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FURTHER RESULTS ON MULTISPLITTING AND TWO-STAGE MULTISPLITTING METHODS

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ABSTRACT. In this paper, we study the regularity of induced splittings from multisplitting and two-stage multisplitting methods of monotone matrices under the assumption that splittings are weak regular, and we also study some comparison theorems for two-stage multisplitting methods of monotone matrices.

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1. Introduction

In this paper, we consider the following linear systems

$$Ax = b, \tag{1}$$

where $A \in R^{n\times n}$ is a nonsingular matrix, and x, $b \in R^n$. $A = M - N$ is called a *splitting* of A if M is nonsingular. The splitting gives rise to a classical iterative method

$$x_i = M^{-1}Nx_{i-1} + M^{-1}b, \quad i = 1, 2, \ldots, \tag{2}$$

where $x_0 \in R^n$ is given as an initial guess.

Wang and Zhao [11] have shown the regularity of induced splittings from two-stage and multisplitting methods of monotone matrices under the assumption that splittings are *regular*. However, in this paper we study the regularity of induced splittings from multisplitting and two-stage multisplitting methods of monotone matrices under the assumption that splittings are *weak regular*.

The purpose of this paper is to show the regularity of induced splittings from multisplitting and two-stage multisplitting methods for monotone matrices, and to also study some comparison theorems for two-stage multisplitting methods of monotone matrices.

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This paper is organized as follows. In Section 2, we present some notation, definitions and preliminary results which we refer to later. In Section 3, we study the regularity of induced splittings from multisplitting methods for monotone matrices. In Section 4, we study the regularity of induced splittings from two-stage multisplitting methods for monotone matrices. In Section 5, we provide some comparison results for two-stage multisplitting methods of monotone matrices.

2. Preliminaries

For a vector $x \in R^n$, $x \geq 0$ ($x > 0$) denotes that all components of x are nonnegative (positive). For two vectors x, $y \in R^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). These definitions carry immediately over to matrices.

A real square matrix $A = (a_{ij})$ is called a *Z-matrix* if $a_{ij} \leq 0$ for $i \neq j$. A real square matrix A is called *monotone* if A is nonsingular with $A^{-1} \geq 0$. A real square matrix $A = (a_{ij})$ is called an *M-matrix* if it is a monotone matrix with $a_{ij} \leq 0$ for $i \neq j$. A square matrix A is called *irreducible* if the directed graph of A is strongly connected [10]. A matrix $A = (a_{ij}) \in R^{n\times n}$ is called *irreducibly diagonally dominant* if A is irreducible,

$$|a_{ii}| \geq \sum_{j\neq i} |a_{ij}| \quad \text{for } i = 1, 2, \ldots, n, \tag{3}$$

and strict inequality holds in (3) for at least one i. It is well known that A is an M-matrix if A is an irreducibly diagonally dominant Z-matrix [10].

A splitting $A = M - N$ is called *regular* if $M^{-1} \geq 0$ and $N \geq 0$, *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, and *convergent* if $\rho(M^{-1}N) < 1$, where $\rho(B)$ denotes the *spectral radius* of a square matrix B. It is well known that if $A = M - N$ is a weak regular splitting of A, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$ [1, 8, 10]. From now on, let I denote the identity matrix of order n.

Let $A = M - N$ be a splitting of A and $M = F - G$ be a splitting of M. Then, the *two-stage iterative method* for solving the linear system (1) is written as

$$x_i = Hx_{i-1} + Rb, \qquad i = 1, 2, \ldots,$$

where

$$H = (F^{-1}G)^s + \left(\sum_{j=0}^{s-1}(F^{-1}G)^j\right) F^{-1}N \quad \text{and} \quad R = \sum_{j=0}^{s-1}(F^{-1}G)^j F^{-1}. \tag{4}$$

It was shown in [4, 5] that the two-stage iterative method converges to the exact solution of the linear system (1) for $A^{-1} \geq 0$ when the outer splitting $A = M - N$ is regular and the inner splitting $M = F - G$ is weak regular.

A collection of triples (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, is called a *multisplitting* of A if $A = M_k - N_k$ is a splitting of A for $k = 1, 2, \ldots, \ell$, and E_k's are nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$. The *multisplitting method* associated with this multisplitting for solving the linear system (1) is as follows.

Algorithm 1 (*Multisplitting method*)

Given an initial vector x_0
For $i = 1, 2, \ldots,$ until convergence
  For $k = 1$ to ℓ
    $M_k y_k = N_k x_{i-1} + b$
  $x_i = \sum_{k=1}^{\ell} E_k y_k$

Algorithm 1 was first introduced by O'Leary and White [7] and was further studied by many authors [3, 6, 11]. The big advantage of Algorithm 1 is that the loop k can be executed completely in parallel by different processors.

Remark 2.1. Multisplitting method (Algorithm 1) using splittings $A = M_k - N_k$ $(k = 1, 2, \ldots, \ell)$ can be written as

$$x_i = \hat{H} x_{i-1} + \hat{R} b, \qquad i = 1, 2, \ldots,$$

where

$$\hat{H} = \sum_{k=1}^{\ell} E_k M_k^{-1} N_k \quad \text{and} \quad \hat{R} = \sum_{k=1}^{\ell} E_k M_k^{-1}.$$

Convergence of Algorithm 1 was established for $A^{-1} \geq 0$ when the splittings $A = M_k - N_k$ are weak regular [7].

When the linear systems in Algorithm 1 are also solved iteratively in each processor using the splittings $M_k = F_k - G_k$, one obtains the following *two-stage multisplitting method.*

Algorithm 2 (*Two-stage multisplitting method*)

Given an initial vector x_0
For $i = 1, 2, \ldots,$ until convergence
  For $k = 1$ to ℓ
    $y_{k,0} = x_{i-1}$
    For $j = 1$ to s
      $F_k y_{k,j} = G_k y_{k,j-1} + N_k x_{i-1} + b$
  $x_i = \sum_{k=1}^{\ell} E_k y_{k,s}$

Remark 2.2. Two-stage multisplitting method (Algorithm 2) using $A = M_k - N_k$ $(k = 1, 2, \ldots, \ell)$ as outer splittings and $M_k = F_k - G_k$ $(k = 1, 2, \ldots, \ell)$ as inner splittings can be written as

$$x_i = \tilde{H}x_{i-1} + \tilde{R}b, \quad i = 1, 2, \ldots,$$

where

$$\tilde{H} = \sum_{k=1}^{\ell} E_k H_k, \quad H_k = (F_k^{-1}G_k)^s + \left(\sum_{j=0}^{s-1}(F_k^{-1}G_k)^j\right) F_k^{-1}N_k, \quad 1 \le k \le \ell$$

and

$$\tilde{R} = \sum_{k=1}^{\ell} E_k R_k, \quad R_k = \sum_{j=0}^{s-1}(F_k^{-1}G_k)^j F_k^{-1}, \quad 1 \le k \le \ell.$$

Convergence of Algorithm 2 for any number of inner iterations $s \ge 1$ was established for $A^{-1} \ge 0$ when the outer splittings $A = M_k - N_k$ are regular and the inner splittings $M_k = F_k - G_k$ are weak regular [9].

In Remark 2.2, it is easy to show that $R_k A = I - H_k$ for $k = 1, 2, \ldots, \ell$.

Remark 2.3. From Remarks 2.1 and 2.2, it can be easily shown that two-stage multisplitting method with $A = M_k - N_k$ ($k = 1, 2, \ldots, \ell$) as outer splittings and $M_k = F_k - G_k$ ($k = 1, 2, \ldots, \ell$) as inner splittings can be viewed as multisplitting method with the splittings $A = R_k^{-1} - R_k^{-1}H_k$ ($k = 1, 2, \ldots, \ell$), where R_k and H_k are defined as in Remark 2.2 and R_k's are assumed to be nonsingular.

Theorem 2.1 ([9]). *Let $A^{-1} \ge 0$, and let $A = M - N$ be a regular splitting and $M = F - G$ be a weak regular splitting. Then, the induced splitting $A = R^{-1} - R^{-1}H$ from two-stage iterative method with $A = M - N$ as an outer splitting and $M = F - G$ as an inner splitting is a weak regular splitting and $\rho(H) < 1$, where R and H are defined as in* (4).

3. Induced splittings from multisplitting methods

In this section, we will give some sufficient conditions which guarantee that the induced splitting from multisplitting method is regular.

Theorem 3.1. *Let $A^{-1} \ge 0$, and let $A = M_k - N_k$ $(1 \le k \le \ell)$ be weak regular splittings, and E_k's be nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$ and $E_k = \alpha_k I$. If there exists a nonnegative matrix U to satisfy*

$$(A + U)M_k^{-1}N_k \ge U, \quad 1 \le k \le \ell, \tag{5}$$

then the induced splitting $A = \hat{R}^{-1} - \hat{R}^{-1}\hat{H}$ from multisplitting method with $A = M_k - N_k$ $(1 \le k \le \ell)$ is a regular splitting, where

$$\hat{H} = \sum_{k=1}^{\ell} E_k M_k^{-1} N_k \quad \textit{and} \quad \hat{R} = \sum_{k=1}^{\ell} E_k M_k^{-1}.$$

Proof. Since $A^{-1} \geq 0$ and the splittings $A = M_k - N_k$ are weak regular, $\hat{H}$ and $\hat{R}$ are nonnegative and $\rho(\hat{H}) < 1$ [7]. Thus, $(I - \hat{H})^{-1}$ exists and

$$(I - \hat{H})^{-1} = \sum_{i=0}^{\infty} \hat{H}^i \geq 0.$$

Since $A = \hat{R}^{-1} - \hat{R}^{-1}\hat{H}$, $A\hat{H} = \hat{R}^{-1}\hat{H}(I - \hat{H})$, which implies that

$$\hat{R}^{-1}\hat{H} = A\hat{H}(I - \hat{H})^{-1}. \tag{6}$$

Using the assumption (5), one obtains

$$\begin{aligned} (A+U)M_k^{-1}N_k \geq U \implies & E_k(A+U)M_k^{-1}N_k \geq E_kU \\ \implies & (A+U)\hat{H} \geq U \\ \implies & A\hat{H}(I - \hat{H})^{-1} \geq U \geq 0. \end{aligned} \tag{7}$$

From (6) and (7), $\hat{R}^{-1}\hat{H} \geq 0$. Thus, the splitting $A = \hat{R}^{-1} - \hat{R}^{-1}\hat{H}$ is regular. □

Corollary 3.2. *Let $A^{-1} \geq 0$, and let $A = M_k - N_k$ $(1 \leq k \leq \ell)$ be weak regular splittings, and E_k's be nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$ and $E_k = \alpha_k I$. If $N_k M_k^{-1} N_k \leq N_k$ $(1 \leq k \leq \ell)$, then the induced splitting $A = \hat{R}^{-1} - \hat{R}^{-1}\hat{H}$ from multisplitting method with $A = M_k - N_k$ $(1 \leq k \leq \ell)$ is a regular splitting, where $\hat{H}$ and $\hat{R}$ are defined as in Theorem 3.1.*

Proof. $N_k M_k^{-1} N_k \leq N_k$ $(1 \leq k \leq \ell)$ implies that

$$AM_k^{-1}N_k = N_k - N_k M_k^{-1} N_k \geq 0, \quad 1 \leq k \leq \ell,$$

which can be obtained directly from $U = 0$ in Theorem 3.1. Hence, this corollary follows. □

Theorem 3.3. *Let $A^{-1} \geq 0$, and let $A = M_k - N_k$ $(1 \leq k \leq \ell)$ be weak regular splittings, and E_k's be nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$ and $E_k = \alpha_k I$. Let $N_a = \sum_{k=1}^{\ell} \alpha_k N_k$ and $N_b \geq N_k$ for $1 \leq k \leq \ell$. If there exists a nonnegative matrix U to satisfy*

$$(N_b - U)M_k^{-1}N_k \leq N_a - U, \quad 1 \leq k \leq \ell, \tag{8}$$

then the induced splitting $A = \hat{R}^{-1} - \hat{R}^{-1}\hat{H}$ from multisplitting method with $A = M_k - N_k$ $(1 \leq k \leq \ell)$ is a regular splitting, where $\hat{H}$ and $\hat{R}$ are defined as in Theorem 3.1.

Proof. From the proof of Theorem 3.1, it was shown that

$$\hat{R} \geq 0, \quad (I - \hat{H})^{-1} \geq 0, \quad \hat{R}^{-1}\hat{H} = A\hat{H}(I - \hat{H})^{-1}.$$

Using these relations, one obtains

$$\hat{R}^{-1}\hat{H} = A\sum_{k=1}^{\ell} E_k M_k^{-1} N_k (I - \hat{H})^{-1} = \sum_{k=1}^{\ell} \alpha_k A M_k^{-1} N_k (I - \hat{H})^{-1}. \tag{9}$$

Since $A^{-1} \geq 0$ and the splittings $A = M_k - N_k$ $(1 \leq k \leq \ell)$ are weak regular, $\rho(M_k^{-1}N_k) < 1$ and thus $(I - M_k^{-1}N_k)^{-1}$ exists for $1 \leq k \leq \ell$. Then, equation (9) can be written as

$$\hat{R}^{-1}\hat{H} = \sum_{k=1}^{\ell} \alpha_k A M_k^{-1} N_k (I - M_k^{-1}N_k)^{-1}(I - M_k^{-1}N_k)(I - \hat{H})^{-1}. \tag{10}$$

Notice that

$$\begin{aligned} \alpha_k A M_k^{-1} N_k (I - M_k^{-1}N_k)^{-1} &= \alpha_k (M_k - N_k) M_k^{-1} N_k (I - M_k^{-1}N_k)^{-1} \\ &= \alpha_k N_k (I - M_k^{-1}N_k)(I - M_k^{-1}N_k)^{-1} \\ &= \alpha_k N_k. \end{aligned} \tag{11}$$

Substituting equation (11) into equation (10), one obtains

$$\begin{aligned} \hat{R}^{-1}\hat{H} &= \sum_{k=1}^{\ell} \alpha_k N_k (I - M_k^{-1}N_k)(I - \hat{H})^{-1} \\ &= \left(N_a - \sum_{k=1}^{\ell} E_k N_k M_k^{-1} N_k \right) (I - \hat{H})^{-1}. \end{aligned} \tag{12}$$

Since $N_k \leq N_b$ $(1 \leq k \leq \ell)$,

$$\sum_{k=1}^{\ell} E_k N_k M_k^{-1} N_k \leq \sum_{k=1}^{\ell} E_k N_b M_k^{-1} N_k = N_b \sum_{k=1}^{\ell} E_k M_k^{-1} N_k = N_b \hat{H}. \tag{13}$$

Substituting (13) into (12), one obtains

$$\hat{R}^{-1}\hat{H} \geq (N_a - N_b\hat{H})(I - \hat{H})^{-1}. \tag{14}$$

Using the assumption (8), one obtains

$$\begin{aligned} (N_b - U)M_k^{-1}N_k \leq N_a - U \implies & E_k(N_b - U)M_k^{-1}N_k \leq E_k(N_a - U) \\ \implies & \sum_{k=1}^{\ell} E_k(N_b - U)M_k^{-1}N_k \leq \sum_{k=1}^{\ell} E_k(N_a - U) \\ \implies & (N_b - U)\hat{H} \leq N_a - U \\ \implies & (N_a - N_b\hat{H})(I - \hat{H})^{-1} \geq U \geq 0. \end{aligned} \tag{15}$$

From (14) and (15), $\hat{R}^{-1}\hat{H} \geq 0$. Thus, the splitting $A = \hat{R}^{-1} - \hat{R}^{-1}\hat{H}$ is regular. □

Notice that Wang and Zhao [11] have shown regularity of induced splittings from multisplitting method under the assumption that $A = M_k - N_k$ are regular splittings. However, this paper shows the regularity of induced splittings from multisplitting method under the assumption that $A = M_k - N_k$ are weak regular splittings.

4. Induced splittings from two-stage multisplitting methods

In this section, we will give some sufficient conditions which guarantee that the induced splitting from two-stage multisplitting method is regular.

Theorem 4.1. *Let $A^{-1} \geq 0$, and let $A = M_k - N_k$ $(1 \leq k \leq \ell)$ be regular splittings, $M_k = F_k - G_k$ $(1 \leq k \leq \ell)$ be weak regular splittings, and E_k's be nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$ and $E_k = \alpha_k I$. If there exists a nonnegative matrix U to satisfy*

$$(A+U)H_k \geq U, \quad 1 \leq k \leq \ell, \tag{16}$$

where

$$H_k = (F_k^{-1}G_k)^s + \left(\sum_{j=0}^{s-1}(F_k^{-1}G_k)^j\right) F_k^{-1}N_k, \quad 1 \leq k \leq \ell,$$

then the induced splitting $A = \tilde{R}^{-1} - \tilde{R}^{-1}\tilde{H}$ from two-stage multisplitting method with $A = M_k - N_k$ as outer splittings and $M_k = F_k - G_k$ as inner splittings is a regular splitting, where

$$\tilde{H} = \sum_{k=1}^{\ell} E_k H_k, \quad \tilde{R} = \sum_{k=1}^{\ell} E_k R_k, \quad R_k = \sum_{j=0}^{s-1}(F_k^{-1}G_k)^j F_k^{-1}, \quad 1 \leq k \leq \ell.$$

Proof. From Theorem 2.1, the splitting $A = R_k^{-1} - R_k^{-1}H_k$ is weak regular for each k. Thus, R_k and H_k are nonnegative for $1 \leq k \leq \ell$, which implies that $\tilde{R}$ and $\tilde{H}$ are nonnegative. Since $A = \tilde{R}^{-1} - \tilde{R}^{-1}\tilde{H}$, one obtains

$$A\tilde{H} = \tilde{R}^{-1}\tilde{H}(I - \tilde{H}). \tag{17}$$

Since $A^{-1} \geq 0$ and $A = \tilde{R}^{-1} - \tilde{R}^{-1}\tilde{H}$ is weak regular, $\rho(\tilde{H}) < 1$ and thus

$$(I - \tilde{H})^{-1} = \sum_{i=0}^{\infty} \tilde{H}^i \geq 0. \tag{18}$$

From the assumption (16), one can obtain

$$A\tilde{H} = A\sum_{k=1}^{\ell} \alpha_k H_k = \sum_{k=1}^{\ell} \alpha_k A H_k \geq U \sum_{k=1}^{\ell} \alpha_k (I - H_k) = U(I - \tilde{H}). \tag{19}$$

From (18) and (19), one obtains

$$A\tilde{H}(I - \tilde{H})^{-1} \geq U \geq 0. \tag{20}$$

From (17) and (20), $\tilde{R}^{-1}\tilde{H} = A\tilde{H}(I - \tilde{H})^{-1} \geq 0$. Thus, the splitting $A = \tilde{R}^{-1} - \tilde{R}^{-1}\tilde{H}$ is regular, which completes the proof. □

Corollary 4.2. *Let $A^{-1} \geq 0$, and let $A = M_k - N_k$ $(1 \leq k \leq \ell)$ be regular splittings, $M_k = F_k - G_k$ $(1 \leq k \leq \ell)$ be weak regular splittings, and E_k's be nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$ and $E_k = \alpha_k I$. If $M_k H_k \geq N_k H_k$ $(1 \leq k \leq \ell)$, then the induced splitting $A = \tilde{R}^{-1} - \tilde{R}^{-1}\tilde{H}$ from* *two-stage multisplitting method with $A = M_k - N_k$ $(1 \le k \le \ell)$ as outer splittings and $M_k = F_k - G_k$ $(1 \le k \le \ell)$ as inner splittings is a regular splitting, where R_k, H_k, $\tilde{R}$ and $\tilde{H}$ are defined as in Theorem 4.1.*

Proof. $M_kH_k \ge N_kH_k$ $(1 \le k \le \ell)$ implies that $AH_k = (M_k - N_k)H_k \ge 0$ $(1 \le k \le \ell)$, which can be obtained directly from $U = 0$ in Theorem 4.1. Hence, this corollary follows. □

Corollary 4.3. *Let $A^{-1} \ge 0$, and let $A = M_k - N_k$ $(1 \le k \le \ell)$ be regular splittings, $M_k = F_k - G_k$ $(1 \le k \le \ell)$ be weak regular splittings, and E_k's be nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$ and $E_k = \alpha_k I$. If $R_k^{-1}H_k(I - H_k) \ge 0$ $(1 \le k \le \ell)$, then the induced splitting $A = \tilde{R}^{-1} - \tilde{R}^{-1}\tilde{H}$ from two-stage multisplitting method with $A = M_k - N_k$ $(1 \le k \le \ell)$ as outer splittings and $M_k = F_k - G_k$ $(1 \le k \le \ell)$ as inner splittings is a regular splitting, where R_k, H_k, $\tilde{R}$ and $\tilde{H}$ are defined as in Theorem 4.1.*

Proof. Since $A = R_k^{-1} - R_k^{-1}H_k$, $AH_k = R_k^{-1}H_k(I - H_k) \ge 0$ by assumption, which can be obtained directly from $U = 0$ in Theorem 4.1. Hence, this corollary follows. □

5. Comparison theorems for two-stage multisplitting methods

In this section, we will provide some comparison results for two-stage multisplitting methods of monotone matrices.

Lemma 5.1 ([2]). *Let $A^{-1} \ge 0$ and $A = M_1 - N_1 = M_2 - N_2$ be weak regular splittings. If $M_1^{-1} \ge M_2^{-1}$, and $M_1^{-1}N_1$ and $M_2^{-1}N_2$ are irreducible matrices, then $\rho(M_1^{-1}N_1) \le \rho(M_2^{-1}N_2)$.*

Theorem 5.2 ([11]). *Let $A^{-1} \ge 0$, and let $A = M_1 - N_1 = M_2 - N_2$ be regular splittings, and $M_1 = F_1 - G_1$ and $M_2 = F_2 - G_2$ be weak regular splittings. If the following conditions are satisfied:*

(1) $(F_1^{-1}G_1)^j$ and $(F_2^{-1}G_2)^j$ are irreducible matrices for $j = 1, 2, \ldots, s$,

(2) $N_1 \le N_2$,

(3) $F_1^{-1} \ge F_2^{-1}$,

then $R_1 \ge R_2$, and H_1 and H_2 are irreducible , where for $i = 1, 2$

$$H_i = (F_i^{-1}G_i)^s + \left(\sum_{j=0}^{s-1}(F_i^{-1}G_i)^j\right) F_i^{-1}N_i \quad \text{and} \quad R_i = \sum_{j=0}^{s-1}(F_i^{-1}G_i)^j F_i^{-1}.$$

Theorem 5.3. *Let $A^{-1} \ge 0$, and let $A = M_k - N_k = \bar{M}_k - \bar{N}_k$ $(1 \le k \le \ell)$ be regular splittings, $M_k = F_k - G_k$ and $\bar{M}_k = \bar{F}_k - \bar{G}_k$ be weak regular splittings for $1 \le k \le \ell$, and E_k's be nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$ and $E_k = \alpha_k I$. If the following conditions are satisfied for $1 \le k \le \ell$:*

(1) $(F_k^{-1}G_k)^j$ and $(\bar{F}_k^{-1}\bar{G}_k)^j$ are irreducible matrices for $j = 1, 2, \ldots, s$,

(2) $N_k \le \bar{N}_k$,

(3) $F_k^{-1} \ge \bar{F}_k^{-1}$,

then $\rho(\tilde{H}) \le \rho(\tilde{H}_*)$, *where*

$$\tilde{H} = \sum_{k=1}^{\ell} E_k H_k, \quad H_k = (F_k^{-1}G_k)^s + \left(\sum_{j=0}^{s-1}(F_k^{-1}G_k)^j\right) F_k^{-1}N_k, \quad 1 \le k \le \ell,$$

$$\tilde{R} = \sum_{k=1}^{\ell} E_k R_k, \quad R_k = \sum_{j=0}^{s-1}(F_k^{-1}G_k)^j F_k^{-1}, \qquad 1 \le k \le \ell$$

and

$$\tilde{H}_* = \sum_{k=1}^{\ell} E_k \bar{H}_k, \quad \bar{H}_k = (\bar{F}_k^{-1}\bar{G}_k)^s + \left(\sum_{j=0}^{s-1}(\bar{F}_k^{-1}\bar{G}_k)^j\right) \bar{F}_k^{-1}\bar{N}_k, \quad 1 \le k \le \ell,$$

$$\tilde{R}_* = \sum_{k=1}^{\ell} E_k \bar{R}_k, \quad \bar{R}_k = \sum_{j=0}^{s-1}(\bar{F}_k^{-1}\bar{G}_k)^j \bar{F}_k^{-1}, \qquad 1 \le k \le \ell.$$

Proof. From the assumptions, it can be easily shown that R_k, $\bar{R}_k$, H_k and $\bar{H}_k$ are nonnegative for $1 \le k \le \ell$. Thus, $\tilde{R}$, $\tilde{R}_*$, $\tilde{H}$ and $\tilde{H}_*$ are also nonnegative. From these facts, we can see that $A = \tilde{R}^{-1} - \tilde{R}^{-1}\tilde{H} = \tilde{R}_*^{-1} - \tilde{R}_*^{-1}\tilde{H}_*$ are weak regular splittings. In addition, it is obvious from Theorem 5.2 that $R_k \ge \bar{R}_k$, and H_k and $\bar{H}_k$ are irreducible matrices for $1 \le k \le \ell$. Thus, $\tilde{R} \ge \tilde{R}_*$, and $\tilde{H}$ and $\tilde{H}_*$ are irreducible matrices. Hence, we have from Lemma 5.1 that $\rho(\tilde{H}) \le \rho(\tilde{H}_*)$. □

Theorem 5.4 ([11]). *Let* $A^{-1} \ge 0$, *and let* $A = M_k - N_k = \bar{M}_k - \bar{N}_k$ $(1 \le k \le \ell)$ *be weak regular splittings, and* E_k*'s be nonnegative diagonal matrices such that* $\sum_{k=1}^{\ell} E_k = I$ *and* $E_k = \alpha_k I$. *If* $M_k^{-1}N_k$ *and* $\bar{M}_k^{-1}\bar{N}_k$ *are irreducible matrices and* $M_k^{-1} \ge \bar{M}_k^{-1}$ *for* $1 \le k \le \ell$, *then* $\rho(\hat{H}) \le \rho(\hat{H}_*)$, *where*

$$\hat{H} = \sum_{k=1}^{\ell} E_k M_k^{-1} N_k \quad \text{and} \quad \hat{H}_* = \sum_{k=1}^{\ell} E_k \bar{M}_k^{-1} \bar{N}_k.$$

Theorem 5.5. *Let* $A^{-1} \ge 0$, *and let* $A = M_k - N_k = \bar{M}_k - \bar{N}_k$ $(1 \le k \le \ell)$ *be regular splittings,* $M_k = F_k - G_k$ *and* $\bar{M}_k = \bar{F}_k - \bar{G}_k$ *be weak regular splittings for* $1 \le k \le \ell$, *and* E_k*'s be nonnegative diagonal matrices such that* $\sum_{k=1}^{\ell} E_k = I$ *and* $E_k = \alpha_k I$. *If* H_k *and* $\bar{H}_k$ *are irreducible matrices and* $R_k \ge \bar{R}_k$ *for* $1 \le k \le \ell$, *then* $\rho(\tilde{H}) \le \rho(\tilde{H}_*)$, *where*

$$\tilde{H} = \sum_{k=1}^{\ell} E_k H_k \quad \text{and} \quad \tilde{H}_* = \sum_{k=1}^{\ell} E_k \bar{H}_k$$

R_k, $\bar{R}_k$, H_k *and* $\bar{H}_k$ *are defined as in Theorem 5.3.*

Proof. Let

$$\tilde{R} = \sum_{k=1}^{\ell} E_k R_k \quad \text{and} \quad \tilde{R}_* = \sum_{k=1}^{\ell} E_k \bar{R}_k.$$

From Remark 2.3, the splittings $A = \tilde{R}^{-1} - \tilde{R}^{-1}\tilde{H}$ and $A = \tilde{R}_*^{-1} - \tilde{R}_*^{-1}\tilde{H}_*$ can be considered as the induced splittings from multisplitting method with $A = R_k^{-1} - R_k^{-1}H_k$ and $A = \bar{R}_k^{-1} - \bar{R}_k^{-1}\bar{H}_k$ for $1 \le k \le \ell$, respectively. From Theorem 2.1, the splittings $A = R_k^{-1} - R_k^{-1}H_k$ and $A = \bar{R}_k^{-1} - \bar{R}_k^{-1}\bar{H}_k$ are weak regular for $1 \le k \le \ell$. Hence, this theorem follows from Theorem 5.4. □

Example 5.6. Consider a matrix $A \in \mathbb{R}^{n\times n}$ of the form

$$A = \begin{pmatrix} 4 & -1 & -1 & & \\ -1 & 4 & -1 & -1 & \\ -1 & -1 & 4 & -1 & -1 \\ & \ddots & \ddots & \ddots & \ddots \\ & & -1 & -1 & 4 \end{pmatrix} \stackrel{\text{let}}{=} (-1,\ -1,\ 4,\ -1,\ -1),$$

where $n \ge 3$. Since A is an irreducibly diagonally dominant Z-matrix, $A^{-1} \ge 0$. For $1 \le k \le \ell$, let $\beta_k = k/100$, $\gamma_k = \beta_k + 0.1$, $\delta_k = -\beta_k/5$ and $\varepsilon_k = \delta_k - 1$. Then $A = M_k - N_k = \bar{M}_k - \bar{N}_k$, $M_k = F_k - G_k$ and $\bar{M}_k = \bar{F}_k - \bar{G}_k$ are splittings for $1 \le k \le \ell$, where

$$\begin{aligned} M_k &= (-1,\ \beta_k - 1,\ 4,\ \beta_k - 1,\ -1), & N_k &= (0,\ \beta_k,\ 0,\ \beta_k,\ 0),\\ \bar{M}_k &= (-1,\ \gamma_k - 1,\ 4,\ \gamma_k - 1,\ -1), & \bar{N}_k &= (0,\ \gamma_k,\ 0,\ \gamma_k,\ 0),\\ F_k &= (\varepsilon_k,\ 2\beta_k - 1,\ \gamma_k + 4,\ 2\beta_k - 1,\ \varepsilon_k), & G_k &= (\delta_k,\ \beta_k,\ \gamma_k,\ \beta_k,\ \delta_k),\\ \bar{F}_k &= (\varepsilon_k,\ \beta_k + \gamma_k - 1,\ \gamma_k + 4,\ \beta_k + \gamma_k - 1,\ \varepsilon_k), & \bar{G}_k &= (\delta_k,\ \beta_k,\ \gamma_k,\ \beta_k,\ \delta_k). \end{aligned}$$

Table 1. Numerical results of Example 5.6

n	s	ℓ	$\rho(\tilde{H})$	$\rho(\tilde{H}_*)$
100	2	8	0.983286	0.989631
		16	0.990816	0.993123
		32	0.995171	0.995894
		45	0.996515	0.996907
	4	8	0.976190	0.987457
		16	0.987356	0.991431
		32	0.993484	0.994764
		45	0.995326	0.996022

Assume that $\ell \le 45$ and $E_k = (1/\ell)I$ for $1 \le k \le \ell$. Using Mathematica which is one of computer algebra systems, it can be easily shown that the splittings satisfy all assumptions of Theorem 5.3 for $1 \le k \le \ell$. From Theorem 5.3,

$\rho(\tilde{H}) < \rho(\tilde{H}_*)$, where $\tilde{H}$ and $\tilde{H}_*$ are defined as in Theorem 5.3. Here the number 45 is chosen to be the largest integer k such that $\bar{F}_k$ is an M-matrix. Numerical experiments for various values of ℓ, $n = 100$ and $s = 2, 4$ are listed in Table 1.

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