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ON THE STRONG CONVERGENCE THEOREMS FOR ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

SHIH-SEN CHANG*, LIANG CAI ZHAO AND DING PING WU

ABSTRACT. Some strong convergence theorems of explicit iteration scheme for asymptotically nonexpansive semi-groups in Banach spaces are established. The results presented in this paper extend and improve some recent results in [T. Suzuki. On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces, Proc. Amer. Math. Soc. 131(2002)2133–2136; H. K. Xu. A strong convergence theorem for contraction semigroups in Banach spaces, Bull. Aust. Math. Soc. 72(2005)371–379; N. Shioji and W. Takahashi. Strong convergence theorems for continuous semigroups in Banach spaces, Math. Japonica. 1(1999)57–66; T. Shimizu and W. Takahashi. Strong convergence to common fixed points of families of nonexpansive mappings, J. Math. Anal. Appl. 211(1997)71–83; N. Shioji and W. Takahashi. Strong convergence theorems for asymptotically nonexpansive mappings in Hilbert spaces, Nonlinear Anal. TMA, 34(1998)87–99; H. K. Xu. Approximations to fixed points of contraction semigroups in Hilbert space, Numer. Funct. Anal. Optim. 19(1998), 157–163.]

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1. Introduction

Throughout this paper, we assume that E is a real Banach space, K is a nonempty bounded closed convex subset of E, E^* is the dual space of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^*, \langle x, f\rangle = ||x|| \cdot ||f||,\ ||x|| = ||f||\},\ \ x \in E. \tag{1.1}$$

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where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued normalized duality mapping by j and denote the fixed point set of a mapping T by $F(T) = \{x \in K : x = Tx\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ and $x_n \rightharpoonup x$ will denote strong and weak convergence of the sequence $\{x_n\}$ to x, respectively.

Recall that a mapping $T : K \to K$ is called uniformly L-Lipschitzian, if there exists $L > 1$ such that for all $x, y \in K$

$$||T^n x - T^n y|| \leq L||x - y||, \qquad \forall n \geq 1.$$

A mapping $T : K \to K$ is called asymptotically nonexpansive mapping, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in K$

$$||T^n x - T^n y|| \leq k_n||x - y||, \qquad \forall n \geq 1.$$

It is clear that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian with $L = \sup_{n\geq 1}\{k_n\}$. The asymptotically nonexpansive mappings were introduced by Goebel and Kirk [6], and they proved that if K is a nonempty bounded closed convex subset of a uniformly convex Banach space E, and $T : K \to K$ is asymptotically nonexpansive, then T has a fixed point in K. Several authors have invesigated iterative methods for approximating fixed points of asymptotically nonexpansive mappings (see [2, 3, 5, 8, 10, 14]).

Let $\{T(t) : t \in [0, \infty)\}$ be a asymptotically nonexpansive semigroup on a bounded closed convex subset K of a Banach space E, if it satisfies the following conditions:

(1) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in K$

$$||T^n(t)x - T^n(t)y|| \leq k_n||x - y||, \quad \forall t \in [0, \infty), \ \forall\, n \geq 0;$$

(2) $T(0)x = x$ for all $x \in K$;

(3) $T(s + t) = T(s) \circ T(t)$ for all $s, t \in [0, \infty)$;

(4) for each $x \in K$, the mapping $T(\cdot)x$ from $[0, \infty)$ into K is continuous.

We put $F := \bigcap_{t\geq 0} \text{Fix}(T(t))$.

In [12], Shioji and Takahashi introduced in Hilbert space the implicit iteration

$$x_n = \alpha_n x + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad n \in N, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{t_n\}$ a sequence of positive real numbers divergent to ∞, and $x \in C$. Under certain restrictions on the sequence $\{\alpha_n\}$, Shioji and Takahashi [12] proved the strong convergence of $\{x_n\}$ to a member of F (see also [17]). In [11], Shimizu and Takahashi studied the strong convergence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad n \in N, \tag{1.3}$$

in Hilbert space where $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings, $t_n \geq 0$ and $t_n \to \infty$. Shioji and Takahashi [13] extended the results of [11]. They studied the strong convergence of (1.3) and the following sequence for an asymptotically nonexpansive semigroup in a Banach space:

$$x_n = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad n \in N. \tag{1.4}$$

In 2002, Suzuki [15] was the first to introduce again in a Hilbert space the following implicit iteration process:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{1.5}$$

for the nonexpansive semigroup case. In 2005, Xu [16] established a Banach space version of the sequence (1.5) of Suzuki [15].

It is our purpose in this paper is to prove, under appropriate conditions, the two iterative sequence given as follows:

$$x_n = \alpha_n x + (1-\alpha_n)T^n(t_n)x_n, \quad \forall\, n \geq 1. \tag{1.6}$$

$$y_{n+1} = \beta_n x + (1-\beta_n)T^n(t_n)y_n, \quad \forall\, n \geq 1. \tag{1.7}$$

converge strongly to $q \in \bigcap_{t\geq 0} \mathrm{Fix}(T(t))$, which is the unique solution in F to the following variational inequality:

$$\langle q - x, j(q-u)\rangle \leq 0 \quad \forall\, u \in F.$$

The results presented in this paper extend and improve some recent results in Shimizu and Takahashi [11], Shioji and Takahashi [12, 13], Suzuki [15], Xu [16, 17].

2. Preliminaries

Let $S := \{x \in E : ||x|| = 1\}$ denote the unit sphere of the Banach space E. The space E is said to have a Gâteaux differentiable norm if the limit

$$\lim_{t\to 0} \frac{||x+ty|| - ||x||}{t}$$

exists for each $x, y \in S$, and we call E smooth when this is the case. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Further, E is said to be uniformly smooth if the limit exists uniformly for $(x, y) \in S \times S$. It is well known that if E is smooth then the duality mapping on E is single-valued, and if E has a uniformly Gâteaux differentiable norm then the duality mapping is norm-to-weak* uniformly continuous on bounded subset of E.

Let K be a nonempty closed convex and bounded subset of a Banach space E and let the diameter of K be defined by $d(K) = \sup\{||x - y|| : x, y \in K\}$. For each $x \in K$, let $r(x, K) = \sup\{||x-y|| : y \in K\}$ and let $r(K) = \inf\{r(x, K) : x \in$

$K\}$ denote the Chebyshev radius of K relative to itself. The normal structure coefficient $N(E)$ of E is defined by

$$N(E) := \inf\{\frac{d(K)}{r(K)} : K \text{ is a closed convex bounded subset of } E \text{ with } d(K) > 0\}.$$

A space E with $N(E) > 1$ is said to have uniform normal structure. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see [5] or [8]).

We denote by LIM the Banach limit. Recall that $LIM \in (l^\infty)^*$ such that $||LIM|| = 1$, $\liminf_{n\to\infty} a_n \le LIM_n a_n \le \limsup_{n\to\infty} a_n$ and $LIM_n a_n = LIM_n a_{n+1}$ for all $\{a_n\} \in l^\infty$.

In the sequel, we shall need the following Lemmas.

Lemma 2.1(Chang [4]). *Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping, then for any $x, y \in E$ the following conclusions holds:*

$$||x+y||^2 \le ||x||^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y);$$
$$||x+y||^2 \ge ||x||^2 + 2\langle y, j(x)\rangle, \quad \forall j(x) \in J(x).$$

Lemma 2.2(Liu [9]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying the following conditions:*

$$a_{n+1} \le (1-\lambda_n)a_n + b_n + c_n, \quad \forall\, n \ge n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\} \subset (0,1)$ with $\sum_{n=0}^\infty \lambda_n = \infty$, $b_n = o(\lambda_n)$, and $\sum_{n=0}^\infty c_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3([8]). *Suppose that E is a Banach space with a uniformly normal structure, K is a nonempty bounded subset of E, and $T : K \to K$ is a uniformly L-Lipschitzian mapping with $L < N(E)^{\frac{1}{2}}$. Suppose also that there exists a nonempty bounded closed convex subset A of K with the following property (P):*

$$x \in A \ \textit{implies}\ \omega_w(x) \subset A, \tag{P}$$

Where $\omega_w(x)$ is the weak $\omega-$limit set of T at x, that is, the set

$$\{y \in E : y = weak\ \omega - \lim T^{n_j}x \ for\ some\ n_j \to \infty\}.$$

Then T has a fixed point in A.

Lemma 2.4([7]). *Let E is a Banach space with uniformly normal structure, K is a nonempty bounded closed convex subset of E, and $T : K \to K$ is an asymptotically nonexpansive mapping. Then T has a fixed point in K.*

3. Main results

Theorem 3.1. *Let E be a real Banach space with uniformly Gâteaux differentiable norm and uniform normal structure, K be a nonempty bounded closed convex subset of E. Let $\{T(t) : t \in [0,\infty)\}$ be a asymptotically nonexpansive semigroup with a sequence $\{k_n\} \subset [1,+\infty), \lim_{n\to\infty} k_n = 1$ on K such that $F := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $x \in K$ be a given point and $t_n \geq 0$. Let $\{\alpha_n\}$ be a sequences in (0, 1) satisfying $\alpha_n \to 0$ (as $n \to \infty$) and $k_n^2 - 1 \leq \alpha_n^2$. Then the following conclusions hold:*

(I) for each $n \geq 1$, there is a unique $x_n \in K$ such that

$$x_n = \alpha_n x + (1-\alpha_n)T^n(t_n)x_n, \tag{3.1}$$

and if, in addition, $\lim_{n\to\infty} ||x_n - T(t)x_n|| = 0$ uniformly in $t \in [0,\infty)$, then

(II) the sequence $\{x_n\}$converges strongly to $q \in F$, which is the unique solution in F to the following variational inequality:

$$\langle q - x, j(q-u)\rangle \leq 0, \quad \forall\, u \in F. \tag{3.2}$$

Proof. The proof of conclusion (I)

For each $n \geq 1$ and $t_n \geq 0$, define a mapping $S_n : K \to K$ by

$$S_n(z) = \alpha_n x + (1-\alpha_n)T^n(t_n)z, \quad z \in K.$$

For all $y, z \in K$, we have

$$\begin{aligned} ||S_n y - S_n z|| &= (1-\alpha_n)||T^n(t_n)y - T^n(t_n)z|| \\ &\leq (1-\alpha_n)k_n||y - z||. \end{aligned}$$

Since $k_n^2 - 1 \leq \alpha_n^2$, for all $n \geq 1$, we have

$$1 - \frac{1}{k_n} \leq \frac{\alpha_n}{k_n(k_n+1)}\alpha_n.$$

Again, since $\{k_n\} \subset [1,+\infty)$ and $\{\alpha_n\} \subset (0,1)$, we have

$$\frac{\alpha_n}{k_n(k_n+1)} < 1.$$

Hence $1 - \frac{1}{k_n} < \alpha_n$, i.e., $1 - \alpha_n < \frac{1}{k_n}$. This implies that

$$(1-\alpha_n)k_n < 1.$$

Thus, it is easy to see that $S_n : K \to K$ is a contractive mapping. By Banach's contraction mapping principle, it yields a unique fixed point $x_n \in K$ such that

$$x_n = \alpha_n x + (1-\alpha_n)T^n(t_n)x_n, \quad \forall\, n \geq 1.$$

The conclusion (I) is proved.

The proof of conclusion (II)

We define the mapping $\phi : K \to \mathfrak{R}$ by

$$\phi(y) = LIM_n||x_n - y||^2, \quad \forall\, y \in K,\ n \geq n_0.$$

Since $\phi(y) \to \infty$ as $||y|| \to \infty$ and ϕ is continuous and convex. Again since E has uniform normal structure, it is reflexive, then there exists $x^* \in K$ such that $\phi(x^*) = \inf_{y\in K} \phi(y)$. this implies that the set

$$C = \{z \in K : \phi(z) = \inf_{y\in K} \phi(y)\} \neq \emptyset.$$

Further, C is bounded closed convex, and C has also property (P) (shown in [8]). By Lemma 2.3 and Lemma 2.4, T has a fixed point $q \in C$. Hence $\bigcap_{t\geq 0} F(T(t)) \bigcap C \neq \emptyset$. Let $q \in F \bigcap C$ and $t \in (0, 1)$, then for any $x \in K$, we have $(1-t)q + tx \in K$. Thus $\phi(q) \leq \phi((1-t)q + tx)$. By Lemma 2.1 we obtain

$$\begin{aligned} 0 &\leq \frac{\phi((1-t)q+tx) - \phi(q)}{t} \\ &= \frac{1}{t} LIM_n(||x_n - ((1-t)q+tx)||^2 - ||x_n - q||^2) \\ &= \frac{1}{t} LIM_n(||(x_n - q) + t(q-x)||^2 - ||x_n - q||^2) \\ &\leq 2LIM_n\langle q - x, j(x_n - q + t(q-x))\rangle. \end{aligned}$$

This implies that

$$LIM_n\langle x - q, j(x_n - q + t(q-x))\rangle \leq 0.$$

Since K is bounded and j is norm-to-weak* uniformly continuous on bounded subset of E, taking limit as $t \to 0$, we have

$$LIM_n\langle x - q, j(x_n - q)\rangle \leq 0. \tag{3.3}$$

Again, it follows from (3.1), we have

$$\alpha_n(x_n - x) = (1-\alpha_n)(T^n(t_n)x_n - x_n),$$

so that for all $u \in F = \bigcap_{t\geq 0} F(T(t))$ and $k_n^2 - 1 \leq \alpha_n^2$, we have

$$\begin{aligned} &\langle x_n - x, j(x_n - u)\rangle \\ &= \frac{1-\alpha_n}{\alpha_n}\langle T^n(t_n)x_n - x_n, j(x_n - u)\rangle \\ &= \frac{1-\alpha_n}{\alpha_n}\{\langle T^n(t_n)x_n - u, j(x_n - u)\rangle \\ &\quad - \langle x_n - u, j(x_n - u)\rangle\} \\ &\leq \frac{1-\alpha_n}{\alpha_n}(||T^n(t_n)x_n - u|| \cdot ||x_n - u|| - ||x_n - u||^2) \\ &\leq \frac{1-\alpha_n}{\alpha_n}(k_n - 1)||x_n - u||^2 \\ &\leq (1-\alpha_n)\alpha_n||x_n - u||^2. \end{aligned}$$

Since $\alpha_n \to 0$ (as $n \to \infty$) and K is bounded, so $\{||x_n - u||\}$ is bounded. Hence we have

$$LIM_n\langle x_n - x, j(x_n - u)\rangle \leq 0. \tag{3.4}$$

Again, since $q \in F = \bigcap_{t \geq 0} F(T(t))$, from (3.4) we have

$$LIM_n \langle x_n - x, j(x_n - q) \rangle \leq 0. \tag{3.5}$$

From inequalities (3.3) and (3.5) we have

$$\begin{aligned} &LIM_n \langle x_n - x, j(x_n - q) \rangle + LIM_n \langle x - q, j(x_n - q) \rangle \\ &= LIM_n \langle x_n - q, j(x_n - q) \rangle \\ &= LIM_n ||x_n - q||^2 \leq 0. \end{aligned}$$

Hence, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \to q \in F$ (as $j \to \infty$). If there exists another subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to p$ (as $k \to \infty$). Again, since $\lim_{n\to\infty} ||x_n - T(t)x_n|| = 0$ uniformly in $t \in [0, \infty)$, then $p \in F$. It follows from (3.3) and (3.5), we have

$$\langle x - q, j(p - q) \rangle \leq 0, \tag{3.6}$$

and

$$\langle p - x, j(p - q) \rangle \leq 0. \tag{3.7}$$

Adding up (3.6) and (3.7) we have

$$\langle p - q, j(p - q) \rangle = ||p - q||^2 \leq 0.$$

This implies that $p = q$. thus, $x_n \to q$ (as $n \to \infty$), and $q \in F$ is unique. Again it follows from (3.4), so that

$$\langle q - x, j(q - u) \rangle \leq 0, \quad \forall\, u \in F = \bigcap_{t \geq 0} F(T(t)).$$

The conclusion (II) is proved. □

Theorem 3.2. *Let E be a real Banach space with uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty bounded closed convex subset of E. Let $\{T(t) : t \in [0, \infty)\}$ be a asymptotically nonexpansive semigroup with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ on K such that $F := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $x \in K$ be a given point and $t_n \geq 0$. Let $\{\alpha_n\}, \{\beta_n\}$ are two sequences in (0, 1) satisfying the following conditions:*

$$(i)\ \alpha_n \to 0, \ \ \beta_n \to 0\ (as\ n \to \infty);$$

$$(ii)\ k_n^2 - 1 \leq \alpha_n^2 \quad and \quad \sum_{n=0}^{\infty} \beta_n = \infty.$$

For any given $y_1 \in K$, define a sequence $\{y_n\}$ by

$$y_{n+1} = \beta_n x + (1 - \beta_n) T^n(t_n) y_n, \quad \forall\, n \geq 1. \tag{3.8}$$

Then the following conclusions hold:

(I) for each $n \geq 1$, there is a unique $x_n \in K$ such that

$$x_n = \alpha_n x + (1 - \alpha_n) T^n(t_n) x_n, \tag{3.9}$$

and if, in addition, $\lim_{n\to\infty} ||x_n - T(t)x_n|| = 0$, $\lim_{n\to\infty} ||y_n - T(t)y_n|| = 0$ uniformly in $t \in [0, \infty)$, then

(II) the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to the same $q \in F$, which is the unique solution in F to the following variational inequality:

$$\langle q - x, j(q-u)\rangle \leq 0, \quad \forall\, u \in F.$$

Proof. From (3.9), we observe that for all $m \geq 1$, $n \geq 1$

$$x_m - y_n = \alpha_m(x - y_n) + (1-\alpha_m)(T^m(t_m)x_m - y_n).$$

Therefore, using Lemma 2.1 we have

$$\begin{aligned}
&||x_m - y_n||^2 \\
&\leq (1-\alpha_m)^2||T^m(t_m)x_m - y_n||^2 + 2\alpha_m\langle x - y_n, j(x_m - y_n)\rangle \\
&= (1-\alpha_m)^2||T^m(t_m)x_m - y_n||^2 + 2\alpha_m\langle x - x_m, j(x_m - y_n)\rangle + 2\alpha_m||x_m - y_n||^2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\langle x - x_m, j(y_n - x_m)\rangle \\
&\leq \frac{1}{2\alpha_m}\{(1-\alpha_m)^2||T^m(t_m)x_m - y_n||^2 + (2\alpha_m - 1)||x_m - y_n||^2\} \\
&\leq \frac{1-2\alpha_m}{2\alpha_m}\{||T^m(t_m)x_m - y_n||^2 - ||x_m - y_n||^2\} + \frac{\alpha_m}{2}||T^m(t_m)x_m - y_n||^2 \\
&\leq \frac{1-2\alpha_m}{2\alpha_m}\{(||T^m(t_m)x_m - T^m(t_m)y_n|| + ||T^m(t_m)y_n - y_n||)^2 - ||x_m - y_n||^2\} \\
&\quad + \frac{\alpha_m}{2}||T^m(t_m)x_m - y_n||^2 \\
&\leq \frac{1-2\alpha_m}{2\alpha_m}\{(k_m||x_m - y_n|| + ||T^m(t_m)y_n - y_n||)^2 - ||x_m - y_n||^2\} \\
&\quad + \frac{\alpha_m}{2}||T^m(t_m)x_m - y_n||^2 \\
&\leq \frac{1-2\alpha_m}{2\alpha_m}\{(k_m^2 - 1)||x_m - y_n||^2 + 2k_m||x_m - y_n||\cdot||T^m(t_m)y_n - y_n|| \\
&\quad + ||T^m(t_m)y_n - y_n||^2\} + \frac{\alpha_m}{2}||T^m(t_m)x_m - y_n||^2.
\end{aligned}$$

Since K is bounded, so that $\{x_n\}$, $\{y_n\}$, $\{||T^m(t_m)x_m - y_n||\}$ are bounded. Let

$$M = \sup_{m\geq 1, n\geq 1}\{||x_m - y_n||, ||x_m - y_n||^2, ||T^m(t_m)x_m - y_n||^2\} < \infty.$$

Therefore, we obtain that

$$\begin{aligned}
&\langle x - x_m, j(y_n - x_m)\rangle \\
&\leq \frac{1-2\alpha_m}{2\alpha_m}\{(k_m^2 - 1)M + 2k_mM||T^m(t_m)y_n - y_n|| \\
&\quad + ||T^m(t_m)y_n - y_n||^2\} + \frac{\alpha_m}{2}M.
\end{aligned} \tag{3.10}$$

By virtue of the assumption that $||y_n - T(t)y_n|| \to 0$ uniformly in $t \in [0,\infty)$ (as $n \to \infty$) and by using induction, we can prove that for all $m \geq 1$

$$||y_n - T^m(t)y_n|| \to 0 \textit{ uniformly in } t \in [0,\infty) \ (\textit{as } n \to \infty).$$

Hence for all $m \geq 1$, we have

$$||y_n - T^m(t_m)y_n|| \to 0 \ (as \ \ n \to \infty). \tag{3.11}$$

By the assumption $k_m^2 - 1 \leq \alpha_m^2$, it follows from (3.10) and (3.11) that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle x - x_m,\ j(y_n - x_m)\rangle \\
&\leq \frac{1-2\alpha_m}{2\alpha_m}(k_m^2 - 1)M + \frac{\alpha_m}{2}M \\
&\leq \frac{1-2\alpha_m}{2\alpha_m}\alpha_m^2 M + \frac{\alpha_m}{2}M \\
&= (1-\alpha_m)\alpha_m M.
\end{aligned} \tag{3.12}$$

Therefore, we have

$$\limsup_{m\to\infty}\ \limsup_{n\to\infty}\langle x - x_m,\ j(y_n - x_m)\rangle \leq 0.$$

But by Theorem 3.1, we have that $x_m \to q \in F := \bigcap_{t\geq 0} F(T(t))$ (as $m \to \infty$). Again since the normal duality mapping j is norm-to-weak* uniformly continuous on bounded subsets of E. Therefore, for any given $\epsilon > 0$, there exists a positive N such that for all $m \geq N$

$$\begin{aligned}
&|\langle x - q,\ j(y_n - q) - j(y_n - x_m)\rangle| < \frac{\epsilon}{3}; \\
&||x - x_m||M < \frac{\epsilon}{3}; \\
&\limsup_{n\to\infty}\langle x - x_m, j(y_n - x_m)\rangle < \frac{\epsilon}{3}.
\end{aligned}$$

This implies that for any $m \geq N$

$$\begin{aligned}
&\limsup_{n\to\infty}\{\langle x - q,\ j(y_n - q)\rangle \\
&= \limsup_{n\to\infty}\{\langle x - q,\ j(y_n - q) - j(y_n - x_m)\rangle \\
&\quad + \langle x - q - (x - x_m),\ j(y_n - x_m)\rangle + \langle x - x_m,\ j(y_n - x_m)\rangle\} \\
&= \limsup_{n\to\infty}\{|\langle x - q,\ j(y_n - q) - j(y_n - x_m)\rangle| + ||x_m - q||M \\
&\quad + \langle x - x_m,\ j(y_n - x_m)\rangle\} \leq \epsilon.
\end{aligned}$$

By the arbitrariness of $\epsilon > 0$, it gets

$$\limsup_{n\to\infty}\langle x - q, j(y_n - q)\rangle \leq 0. \tag{3.13}$$

Let

$$\gamma_n = \max\{\langle x - q, j(y_n - q)\rangle, 0\}, \quad \forall n \geq 1.$$

Next we prove that

$$\lim_{n\to\infty}\gamma_n = 0.$$

Indeed, for any given $\epsilon > 0$ it follows from (3.13) that there exists a positive integer n_1 such that

$$\langle x - q, j(y_n - q)\rangle \leq \epsilon, \quad \forall n \geq n_1.$$

This implies that

$$0 \leq \gamma_n \leq \epsilon \quad \forall n \geq n_1.$$

By the arbitrariness of $\epsilon > 0$, we have that $\lim_{n\to\infty} \gamma_n = 0$.

Again, let $q \in F = \bigcap_{t\geq 0} F(T(t))$, from the sequence $\{y_n\}$ defined by (3.8) and Lemma 2.1 for all $n \geq 1$ we have

$$\begin{aligned} ||y_n - q||^2 &= ||\beta_n(x-q) + (1-\beta_n)(T^n(t_n)y_n - q)||^2 \\ &\leq (1-\beta_n)^2||T^n(t_n)y_n - q||^2 + 2\beta_n\langle x - q, j(y_{n+1} - q)\rangle \\ &\leq (1-\beta_n)k_n^2||y_n - q||^2 + 2\beta_n\langle x - q, j(y_{n+1} - q)\rangle \\ &\leq (1-\beta_n)||y_n - q||^2 + (k_n^2 - 1)||y_n - q||^2 + 2\beta_n\gamma_{n+1} \\ &\leq (1-\beta_n)||y_n - q||^2 + (k_n - 1)M_1 + 2\beta_n\gamma_{n+1}. \end{aligned} \tag{3.14}$$

where $M_1 = \sup_{n\geq 1}\{(k_n+1)\cdot ||y_n - q||^2\}$.

Take $a_n = ||y_n - q||^2$, $\lambda_n = \beta_n$, $b_n = 2\beta_n\gamma_{n+1}$ and $c_n = (k_n - 1)M_1$, in Lemma 2.2. By the assumptions, it is easy to prove that $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, hence the conditions in Lemma 2.2 are satisfied, and so we have

$$\lim_{n\to\infty} ||y_n - q|| = 0, \quad \text{i.e.,} \quad y_n \to q \in F.$$

Further, since $\lim_{n\to\infty} y_n = q \in F$, then by Theorem 3.1 we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = q \in F = \bigcap_{t\geq 0} F(T(t)).$$

and q is the unique solution in F to the following variational inequality:

$$\langle q - x, j(q-u)\rangle \leq 0, \quad \forall\, u \in F.$$

The proof of theorem 3.2 is completed.

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Shih-sen Chang is a professor of Mathematics of Yibin University and Sichuan University. During 1992 to 2007, he visited Swiss, America, Canada, Australia, Italy, France, Taiwan, Hong Kong etc. as a visiting Professor. He visited Korea as a visiting Professor more than twenty times. His research interests focus on Variational Inequality theory, fixed point theory, Integral equation theory, and random Analysis theory. 1995. From 1985-2000 he received The Foundation of National Natural Science of China five times.

Department of Mathematics, Yibin University, Yibin, Sichuan 644007, China
e-mail: sszhang_1@yahoo.com.cn

Liang Cai Zhao
Department of Mathematics, Yibin University, Yibin, Sichuan 644007, China
e-mail: zhaolcyb@yahoo.com.cn

Ding Ping Wu
Department of Mathematics, Yibin University, Yibin, Sichuan 644007, China
e-mail: wdp68@163.com