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ON HOMOMORPHISMS ON CSÁSZÁR FRAMES

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Abstract. We introduce a concept of continuous homomorphisms between Császár frames and show that the Cauchy completion in CsFrm gives rise to a coreflection in the category PCsFrm (resp. UCsFrm) consisting of proximal Császár frames and uniform continuous homomorphisms (resp. uniform Császár frames and uniform continuous homomorphisms).

1. Introduction

Many authors have focused attention on the fact that the important aspect of a topological space is not its set of points but its lattice of open subsets. The study of topological properties from a lattice-theoretic viewpoint was initiated by Wallman. Theory of frames was introduced by C. Ehresmann and J. Bénabou (*cf.* [10]). A *frame* is a complete lattice L satisfying the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

for all $x \in L$ and $S \subseteq L$.

In view of the fact that in a set, uniformities, proximities, nearness structures and syntopogenous structures determine a topology on the set, such structures must appear as additional structures on frames. For such a reason, the study of structured frames was introduced by Isbell ([9]) and in recent years, various authors are studying structured frames (*cf.* [4, 5, 8, 11]).

This paper is a continuation of the previous paper [5] in which we introduce Császár frames generalizing syntopogenous spaces [6]. In this paper, we introduce a concept of continuous homomorphisms preserving Császár frame structures and investigate some properties of continuous homomorphisms. Moreover, we show that the Cauchy completion in CsFrm gives rise to a coreflection in the category PCsFrm (resp. UCsFrm) consisting of proximal Császár frames

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and uniform continuous homomorphisms (resp. uniform Császár frames and uniform continuous homomorphisms). Now we briefly recall some of basic concepts of frame theory and introduce notation and terminology. For the general background of frames and Császár frames, we refer to ([10], [5], resp.) and for the category theory, we refer to [1].

In a frame L, we have the relation $a \prec b$ by $a^* \vee b = e$, where a^* is the *pseudocomplement* of a in L, given by

$$a^* = \bigvee\{x \in L \mid x \wedge a = 0\}.$$

A *Császár order* $\lhd$ is a binary relation on L satisfying the following:

(1) $0 \lhd 0$ and $e \lhd e$,
(2) $x \lhd y$ implies $x \leq y$,
(3) $x \leq a \lhd b \leq y$ implies $x \lhd y$.

A Császár order $\lhd$ is *symmetric* if $x \lhd y$ implies $y^* \lhd x^*$. A *Császár frame* is a pair $(L, \mathcal{L})$, where L is a frame and $\mathcal{L}$ is a set of Császár orders on L with the following properties:

(1) $\mathcal{L}$ is up-directed.
(2) every member of $\mathcal{L}$ is a meet-sublattice of $L \times L$.
(3) $\mathcal{L}$ is admissible, i.e., for any $x \in L$, $x = \bigvee\{y \in L \mid y \lhd_{\mathcal{L}} x\}$ where $\lhd_{\mathcal{L}} = \cup\{\lhd \in \mathcal{L}\}$.

A *frame homomorphism* is a map $h : M \to L$ between frames preserving arbitrary finite meets (including the top e) and arbitrary joins (including the bottom 0), and has the right adjoint $h_* : L \to M$, i.e.,

$$h(x) \leq y \text{ if and only if } x \leq h_*(y).$$

A frame homomorphism $h : M \to L$ is *dense* if $h(a) = 0$ implies $a = 0$.
Let $h : M \to L$ be a frame homomorphism and let $\lhd_m$ and $\lhd_l$ be Császár orders on M and L, respectively. Then we have the two binary relations as follows:

$$xh_*(\lhd_l)y \text{ if and only if there are elements } a, b \in L \text{ such that } h(x) \leq a \lhd_l b,\ h_*(b) \leq y$$

and

$$xh(\lhd_m)y \text{ if and only if there are elements } a, b \in M \text{ such that } x \leq h(a),\ a \lhd_m b,\ h(b) \leq y.$$

If a frame homomorphism $h : M \to L$ is dense, then $h_*(\lhd)$ is a Császár order on M, and for any frame homomorphism $h : M \to L$, $h(\lhd)$ is a Császár order on L.

2. Continuous homomorphisms on Császár frames

In this section we introduce a concept of continuous homomorphisms between Császár frames and show that every dense onto continuous homomorphism is a uniform continuous homomorphism and hence a Cauchy frame homomorphism.

Definition. A Császár frame $(L, \mathcal{L})$ is said to be:

(1) *strong* if for each $\lhd \in \mathcal{L}$, there is $\lhd_o \in \mathcal{L}$ such that $a \lhd b$ implies $a \lhd_o c \lhd_o b$ for some $c \in L$.
(2) *symmetric* if every member of $\mathcal{L}$ is symmetric.
(3) *regular* if every member of $\mathcal{L}$ is coarser than $\prec$.
(4) *proximal* if it is strong, symmetric and regular.
(5) *uniform* if it is proximal and every member of $\mathcal{L}$ is a meet-complete sublattice of $L \times L$.

If $h : M \to (L, \mathcal{L})$ is a dense onto frame homomorphism and $(L, \mathcal{L})$ is symmetric and strong, then $(M, h_*(\mathcal{L}))$ is also symmetric and strong, where $h_*(\mathcal{L})$ denotes the set $\{h_*(\lhd) : \lhd \in L\}$.

As noted in the previous section, $\lhd_{\mathcal{L}}$ denotes $\cup\{\lhd \in \mathcal{L}\}$ for a Császár frame $(L, \mathcal{L})$.

Proposition 2.1. *If $(L, \mathcal{L})$ is a proximal Császár frame, then $\lhd_{\mathcal{L}}$ satisfies the following:*

(1) *$\lhd_{\mathcal{L}}$ is a symmetric Császár order.*
(2) *$\lhd_{\mathcal{L}}$ is strong.*
(3) *$\lhd_{\mathcal{L}}$ is a sublattice of $L \times L$.*

Proof. (1) It is enough to show that $\lhd_{\mathcal{L}}$ is symmetric. Suppose $x \lhd_{\mathcal{L}} y$, then $x \lhd y$ for some $\lhd \in \mathcal{L}$. Since $\lhd$ is symmetric, $y^* \lhd x^*$ and hence $y^* \lhd_{\mathcal{L}} x^*$.

(2) Suppose $x \lhd_{\mathcal{L}} y$, then $x \lhd y$ for some $\lhd \in \mathcal{L}$. Since $\mathcal{L}$ is strong, $x \lhd_0 z \lhd_0 y$ for some $z \in L$ and $\lhd_0 \in \mathcal{L}$, and hence $x \lhd_{\mathcal{L}} z \lhd_{\mathcal{L}} y$. Thus $\lhd_{\mathcal{L}}$ is strong.

(3) Suppose $a \lhd_{\mathcal{L}} b$ and $a \lhd_{\mathcal{L}} c$, then there are $\lhd_1, \lhd_2 \in \mathcal{L}$ such that $a \lhd_1 b$ and $a \lhd_2 c$. Since $\mathcal{L}$ is up-directed, there is $\lhd_0 \in \mathcal{L}$ such that $\lhd_1 \cup \lhd_2 \subseteq \lhd_0$. Since $\lhd_0$ is a meet-sublattice of $L \times L$, $a \lhd_0 b \wedge c$ and hence $a \lhd_{\mathcal{L}} b \wedge c$. Suppose $b \lhd_{\mathcal{L}} a$ and $c \lhd_{\mathcal{L}} a$, then there are $x, y, u, v \in L$ such that $b \lhd_{\mathcal{L}} x \prec u \lhd_{\mathcal{L}} a$ and $c \lhd_{\mathcal{L}} y \prec v \lhd_{\mathcal{L}} a$ because $\mathcal{L}$ is strong and $\lhd_{\mathcal{L}} \subseteq \prec$. Since $\lhd_{\mathcal{L}}$ is symmetric and a meet-sublattice of $L \times L$, $(b \vee c)^{**} \lhd_{\mathcal{L}} (x \vee y)^{**}$ and so $(b \vee c) \lhd_{\mathcal{L}} (x \vee y)^{**}$. Since $(x \vee y)^{**} \prec (u \vee v)$ and $(u \vee v) \leq a$, $(b \vee c) \lhd_{\mathcal{L}} a$. □

Remark 2.2. If $(L, \mathcal{L})$ is a proximal Császár frame, then $\lhd_{\mathcal{L}}$ is a strong inclusion. Therefor, if $(L, \mathcal{L})$ is a proximal Császár frame or a uniform Császár frame, it has a compactification (*cf.* [2]).

Now we introduce a concept of continuous homomorphisms and surjections between Császár frames.

Definition. Let $(L, \mathcal{L})$ and $(M, \mathcal{M})$ be Császár frames. A frame homomorphism $h : M \to L$ is said to be:

(1) a *continuous homomorphism* if for each $\lhd_m \in \mathcal{M}$, there is a $\lhd_l \in \mathcal{M}$ with $h(\lhd_m) \subseteq \lhd_l$, or equivalently $(h \times h)(\lhd_m) \subseteq \lhd_l$.
(2) a *surjection* if it is onto dense and $\mathcal{M} = \{h_*(\lhd) : \lhd \in \mathcal{L}\}$.

Remark 2.3. It is now clear that the class of Császár frames and continuous homomorphisms forms a category.

If $h : M \to L$ is an onto dense frame homomorphism, then for any Császár order $\lhd$ on L, $h(h_*(\lhd)) = \lhd$ and hence every surjection is a continuous homomorphism.

Definition ([5]). A filter F on a Császár frame $(L, \mathcal{L})$ is said to be :

(1) *Cauchy* if $a \lhd_{\mathcal{L}} b$ implies $a^* \in F$ or $b \in F$.
(2) $\lhd_{\mathcal{L}}$*-regular*(or simply regular) if for any $a \in F$, there is $b \in F$ with $b \lhd_{\mathcal{L}} a$.

For a filter F on a Császár frame $(L, \mathcal{L})$, we write

$$F^\circ = \{a \in L : b \lhd_{\mathcal{L}} a \text{ for some } b \in F\}.$$

Definition. Let L be a frame and $\lhd$ a Császár order on L. An element $a \in L$ is called $\lhd$*-small* if $a \leq x^*$ or $a \leq y$ whenever $x \lhd y$.

Denote by $S(\lhd)$ the set of $\lhd$-small elements. Let U be a set and $\sqsubset$ an order on U. Then for $A, B \subseteq U$, A is said to $\sqsubset$*-refine* B if for any $a \in A$, there is $b \in B$ with $a \sqsubset b$ and we write $A \sqsubset B$.

Remark 2.4. A filter F on a Császár frame $(L, \mathcal{L})$ is Cauchy if and only if for any $\lhd \in \mathcal{L}$, $F \cap S(\lhd) \neq \emptyset$.

Definition ([5]). A frame homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ between Császár frames is called a *uniform homomorphism* if for each $\lhd_m \in \mathcal{M}$ there is a $\lhd_l \in \mathcal{L}$ with $S(\lhd_l) \leq h(S(\lhd_m))$.

Recall that if $h : M \to L$ is a dense frame homomorphism, then for each $x \in M$, $h_*(h(x)) \leq x^{**}$.

Theorem 2.5. *If $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ is a dense onto continuous homomorphism between proximal Császár frames, then h is uniform.*

Proof. Take any $\lhd_m \in \lhd_{\mathcal{M}}$. Since $(M, \mathcal{M})$ is proximal, there is $\lhd \in \lhd_{\mathcal{M}}$ such that $s \lhd_m t$ implies $s \lhd u^{**} \lhd t$. Since h is continuous, there is $\lhd_l \in \lhd_{\mathcal{L}}$ with $h(\lhd) \subseteq \lhd_l$ and hence $S(\lhd_l) \subseteq S(h(\lhd))$. Now, it is enough to show that $S(h(\lhd)) \subseteq h(S(\lhd_m))$. Take any $a \in S(h(\lhd))$ and suppose $x \lhd_m y$. Then $x \lhd z \lhd y$ for some $z \in M$ with $z = z^{**}$ and hence $h(x)\ h(\lhd)\ h(z)\ h(\lhd)\ h(y)$. Then $a \leq [h(x)]^*$ or $a \leq h(z)$ for $a \in S(h(\lhd))$. Since h is dense, $h_*(a) \wedge x = 0$ or $h_*(a) \leq h_*(h(z)) \leq z \leq y$. Hence $h_*(a) \in S(\lhd_m)$ and hence $h_*(S(h(\lhd)) \subseteq S(\lhd_m)$. Since h is onto, $S(h(\lhd)) \subseteq h(S(\lhd_m))$. Thus $S(\lhd_l) \subseteq h(S(\lhd_m))$. This completes the proof. □

Definition ([5]). A frame homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ between Császár frames is a *Cauchy frame homomorphism* if for any regular Cauchy filter F in L, there is a regular Cauchy filter G in M with $h(G) \subseteq F$.

Theorem 2.6 ([5]). *Every uniform homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ between proximal Császár frames is a Cauchy homomorphism.*

Remark 2.7. Using the above theorems, one has the following:

(1) Every dense onto continuous homomorphism between proximal Császár frames is a Cauchy homomorphism.
(2) Every surjection is a Cauchy homomorphism.

Proposition 2.8. *If a frame homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ is a surjection and F is a regular Cauchy filter on M, then $h(F)$ is a regular Cauchy filter on L.*

Proof. Since h is dense, $h(F)$ is a filter base. Take any $a \in h(F)$ and suppose $a \leq b$. Then there is $x \in F$ with $h(x) = a$. Then $x \leq h_*(b)$ and so $b \in h(F)$ for h is onto. Thus $h(F)$ is a filter. Suppose $a \lhd b$ in L, then $h_*(a)\ h_*(\lhd)\ h_*(b)$. Since F is Cauchy, $[h_*(a)]^* \in F$ or $h_*(b) \in F$. Hence $h([h_*(a)])^* \in h(F)$ or $h(h_*(b)) \in F$ and hence $a^* \in h(F)$ or $b \in h(F)$ for h is onto. Further, there are $s, t \in F$ such that $f(t) = a$ and $sh_*(\lhd)t$ for some $\lhd \in \mathcal{L}$ and hence $h(s) \lhd a$ and $h(s) \in h(F)$. Thus $h(F)$ is regular. □

Definition. A frame homomorphism between proximal Császár frames is said to be *uniform continuous* if it is uniform and continuous.

PCsFrm denotes the Category of proximal Császár frames and uniform continuous homomorphisms and CPCsFrm denotes the full subcategory of PCsFrm determined by Cauchy complete proximal Császár frames.

3. Cauchy completions of Császár frames

Definition ([3]). A strict extension of a frame L is an onto dense frame homomorphism $h : M \to L$ satisfying the following:

$$M = \{\bigvee A : A \subseteq h_*(L)\}.$$

For a Császár frame $(L, \mathcal{L})$, let X be the set of regular Cauchy filters on L and $L \times P(X)$ the product frame of L and $P(X)$. Then $S_X L = \{(a, \Sigma_a) : a \in L\}$ is a subframe of $L \times P(X)$, where $\Sigma_a = \{F \in X : a \in F\}$. Let $s : S_X L \to L$ be the restriction of the first projection of $Pr_L : L \times P(X) \to L$ and cL denotes $\{(\bigvee A, \Sigma_A) : A \subseteq s_*(L)\}$. Then $c_L : cL \to L$ is a strict extension of L associated with X(see [3, 8] for a more detail).

Remark 3.1. For any $a \in L$, $\Sigma_a = \{F \in X : b \in F \text{ for some } b \lhd_{\mathcal{L}} a\}$.

Recall that in any frame L, a *cover* of L is any subset whose join is e and a filter F on L is *convergent* if for any cover A of L, $F \cap A \neq \emptyset$ (see [7]).

Definition ([5]). A Császár frame is *Cauchy complete* if every regular Cauchy filter is convergent.

The proof of the following can be found in [5].

Proposition 3.2. *For any proximal Császár frame* $(L, \mathcal{L})$, $(cL, \mathcal{L}^*)$ *is a Cauchy complete proximal Császár frame, where* $\mathcal{L}^* = \{c_{L*}(\lhd) : \lhd \in \mathcal{L}\}$.

Lemma 3.3 ([5]). (1) *Let* $(L, \mathcal{L})$ *be a proximal Császár frame. Then for any* $\lhd \in \mathcal{L}$, *there is* $\lhd_\circ \in \mathcal{L}$ *with* $S(\lhd_\circ) \lhd_\circ S(\lhd)$.

(2) *If* $h : M \to L$ *is an onto dense frame homomorphism and* $\lhd$ *is a Császár order on* L, *then* $S(h_*(\lhd)) \leq h_*(S(\lhd))$.

Let $u : PCsFrm \to Frm$ be the forgetful functor, then one has the following:

Lemma 3.4. *Every surjection* $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ *between proximal Császár frames is u-initial.*

Proof. Take any proximal Császár frame $(K, \mathcal{K})$ and a frame homomorphism $k : K \to M$ such that $h \circ k : (k, \mathcal{K}) \to (L, \mathcal{L})$ is a uniform continuous homomorphism. We first show that $k : (K, \mathcal{K}) \to (M, \mathcal{M})$ is continuous. Take any $\lhd_1 \in \mathcal{K}$ and suppose $x \lhd_1 y$. Then $x \lhd a \lhd b \lhd y$ for some $a, b \in K$ and $\lhd \in \mathcal{K}$. Since $(K, \mathcal{K})$ is a regular proximal Császár frame and k is a frame homomorphism, $k(x) \prec k(a) \prec k(b)^{**} \prec k(y)$. Since $h \circ k$ is continuous, there is $\lhd_2 \in \mathcal{L}$ with $h(k(x)) \lhd_2 h(k(a)) \lhd_2 h(k(b)) \lhd_2 h(k(y))$. Since h is a surjection,

$$k(x)\ h_*(\lhd_2)\ h_*[h(k(a))]\ h_*(\lhd_2)\ h_*[h(k(b))] \prec k(b)^{**} \leq k(y)$$

and hence $k(x)\ h_*(\lhd_2)\ k(y)$. Thus k is continuous. Now we show that k is uniform. Take any $\lhd_1 \in \mathcal{K}$. Since $(k, \mathcal{K})$ is a proximal Császár frame, there is $\lhd_\circ \in \mathcal{K}$ with $S(\lhd_\circ) \lhd_\circ S(\lhd_2)$. Since $h \circ k$ is uniform, there is $\lhd_2 \in \mathcal{L}$ with $S(\lhd_2) \leq h \circ k(S(\lhd_\circ))$ and hence $h_*(S(\lhd_2)) \leq h_* \circ h[k(S(\lhd_\circ))]$. Since $S(\lhd_\circ) \lhd_\circ S(\lhd_1)$ for any $a \in S(\lhd_\circ)$, there is $b \in S(\lhd_1)$ with $h_* \circ h(k(a)) \leq k(b)$ and hence $h_* \circ h[k(S(\lhd_\circ))] \leq k(S(\lhd_1))$. Since h is onto dense, $S(h_*(\lhd_2)) \leq h_*(S(\lhd_2))$ and so $S(h_*(\lhd_2)) \leq k(S(\lhd_1))$. Thus k is uniform. □

Recall that for any Cauchy complete Császár frame $(L, \mathcal{L})$, $c_L : (L, \mathcal{L}^*) \to (L, \mathcal{L})$ is an isomorphism. Using this and the above lemma, we show that $c_L : (cL, \mathcal{L}^*) \to (L, \mathcal{L})$ is a coreflection of $(L, \mathcal{L})$ in the category PCsFrm.

Theorem 3.5. *The category CPCsFrm is coreflective in the category PCsFrm.*

Proof. Take any proximal Császár frame $(L, \mathcal{L})$. Clearly $c_L : (cL, \mathcal{L}^*) \to (L, \mathcal{L})$ is a surjection and hence a uniform continuous homomorphism. Take any Cauchy complete proximal Császár frame $(M, \mathcal{M})$ and a uniform continuous homomorphism $h : M \to L$. We define $\bar{h} : cM \to cL$ by $\bar{h}(a, \Sigma_a) = (h(a), \cup\{\Sigma_{h(x)} : x \lhd_{\mathcal{M}} a\})$. Then $\bar{h}$ is a map with $h \circ c_M = c_L \circ \bar{h}$. Let $k = \bar{h} \circ c_{M*}$. Since $h : M \to L$ is a uniform continuous homomorphism, it is a Cauchy homomorphism and hence k is a frame homomorphism by

Theorem 3.1.7 in [5]. By Lemma 3.4, k is a uniform continuous homomorphism. Since c_L is dense and hence monomorphism, such a k is unique. Thus $c_L : (cL, \mathcal{L}^*) \to (L, \mathcal{L})$ is a coreflection of $(L, \mathcal{L})$ in the category PCsFrm. □

Lemma 3.6. *If $h : M \to L$ is a dense frame homomorphism and $\lhd$ a meet-complete sublattice of $L \times L$, then $h_*(\lhd)$ is also a meet-complete sublattice of $M \times M$.*

Proof. Let $S \subseteq L$ and suppose $ah_*(\lhd)s$ for all $s \in S$. Then there are $x_s, y_s \in M$ such that $h(a) \leq x_s \lhd y_s$ and $h_*(y_s) \leq b_s$. Since $\lhd$ is a meet-complete sublattice of $L \times L$ and h_* preserves arbitrary meets, $h(a) \leq \bigwedge_{s\in S} x_s \lhd \bigwedge_{s\in S} y_s$ and $h_*(\bigwedge_{s\in S} y_s) \leq \bigwedge_{s\in S} b_s$. Hence $ah_*(\lhd) \bigwedge_{s\in S} b_s$. □

Collecting the above, we have the following:

Theorem 3.7. *The category of Cauchy complete uniform Császár frames and uniform continuous homomorphisms is a coreflective subcategory of the category of uniform Császár frames and uniform continuous homomorphisms.*

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