(A), j\neq i} |a_{ij}|\,, \text{ and if } N_0(A)=\{i\} \text{ or } N_0(A)=\phi, \text{ we set } \gamma_i = 0,$$

$$\alpha_i = \sum_{j\in N_1(A), j\neq i} |a_{ij}|\,, \text{ and if } N_1(A)=\{i\} \text{ or } N_1(A)=\phi, \text{ we set } \alpha_i = 0,$$

$$\beta_i = \sum_{j\in N_2(A), j\neq i} |a_{ij}|\,, \text{ and if } N_2(A)=\{i\} \text{ or } N_2(A)=\phi, \text{ we set } \beta_i = 0,$$

$$r_0 = \max_{i\in N_1(A)} \left(\frac{\gamma_i+\beta_i}{|a_{ii}|-\alpha_i}\right), \quad P_i = \gamma_i + r_0\alpha_i + \beta_i\,, \quad \forall i \in N_1(A),$$

$$h = \max_{i\in N_1(A)} \left(\frac{\gamma_i+\beta_i}{P_i - \sum\limits_{t\in N_1(A), t\neq i} |a_{it}|\frac{P_t}{|a_{tt}|}}\right).$$

Algorithm A(L. Li et al. in [2]). Suppose $A=(a_{ij})\in C^{n\times n}$, $a_{ii}\neq 0$, is an irreducible matrix. Let $N_1(A)\neq\phi$.

(A) $N_1(A)=0$, $N_2(A)=0$. For $i=1,2,\ldots,n$, do {
  (A1) Compute $R_i(A)=\sum_{j\neq i}|a_{ij}|$,
  (A2) If $|a_{ii}|>R_i(A)$, then { $N_1(A)=1$, $d_i=\frac{R_i(A)}{|a_{ii}|}$,

$$a_{ji}=a_{ji}*d_i\,, \qquad j=1,2,\ldots,n\}$$

    else if $|a_{ii}|<R_i(A)$, then $N_2(A)=1$,
else, }
(B) If $N_1(A)=0$, then print 'A is not a GDDM', go to (C).
  else if $N_2(A)=0$, then print 'A is a GDDM', go to (C).
  else return to (A).
(C) End.

Algorithm A'(Liu and He in [1]). Input: a given irreducible matrix $A=(a_{ij})\in C^{n\times n}$.
Output: $D=D^{(1)}D^{(2)}\cdots D^{(m)}\in\mathcal{D}_A$ if A is an H-matrix.
1. if $N_1(A)=\phi$ or $a_{ii}=0$ for some $i\in N$, 'A is not an H-matrix', stop; otherwise,
2. set $m=1$, $A^{(0)}=A$, $D^{(0)}=I$,
3. compute $A^{(m)}=A^{(m-1)}D^{(m-1)}=(a_{ij}^{(m)})$,
4. if $N_1(A^{(m)})=\phi$, 'A is not an H-matrix', stop; if $N_1(A^{(m)})\bigcup N_0(A^{(m)})=N$, '$A$ is an
  H-matrix', stop; otherwise,
5. compute $\alpha_i^{(m)}$, $\beta_i^{(m)}$, $\gamma_i^{(m)}$, $i\in N$,
6. set $r_2=\max\limits_{i\in N_1(A^{(m)})}\frac{\beta_i^{(m)}+\gamma_i^{(m)}}{|a_{ii}^{(m)}|-\alpha_i^{(m)}}$,
7. set $d=(d_i)$, where

If m is an odd number,then $d_i = \begin{cases} \frac{R_i(A^{(m)})}{|a_{ii}^{(m)}|}, & \text{if } i \in N_1(A^{(m)}), \\ 1, & \text{if } i \in N_0(A^{(m)}) \bigcup N_2(A^{(m)}). \end{cases}$

If m is an even number, then $d_i = \begin{cases} r_2, & \text{if } i \in N_1(A^{(m)}), \\ 1, & \text{if } i \in N_0(A^{(m)}) \bigcup N_2(A^{(m)}). \end{cases}$

8. set $D^{(m)} = diag(d), m = m+1$, go to step 3.

Next, we provide a new improved interleaved iteration algorithm.

Algorithm I. Input: a given irreducible matrix $A = (a_{ij}) \in C^{n\times n}$.
Output: $D = D^{(1)}D^{(2)}\cdots D^{(m)} \in \mathcal{D}_A$ if A is an H-matrix.
1. if $N_1(A) = \phi$ or $a_{ii} = 0$ for some $i \in N$, 'A is not an H-matrix', stop; otherwise,
2. set $m = 1, A^{(0)} = A, D^{(0)} = I$,
3. compute $A^{(m)} = A^{(m-1)}D^{(m-1)} = (a_{ij}^{(m)})$,
4. if $N_1(A^{(m)}) = \phi$, 'A is not an H-matrix', stop; if $N_1(A^{(m)}) \bigcup N_0(A^{(m)}) = N$, '$A$ is an
H-matrix', stop; otherwise,
5. compute $\alpha_i^{(m)}, \beta_i^{(m)}, \gamma_i^{(m)}, \forall i \in N$,
6. compute $r_0^{(m)}, P_i^{(m)}, h^{(m)}, \forall i \in N_1(A^{(m)})$,
7. set $r_i = \frac{h^{(m)}P_i^{(m)}}{|a_{ii}^{(m)}|}$,
8. set $d = (d_i)$, where

If m is an odd number, then $d_i = \begin{cases} \frac{R_i(A^{(m)})}{|a_{ii}^{(m)}|}, & \text{if } i \in N_1(A^{(m)}), \\ 1, & \text{if } i \in N_0(A^{(m)}) \bigcup N_2(A^{(m)}). \end{cases}$

If m is an even number, then $d_i = \begin{cases} r_i, & \text{if } i \in N_1(A^{(m)}), \\ 1, & \text{if } i \in N_0(A^{(m)}) \bigcup N_2(A^{(m)}). \end{cases}$

9. set $D^{(m)} = diag(d), m = m+1$, go to step 3.

Remark 1. In Algorithm I, A is an irreducible matrix, so for all $i \in N_1(A^{(m)})$, we have

$$0 < r_0^{(m)} < 1, \ 0 < \frac{P_i^{(m)}}{|a_{ii}^{(m)}|} < 1,$$

and

$$r_0^{(m)} \geq \frac{\gamma_i^{(m)} + \beta_i^{(m)}}{|a_{ii}^{(m)}| - \alpha_i^{(m)}},$$

$$r_0^{(m)}|a_{ii}^{(m)}| \geq \gamma_i^{(m)} + r_0^{(m)}\alpha_i^{(m)} + \beta_i^{(m)} = P_i^{(m)}, \quad r_0^{(m)} \geq \frac{P_i^{(m)}}{|a_{ii}^{(m)}|},$$

and

$$\frac{\gamma_i^{(m)}+\beta_i^{(m)}}{P_i^{(m)}-\sum\limits_{t\in N_1(A^{(m)}),t\neq i}|a_{it}^{(m)}|\frac{P_t^{(m)}}{|a_{tt}^{(m)}|}} = \frac{P_i^{(m)}-r_0^{(m)}\alpha_i^{(m)}}{P_i^{(m)}-\sum\limits_{t\in N_1(A^{(m)}),t\neq i}|a_{it}^{(m)}|\frac{P_t^{(m)}}{|a_{tt}^{(m)}|}} \le \frac{P_i^{(m)}-\sum\limits_{t\in N_1(A^{(m)}),t\neq i}|a_{it}^{(m)}|\frac{P_t^{(m)}}{|a_{tt}^{(m)}|}}{P_i^{(m)}-\sum\limits_{t\in N_1(A^{(m)}),t\neq i}|a_{it}^{(m)}|\frac{P_t^{(m)}}{|a_{tt}^{(m)}|}} = 1,$$

then

$$0<h^{(m)}\le 1\,,$$

$$0<\frac{h^{(m)}P_i^{(m)}}{|a_{ii}^{(m)}|}=r_i\le\frac{P_i^{(m)}}{|a_{ii}^{(m)}|}\le r_0^{(m)}=r_2<1\,,\ \forall i\in N_1(A^{(m)})\,.$$

Therefore we have that this algorithm needs fewer number of iterations than Algorithm A and A'. The theoretical analysis of Algorithm I as a characterization of H-matrices is presented by the following theorem:

Theorem 1. $A=(a_{ij})\in C^{n\times n}$ *is an irreducible H-matrix if and only if Algorithm I stops after a finite number of iterations by producing a strictly diagonally dominant matrix.*

Proof. Sufficiency: Suppose that Algorithm I stops after m iterations. That means, we have obtained a strictly diagonally dominant matrix $A^{(m)}=A^{(0)}D^{(1)}D^{(2)}\cdots D^{(m-1)}=AD$, where $D=D^{(0)}D^{(1)}\cdots D^{(m-1)}$ is a positive diagonal matrix. Thus, A is an irreducible H-matrix.

Necessity: Let A be an irreducible H-matrix. For notational convenience, we assume A is a nonnegative matrix. By using way of contradiction, suppose that Algorithm I doesn't stop after a finite number of iterations. From Algorithm I, we have $A^{(m)}=A^{(1)}D^{(1)}D^{(2)}\cdots D^{(m-1)}=AD$, where $D=D^{(1)}D^{(2)}\cdots D^{(m-1)}$ is a positive diagonal matrix, then it is obvious that

$$A=A^{(1)}\ge\cdots\ge A^{(m)}\ge\cdots\ge 0\,.$$

The infinite matrix sequence $\{A^{(m)}\}$ is bounded and monotone decreasing, then we have

$$\lim_{m\to\infty}A^{(m)}=B\ge 0\,,$$

where $B=AF$, $F=D^{(1)}D^{(2)}\cdots D^{(m)}\cdots$ is a positive diagonal matrix.

Next, we want to prove

$$\lim_{m\to\infty}N_1(A^{(m)})=N_1(B)=\phi.$$

By using way of contradiction again, we assume $\lim\limits_{m\to\infty}N_1(A^{(m)})\neq\phi$, then $1-r_i>0$, $\forall i\in N_1(A^{(m)})$ and there exist some i and ε_1, ε_2 such that

$$a_{ii}^{(m)}-R_i(A^{(m)})>\varepsilon_1\,,\quad a_{ii}^{(m)}(1-r_i)>\varepsilon_2\,,\quad m=1,2,\ldots.$$

We set $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$.

When m is an odd number, from Algorithm I, we have

$$\begin{aligned} 0 < a_{ii}^{(m+1)} &= a_{ii}^{(m)} \frac{R_i(A^{(m)})}{a_{ii}^{(m)}} \\ &= a_{ii}^{(m)} - \left(a_{ii}^{(m)} - R_i(A^{(m)})\right) \\ &< a_{ii}^{(m)} - \varepsilon_0 \,. \end{aligned}$$

When m is an even number, from Algorithm I, we have

$$\begin{aligned} 0 < a_{ii}^{(m+1)} &= a_{ii}^{(m)} r_i \\ &< a_{ii}^{(m)} - \varepsilon_2 \\ &< a_{ii}^{(m)} - \varepsilon_0 \,. \end{aligned}$$

Note that ε_0 is positive and therefore

$$a_{ii}^{(0)} = a_{ii}^{(1)} > a_{ii}^{(2)} + \varepsilon_0 > \cdots > a_{ii}^{(m)} + (m-1)\varepsilon_0.$$

Let $m \to \infty$. Then $a_{ii}^{(0)} \to \infty$, we obtain a contradiction. Thus,

$$\lim_{m\to\infty} N_1(A^{(m)}) = N_1(B) = \phi.$$

That means B is not an H-matrix. On the other hand there exists a positive diagonal matrix E such that $AE = B(F^{-1}E)$ is strictly diagonally dominant. We know that $F^{-1}E$ is still a positive diagonal matrix, so B is an H-matrix. Then we obtain another contradiction, completing the proof of this theorem. □

The drawback of Algorithms I is that when A is not an H-matrix, it requires a large number of iterations. Kohno *et al.* in [3] proposed a new algorithmic procedure to conquer this drawback, and Liu *et al.* in [1] have improved it.

Algorithm B(T. Kohno et al. in [3]). Input: a given matrix $A = (a_{ij}) \in C^{n\times n}$. Output: $D = D^{(1)}D^{(2)}\cdots D^{(m)} \in \mathcal{D}_A$ if A is an H-matrix.

1. if $N_1(A) = \phi$ or $a_{ii} = 0$ for some $i \in N$, 'A is not an H-matrix', stop;otherwise,
2. set $m = 1$, $A^{(0)} = A$, $D^{(0)} = I$,
3. compute $A^{(m)} = A^{(m-1)}D^{(m-1)} = (a_{ij}^{(m)})$,
4. compute
$$d_i^{(m)} = \frac{\sum_{j=1}^{n} |\, a_{ij}^{(m)} \,|}{|\, a_{ii}^{(m)} \,|}, \quad i \in N.$$
5. if $d_i^{(m)} < 2$ for all i, 'A is an H-matrix', stop;
   if $d_i^{(m)} \geq 2$ for all i, 'A is not an H-matrix', stop; otherwise,
6. set $D^{(m)} = diag(d_i^{(m)})$, $m = m + 1$, go to step 3.

Algorithm B' (Liu and He in [1]). Input: a given irreducible matrix $A = (a_{ij}) \in C^{n\times n}$.

Output: $D = D^{(1)}D^{(2)}\cdots D^{(m)} \in \mathcal{D}_A$ if A is an H-matrix.
1. if $N_1(A) = \phi$ or $a_{ii} = 0$ for some $i \in N$, 'A is not an H-matrix', stop; if $N_2(A) = \phi$, 'A is an H-matrix', stop; otherwise,
2. set $m = 1$, $A^{(0)} = A$, $D^{(0)} = I$,
3. compute $A^{(m)} = A^{(m-1)}D^{(m-1)} = (a_{ij}^{(m)})$,
4. compute $\alpha_i^{(m)}$, $\beta_i^{(m)}$, $\gamma_i^{(m)}$, $i \in N$,
5. set $r_1 = \min\limits_{i\in N_0(A^{(m)})\cup N_1(A^{(m)})} \frac{\beta_i^{(m)}}{|a_{ii}^{(m)}|-\alpha_i^{(m)}-\gamma_i^{(m)}}$,
6. if $|a_{ii}^{(m)}| \le r_1\,\alpha_i^{(m)} + \beta_i^{(m)} + r_1\,\gamma_i^{(m)}$ for all $i \in N_2(A^{(m)})$, 'A is not an H-matrix', stop; otherwise,
7. set $r_2 = \max\limits_{i\in N_1(A^{(m)})} \frac{\beta_i^{(m)}+\gamma_i^{(m)}}{|a_{ii}^{(m)}|-\alpha_i^{(m)}}$,
8. if $N_0(A^{(m)})\bigcup N_1(A^{(m)}) = N$ or $|a_{ii}^{(m)}| > r_2\alpha_i^{(m)} + \beta_i^{(m)} + \gamma_i^{(m)}$ for all $i \in N_0(A^{(m)})\bigcup N_2(A^{(m)})$, '$A$ is an H-matrix', stop; otherwise,
9. set $d = (d_i)$, where

$$\text{If } m \text{ is an odd number, then } d_i = \begin{cases} \frac{R_i(A^{(m)})}{|a_{ii}^{(m)}|}, & \text{if } i \in N_1(A^{(m)}),\\ 1, & \text{if } i \in N_0(A^{(m)})\bigcup N_2(A^{(m)}). \end{cases}$$

$$\text{If } m \text{ is an even number, then } d_i = \begin{cases} r_2, & \text{if } i \in N_1(A^{(m)}),\\ 1, & \text{if } i \in N_0(A^{(m)})\bigcup N_2(A^{(m)}). \end{cases}$$

10. set $D^{(m)} = diag(d), m = m+1$, go to step 3.

Next, we give a new improved algorithm for general irreducible matrices on the basis of Algorithm B and B'.

Algorithm II. Input: a given irreducible matrix $A = (a_{ij}) \in C^{n\times n}$.
Output: $D = D^{(1)}D^{(2)}\cdots D^{(m)} \in \mathcal{D}_A$ if A is an H-matrix.
1. if $N_1(A) = \phi$ or $a_{ii} = 0$ for some $i \in N$, 'A is not an H-matrix', stop; if $N_2(A) = \phi$, 'A is an H-matrix', stop; otherwise,
2. set $m = 1$, $A^{(0)} = A$, $D^{(0)} = I$,
3. compute $A^{(m)} = A^{(m-1)}D^{(m-1)} = (a_{ij}^{(m)})$,
4. compute $\alpha_i^{(m)}$, $\beta_i^{(m)}$, $\gamma_i^{(m)}$, $i \in N$,
5. compute $r_0^{(m)}$, $P_i^{(m)}$, $h^{(m)}$, $\forall i \in N_1(A^{(m)})$,
6. set $r_1' = \min\limits_{i\in N_0(A^{(m)})\cup N_1(A^{(m)})} \frac{\beta_i^{(m)}}{|a_{ii}^{(m)}| - \sum\limits_{t\in N_1(A^{(m)}),t\neq i} |a_{it}^{(m)}| \frac{P_t^{(m)}}{|a_{tt}^{(m)}|} - \gamma_i^{(m)}}$,

7. if $|a_{ii}^{(m)}| \le r_1'\alpha_i^{(m)} + \beta_i^{(m)} + r_1'\gamma_i^{(m)}$ for all $i \in N_2(A^{(m)})$, 'A is not an H-matrix', stop;

otherwise,

8. set $r_i = \frac{h^{(m)}P_i^{(m)}}{|a_{ii}^{(m)}|}$,

9. if $N_0(A^{(m)}) \bigcup N_1(A^{(m)}) = N$ or $|a_{ii}^{(m)}| > \sum_{t\in N_1(A^{(m)}),t\neq i} |a_{it}^{(m)}| \frac{h^{(m)}P_t^{(m)}}{|a_{tt}^{(m)}|} + \beta_i^{(m)} + \gamma_i^{(m)}$ for all $i \in N_0(A^{(m)}) \bigcup N_2(A^{(m)})$, '$A$ is an H-matrix', stop; otherwise,

10. set $d = (d_i)$, where

If m is an odd number, then $d_i = \begin{cases} \frac{R_i(A^{(m)})}{|a_{ii}^{(m)}|}, & \text{if } i \in N_1(A^{(m)}), \\ 1, & \text{if } i \in N_0(A^{(m)}) \bigcup N_2(A^{(m)}). \end{cases}$

If m is an even number, then $d_i = \begin{cases} r_i, & \text{if } i \in N_1(A^{(m)}), \\ 1, & \text{if } i \in N_0(A^{(m)}) \bigcup N_2(A^{(m)}). \end{cases}$

11. set $D^{(m)} = diag(d), m = m+1$, go to step 3.

We prove the following theorem for Algorithm Ⅱ.

Theorem 2. *Let $A = (a_{ij}) \in C^{n\times n}$ be an irreducible matrix, if Algorithm Ⅱ stops after k iterations,*

(1) *When $|a_{ii}^{(k)}| \le r_1'\alpha_i^{(k)} + \beta_i^{(k)} + r_1'\gamma_i^{(k)}$, where $r_1' = \min_{i\in N_0(A^{(k)})\cup N_1(A^{(k)})} \frac{\beta_i^{(k)}}{|a_{ii}^{(k)}| - \sum_{t\in N_1(A^{(k)}),t\neq i} |a_{it}^{(k)}| \frac{P_t^{(k)}}{|a_{tt}^{(k)}|} - \gamma_i^{(k)}}$, then A is not an H-matrix;*

(2) *When $N_0(A^{(k)}) \bigcup N_1(A^{(k)}) = N$ or $|a_{ii}^{(k)}| > \sum_{t\in N_1(A^{(k)}),t\neq i} r_t|a_{it}^{(k)}| + \beta_i^{(k)} + \gamma_i^{(k)}$ for all $i \in N_0(A^{(k)}) \bigcup N_2(A^{(k)})$, where $r_i = \frac{h^{(k)}P_i^{(k)}}{|a_{ii}^{(k)}|}$, then A is an H-matrix.*

Proof. For $\forall i \in N_1(A^{(m)})$, when m is an odd number, from Algorithm Ⅱ, as $\frac{R_i(A^{(m)})}{|a_{ii}^{(m)}|} < 1$, we have

$$\begin{aligned} &|a_{ii}^{(m+1)}| - R_i(A^{(m+1)}) \\ &= |a_{ii}^{(m)}| \frac{R_i(A^{(m)})}{|a_{ii}^{(m)}|} - \sum_{i\in N_1(A^{(m)})} |a_{ij}^{(m)}| \frac{R_j(A^{(m)})}{|a_{jj}^{(m)}|} - \beta_i^{(m)} - \gamma_i^{(m)} \\ &\ge R_i(A^{(m)}) - \alpha_i^{(m)} - \beta_i^{(m)} - \gamma_i^{(m)} = 0. \end{aligned}$$

When m is an even number, from Algorithm Ⅱ, we have

$$\begin{aligned} &|a_{ii}^{(m+1)}| - R_i(A^{(m+1)}) \\ =\ & r_i|a_{ii}^{(m)}| - \sum_{t\in N_1(A^{(m)}),t\neq i} r_t|a_{it}^{(m)}| - \beta_i^{(m)} - \gamma_i^{(m)} \end{aligned}$$

$$\begin{aligned}
&= h^{(m)}\left(P_i^{(m)} - \sum_{t\in N_1(A^{(m)}),t\neq i} |a_{it}^{(m)}|\frac{P_t^{(m)}}{|a_{tt}^{(m)}|}\right) - \beta_i^{(m)} - \gamma_i^{(m)} \\
&\geq \frac{\beta_i^{(m)}+\gamma_i^{(m)}}{P_i^{(m)} - \sum_{t\in N_1(A^{(m)}),t\neq i} |a_{it}^{(m)}|\frac{P_t^{(m)}}{|a_{tt}^{(m)}|}}\left(P_i^{(m)} - \sum_{t\in N_1(A^{(m)}),t\neq i} |a_{it}^{(m)}|\frac{P_t^{(m)}}{|a_{tt}^{(m)}|}\right) \\
&\quad -\beta_i^{(m)} - \gamma_i^{(m)} = (\beta_i^{(m)}+\gamma_i^{(m)}) - \beta_i^{(m)} - \gamma_i^{(m)} = 0\,.
\end{aligned}$$

Thus $\left(N_0(A^{(1)})\bigcup N_1(A^{(1)})\right) \subseteq \left(N_0(A^{(2)})\bigcup N_1(A^{(2)})\right) \subseteq \cdots \subseteq \left(N_0(A^{(k)})\bigcup N_1(A^{(k)})\right)$. This means that multiplication with $D^{(m-1)}$ from the right doesn't change the diagonally dominant rows of $A^{(m-1)}$. Then,

(1) First we denote $\widetilde{N}_0(A) = \{i \in N_1(A) \,|\, \beta_i = 0\}$, $\xi_i = \sum_{j\in\widetilde{N}_0,\neq i} |a_{ij}|$, and if $\widetilde{N}_0(A) = \{i\}$ or $\widetilde{N}_0(A) = \phi$, we set $\xi_i = 0$.

If $N_1(A^{(k)}) = \phi$, then $r_1' = 1$. Furthermore $|a_{ii}^{(k)}| \leq \beta_i^{(k)} + \gamma_i^{(k)} = R_i(A^{(k)})$ for all $i \in N_2(A^{(k)})$, it is obvious that A is not an H-matrix, then we always assume $N_1(A^{(k)}) \neq \phi$ in the following.

When $N_1(A^{(k)}) \neq \phi$ and $\widetilde{N}_0(A^{(k)}) = \phi$, then $r_1' > 0$. We construct a positive diagonal matrix $\widetilde{D} = \mathtt{diag}\{\, \widetilde{d}_i \,|\, \widetilde{d}_i = r_1',\, i \in N_0(A^{(k)})\bigcup N_1(A^{(k)});\ \widetilde{d}_i = 1,\, i \in N_2(A^{(k)})\,\}$, and write $\widetilde{A} = A^{(k)}\,\widetilde{D} = (\widetilde{a}_{ij})$.

For $i \in N_0(A^{(k)})\bigcup N_1(A^{(k)})$, as $\frac{P_i^{(k)}}{|a_{ii}^{(k)}|} < 1, i \in N_1(A^{(k)})$, we have

$$\begin{aligned}
& |\widetilde{a}_{ii}| - \widetilde{\alpha} - \widetilde{\beta} - \widetilde{\gamma} \\
=\ & r_1'|a_{ii}^{(k)}| - r_1'\alpha_i^{(k)} - \beta_i^{(k)} - r_1'\gamma_i^{(k)} \\
=\ & r_1'(|a_{ii}^{(k)}| - \alpha_i^{(k)} - \gamma_i^{(k)}) - \beta_i^{(k)} \\
\leq\ & \frac{\beta_i^{(k)}}{|a_{ii}^{(k)}| - \sum_{t\in N_1(A^{(k)}),t\neq i} |a_{it}^{(k)}|\frac{P_t^{(k)}}{|a_{tt}^{(k)}|} - \gamma_i^{(k)}}(|a_{ii}^{(k)}| - \alpha_i^{(k)} - \gamma_i^{(k)}) - \beta_i^{(k)} \\
<\ & \frac{\beta_i^{(k)}}{|a_{ii}^{(k)}| - \sum_{t\in N_1(A^{(k)}),t\neq i} |a_{it}^{(k)}|\frac{P_t^{(k)}}{|a_{tt}^{(k)}|} - \gamma_i^{(k)}}(|a_{ii}^{(k)}| \\
& \quad - \sum_{t\in N_1(A^{(k)}),t\neq i} |a_{it}^{(k)}|\frac{P_t^{(k)}}{|a_{tt}^{(k)}|} - \gamma_i^{(k)}) - \beta_i^{(k)} = \beta_i^{(k)} - \beta_i^{(k)} = 0\,.
\end{aligned}$$

For $i \in N_2(A^{(k)})$, we have

$$|\widetilde{a}_{ii}| - \widetilde{\alpha} - \widetilde{\beta} - \widetilde{\gamma} = |a_{ii}^{(k)}| - r_1'\alpha_i^{(k)} - \beta_i^{(k)} - r_1'\gamma_i^{(k)} \leq 0.$$

So $\widetilde{A}$ has no diagonally dominant row. From Algorithm II, we obtain $\widetilde{A} = A^{(k)}\widetilde{D} = AD^{(1)}D^{(2)}\cdots D^{(k-1)}\widetilde{D}$, where $D^{(1)}D^{(2)}\cdots D^{(k-1)}\widetilde{D}$ is a positive diagonal matrix, thus A is not an H-matrix.

When $N_1(A^{(k)}) \neq \phi$ and $\widetilde{N}_0(A^{(k)}) \neq \phi$, then $r_1' = 0$. Furthermore,

$$|a_{ii}^{(k)}| \le r_1' \alpha_i^{(k)} + \beta_i^{(k)} + r_1' \gamma_i^{(k)} = \beta_i^{(k)}, \qquad \forall i \in N_2(A^{(k)}),$$

thus for any positive number d, we have

$$d\,|a_{ii}^{(k)}| \le d\,\beta_i^{(k)}, \qquad \forall i \in N_2(A^{(k)}).$$

Notice that $\beta_i^{(k)} \neq 0$, $i \in (N_0(A^{(k)}) \bigcup N_1(A^{(k)}))\backslash \widetilde{N}_0(A^{(k)})$, we construct a positive diagonal matrix $\widehat{D} = diag\{\,\widehat{d}_i \mid \widehat{d}_i = c,\ i \in (N_0(A^{(k)}) \bigcup N_1(A^{(k)}))\backslash \widetilde{N}_0(A^{(k)})\,;\ \widehat{d}_i = 1,\ i \in \widetilde{N}_0(A^{(k)})\,;\ \widehat{d}_i = d,\ i \in N_2(A^{(k)})\,\}$, where

$$c = \max_{i \in N_0(A^{(k)}) \bigcup N_1(A^{(k)})} \frac{|a_{ii}^{(k)}| - \xi_i^{(k)}}{\alpha_i^{(k)} + \gamma_i^{(k)} - \xi_i^{(k)}} \ge 1\,,$$

$$d = \max_{i \in (N_0(A^{(k)}) \bigcup N_1(A^{(k)}))\backslash \widetilde{N}_0(A^{(k)})} \frac{c(|a_{ii}^{(k)}| - \alpha_i^{(k)} - \gamma_i^{(k)} + \xi_i^{(k)}) - \xi_i^{(k)}}{\beta_i^{(k)}} > 0\,,$$

and write $\widehat{A} = A^{(k)}\,\widehat{D} = (\widehat{a}_{ij})$.

For $i \in \widetilde{N}_0(A^{(k)})$, we have

$$\begin{aligned}
& |\widehat{a}_{ii}| - \widehat{\alpha} - \widehat{\beta} - \widehat{\gamma} \\
= \;& |a_{ii}^{(k)}| - c\,(\alpha_i^{(k)} + \gamma_i^{(k)} - \xi_i^{(k)}) - \xi_i^{(k)} - d\,\beta_i^{(k)} \\
= \;& |a_{ii}^{(k)}| - \xi_i^{(k)} - c\,(\alpha_i^{(k)} + \gamma_i^{(k)} - \xi_i^{(k)}) \\
\le \;& |a_{ii}^{(k)}| - \xi_i^{(k)} - \frac{|a_{ii}^{(k)}| - \xi_i^{(k)}}{\alpha_i^{(k)} + \gamma_i^{(k)} - \xi_i^{(k)}}(\alpha_i^{(k)} + \gamma_i^{(k)} - \xi_i^{(k)}) = 0.
\end{aligned}$$

For $i \in \big(N_0(A^{(k)}) \bigcup N_1(A^{(k)})\big) \backslash \widetilde{N}_0(A^{(k)})$, we have

$$\begin{aligned}
& |\widehat{a}_{ii}| - \widehat{\alpha} - \widehat{\beta} - \widehat{\gamma} \\
= \;& c\,|a_{ii}^{(k)}| - c\,(\alpha_i^{(k)} + \gamma_i^{(k)} - \xi_i^{(k)}) - \xi_i^{(k)} - d\,\beta_i^{(k)} \\
\le \;& c\,(|a_{ii}^{(k)}| - \alpha_i^{(k)} - \gamma_i^{(k)} + \xi_i^{(k)}) - \xi_i^{(k)} \\
& - \frac{c\,(|a_{ii}^{(k)}| - \alpha_i^{(k)} - \gamma_i^{(k)} + \xi_i^{(k)}) - \xi_i^{(k)}}{\beta_i^{(k)}}\beta_i^{(k)} = 0.
\end{aligned}$$

For $i \in N_2(A^{(k)})$, as $c\,(\alpha_i^{(k)} + \gamma_i^{(k)} - \xi_i^{(k)}) + \xi_i^{(k)} \ge \alpha_i^{(k)} + \gamma_i^{(k)} > 0$, and $d\,|a_{ii}^{(k)}| \le d\beta_i^{(k)}$, we have

$$|\widehat{a}_{ii}| - \widehat{\alpha} - \widehat{\beta} - \widehat{\gamma} = d\,|a_{ii}^{(k)}| - c\,(\alpha_i^{(k)} + \gamma_i^{(k)} - \xi_i^{(k)}) - \xi_i^{(k)} - d\,\beta_i^{(k)} < 0.$$

So $\widehat{A}$ has no diagonally dominant row. From Algorithm II, we obtain $\widehat{A} = A^{(k)}\widehat{D} = AD^{(1)}D^{(2)}\cdots D^{(k-1)}\widehat{D}$, where $D^{(1)}D^{(2)}\cdots D^{(k-1)}\widehat{D}$ is a positive diagonal matrix, thus A is not an H-matrix.

(2) We construct a positive diagonal matrix $D = \mathtt{diag}\{d_i \,|\, d_i = r_i, i \in N_1(A^{(k)}); d_i = 1, i \in N_0(A^{(k)}) \bigcup N_2(A^{(k)})\}$, and write $A' = A^{(k)}D = (a'_{ij})$.

For $i \in N_1(A^{(k)})$, we have

$$\begin{aligned} |a'_{ii}| - \alpha' - \beta' - \gamma' &= r_i|a_{ii}^{(k)}| - \sum_{t\in N_1(A^{(k)}),t\neq i} r_t|\,a_{it}^{(k)}\,| - \beta_i^{(k)} - \gamma_i^{(k)} \\ &= h^{(k)}\left(P_i^{(k)} - \sum_{t\in N_1(A^{(k)}),t\neq i} |\,a_{it}^{(k)}\,|\frac{P_t^{(k)}}{|\,a_{tt}^{(k)}\,|}\right) - \beta_i^{(k)} - \gamma_i^{(k)} \\ &\geq (\beta_i^{(k)} + \gamma_i^{(k)}) - \beta_i^{(k)} - \gamma_i^{(k)} = 0\,. \end{aligned}$$

For $i \in N_0(A^{(k)}) \bigcup N_2(A^{(k)})$, we have

$$|a'_{ii}| - \alpha' - \beta' - \gamma' = |a_{ii}^{(k)}| - \sum_{t\in N_1(A^{(k)}),t\neq i} r_t|\,a_{it}^{(k)}\,| - \beta_i^{(k)} - \gamma_i^{(k)} > 0.$$

So A' is an irreducibly diagonally dominant matrix. From Algorithm II, we obtain $A' = A^{(k)}D = AD^{(1)}D^{(2)}\cdots D^{(k-1)}D$, where $D^{(1)}D^{(2)}\cdots D^{(k-1)}D$ is a positive diagonal matrix, thus A is an H-matrix. □

3. Examples

We give the following examples to show the effectiveness of the proposed algorithms:

Example 1. Let

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 & 2 \\ 2 & 4 & 1 & 1 & 1 \\ 0.5 & 0.5 & 3 & 1 & 0.5 \\ 0.5 & 0.25 & 3 & 4 & 0 \\ 1 & 0 & 4 & 0 & 20 \end{pmatrix},$$

we have that Algorithm I needs only one iteration for identifying A is an H-matrix, while both Algorithm A and Algorithm A' require three iterations.

Example 2. Let

$$A = \begin{pmatrix} 1 & 0.1 & 0.05 & 0 \\ 0.3 & 1 & 0 & 0.05 \\ 0 & 0.05 & 1 & 1.05 \\ 0.05 & 0.1 & 1.05 & 1 \end{pmatrix},$$

we have that Algorithm II needs only one iteration for identifying A is not an H-matrix, while Algorithm B requires eleven iterations.

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