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EXAMPLES AND FUNCTION THEOREMS AROUND AP AND WAP SPACES

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Abstract. We provide some examples around AP and WAP spaces which are connected with higher convergence properties-radiality, semiradiality and pseudoradiality. We also prove a theorem (Theorem 3.2) that (a) any pseudo-open continuous image of an AP-space is an AP-space and (b) any pseudo-open continuous image of an WAP-space is an WAP-space. This answers the question posed by V. V. Tkachuk and I. V. Yaschenko [10].

1. Introduction

All spaces under consideration are assumed to be Hausdorff. For all undefined terminologies, see [1] and [5].

A space X is said to have the property of *Approximation by Points* (*Weak Approximation by Points*), for short, AP(WAP), if for every non-closed set A and every (some) point $x \in \overline{A} - A$ there is a subset $B \subset A$ such that $\overline{B} - A = \{x\}$. Such a set B is also called *almost closed.* The above definitions were originated in categorical topology by A. Pultr and A. Tozzi ([7]). P. Simon ([8]) was first to study these properties from a general topological point of view. We say that a subset A of a space X is AP-*closed* if for every $F \subset A$ the relation $|\overline{F} - A| \neq 1$ holds. It is clear that X is a WAP space if and only if every AP-closed subset of X is closed.

A. Bella and I. V. Yaschenko ([2], [3]) discovered strong connections of these properties with higher convergence properties-radiality, semiradiality, and pseudoradiality. They showed that every semiradial space is WAP, every compact WAP space is pseudoradial, and a product of compact WAP and compact semiradial space is WAP.

W. Hong [6] defined the space having the property of *Approximation by Countable Points*, for short, ACP, provided that for every non-closed set A and every point $x \in \overline{A} - A$ there is a countable subset $B \subset A$ such that $\overline{B} - A = \{x\}$.

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He also defined a WACP space as a generalization of a ACP space. It has shown that WACP implies WAP.

We define a subset A of a space X to be ACP-*closed* if for every countable $F \subset A$ the relation $|\overline{F} - A| \neq 1$ holds. Then it is easy to show that X is a WACP space if and only if every ACP-closed subset of X is closed.

A space in which every point has a local base linearly ordered by set inclusion is called a *lob* space ([4]).

A space X is *Fréchet-Urysohn* if for any $A \subset X$ and any $x \in \overline{A}$ there is a sequence S in A which converges to x. A space X is *sequential* if for any non-closed $A \subset X$ there is a sequence S in A which converges to some $x \in \overline{A} - A$. A space X has *countable tightness*, i.e., $t(X) \leq \omega$, if whenever $A \subset X$ and $x \in \overline{A}$, there exists a countable set $B \subset A$ such that $x \in \overline{B}$. A space X is *radial* (*pseudoradial*) if for every non-closed set A and every (some) point $x \in \overline{A} - A$ there is a transfinite sequence $S \subset A$ which converges to x. Recall that a subset A of a space X is κ-closed whenever $B \subset A$ and $|B| \leq \kappa$ imply $\overline{B} \subset A$. A space X is *semiradial* if for every non-κ-closed set A there is a transfinite sequence $S \subset A$ which converges to a point outside A and satisfies $|S| \leq \kappa$.

The relations radial $\to$ semiradial $\to$ pseudoradial always hold and in general the arrows cannot be reversed even for compact spaces. For more details on pseudoradial and related spaces see [2].

2. Some examples

There are some strange situations with evident questions about AP, WAP, WACP spaces ([2], [3], and [6]). Also, there are still lots of unsolved problems on how these spaces related to those spaces with radiality, semiradiality, pseudoradiality. We provide some examples here which may be already known, but it is useful for us to see what is going on AP and WAP spaces. At least, we could only do it under CH and the example is not evident at all.

The following is an example of a radial space.

Example 2.1. Let D be any infinite set with $|D| = \omega_1$, and let $p \notin D$. We define a basic open set $B(x)$ for $x \in X = \{p\} \cup D$ as follows:

(i) if $x \in D$, then $B(x) = \{x\}$;

(ii) if $x = p$, then $B(p) = \{p\} \cup G$, where $G \subset D$ and $D - G$ is countable.

Claim 1 : X is radial.

Let A be a non-closed subset of X. Since every countable subset of X is closed, $|A| = \omega_1$ and $p \in \overline{A} - A$. We enumerate $A = \{x_\alpha : \alpha \in \omega_1\}$. We must show that the well-ordered net A converges to p. Let U be an open neighborhood of p. Then there exists a subset $G \subset D$ such that $\{p\} \cup G \subset U$ where $D - G$ is countable. Since $A = (A \cap G) \cup (A - G)$ and $A - G$ is countable, we can find $\beta \in \omega_1$ such that if $\alpha > \beta$, then $x_\alpha \notin A - G$. Therefore $x_\alpha \in A \cap G \subset A \cap U$ for all $\alpha > \beta$. We have shown that X is radial. □

Theorem 2.2 ([4]). *Every radial space X with $t(X) \leq \omega$ is Fréchet-Urysohn.*

We now give an example of a Hausdorff sequential semiradial space which is neither AP nor radial.

Example 2.3. Let $X = \mathbb{N} \times \mathbb{N} \cup \{y_n : n \in \mathbb{N}\} \cup \{z\}$. We define a basic open set on X as follows:

(i) (m, n) is an isolated point of X for each $m, n \in \mathbb{N}$;

(ii) $V_k(y_n) = \{y_n\} \cup \{(m, n) : m \geq k\}$ is open in X for each $k \in \mathbb{N}$;

(iii) for each $p \in \mathbb{N}$,

$$W_p(z) = \bigcup_{n \geq p} (V_1(y_n) - F_n) \cup \{y_n : n \geq p\} \cup \{z\}$$

is open in X where each F_n is a finite subset of $V_1(y_n)$.

Then the space X is Hausdorff sequential ([1]) and X is not AP ([6]). One can see easily that X is not radial by Theorem 2.2. We shall show that X is semiradial. Note that every non closed subset A of X is not ω-closed. Suppose that A is a non ω-closed subset of X. Then there is a subset B of A such that $|B| \leq \omega$ and $\overline{B} \not\subset A$. Since y_n's and z are the only non-isolated points of X, we may take the following two cases:

Case 1 : $y_n \in \overline{B} - A$ for some $n \in \mathbb{N}$.

Since $|V_1(y_n) \cap B| = \omega$, the sequence $\{(m, n) \in B : (m, n) \in V_1(y_n)\}$ converges to $y_n \notin A$.

Case 2 : $z \in \overline{B} - A$.

If $A \cap \{y_n : n \in \mathbb{N}\}$ is finite, then take $p = 1 + \max\{n \in \mathbb{N} : y_n \in A\}$ and denote

$$W_p(z) = \bigcup_{n \geq p} (V_1(y_n) - F_n) \cup \{y_n : n \geq p\} \cup \{z\},$$

where each F_n is a finite subset of $V_1(y_n)$. Since $|(V_1(y_n) - F_n) \cap B| = \omega$ for some $n \geq p$, the sequence $\{(m, n) \in B : (m, n) \in (V_1(y_n) - F_n)\}$ converges to y_n.

If $A \cap \{y_n : n \in \mathbb{N}\}$ is infinite, then the sequence $\{y_n \in A : n \in \mathbb{N}\}$ converges to $z \in \overline{A} - A$. Therefore X is semiradial.

The following is an example of a Fréchet-Urysohn space which is not lob.

Example 2.4. Consider the one-point compactification $X = D \cup \{\star\}$ where $|D| > \omega$. It is well-known that X is Fréchet-Urysohn, but not first countable. Moreover, it is easy to see that X is not a lob space.

The following example was given in [6] and [9] which was originally from the well-known book of Counterexamples in topology written by L. A. Steen and J. A. Seebach, Jr.. We can say something more on this example. Precisely, this is an example of a space of countably Fréchet-Urysohn which is not WAP or even not pseudoradial under CH.

Example 2.5. Let $X = \mathbb{R}$ and $\mathcal{T}_1$ be the usual topology on $\mathbb{R}$. Then $\mathcal{T} = \{O - K : O \in \mathcal{T}_1, K \subset X \text{ is countable}\}$ is a topology on X which is called the countable complement extension topology with $t(X) > \omega$. Then X is countably Fréchet-Urysohn, and hence countably AP, but not AP ([6]). Furthermore, we can prove that X is not WAP with the similar argument as in [6] and X is not pseudoradial under CH.

Claim 1 : X is not WAP.

Suppose X is WAP. Since $\overline{A} = [0,1]$ for a subset $A = [0,1] - \mathbb{Q}$, $[A]_{AP} - A \neq \emptyset$, where $[A]_{AP} = A \cup \{x \in \overline{A} - A : \exists F \subset A \text{ such that } \overline{F} - F = \{x\}\}$. So there exists a subset B of A such that $\overline{B} = B \cup \{p\}$ for some point $p \in [A]_{AP} - A$. Since every countable subset is closed in X, B is uncountable. For each $x \in [0,1] \cap \mathbb{Q} - \{p\}$, we have $x \notin \overline{B}$. Hence there are $\epsilon_x > 0$ and a countable subset $K_x \subset X$ such that

$$((x - \epsilon_x, x + \epsilon_x) - K_x) \cap B = \emptyset.$$

Thus

$$\left(\bigcup\{(x - \epsilon_x, x + \epsilon_x) - K_x : x \in [0,1] \cap \mathbb{Q} - \{p\}\}\right) \cap B = \emptyset.$$

Since $\bigcup\{(x - \epsilon_x, x + \epsilon_x) - K_x : x \in [0,1] \cap \mathbb{Q} - \{p\}\} \supset A - \bigcup\{K_x : x \in [0,1] \cap \mathbb{Q} - \{p\}\}$,

$$\left(A - \bigcup\{K_x : x \in [0,1] \cap \mathbb{Q} - \{p\}\}\right) \cap B = \emptyset.$$

Therefore $B \subset \bigcup\{K_x : x \in [0,1] \cap \mathbb{Q} - \{p\}\}$, which is a contradiction to the cardinality of B.

Claim 2 : X is not pseudoradial under CH.

We consider the same subset A in Claim 1. Suppose that a well-ordered net $\{x_\alpha : a \in I\}$ in A converges to a point $x \in \overline{A} - A$. Since every countable subset of X is closed, $\omega < |I| \leq \mathfrak{c}$. Then for any basic open neighborhood $O - K$ of x, $(O - K) \cap \{x_\alpha : \alpha \in I\} \neq \emptyset$. In fact, $(O - K) \cap \{x_\alpha : \alpha \in I\}$ must be uncountable. Denote $O_n = (x - \frac{1}{n}, x + \frac{1}{n})$ and $U_n = O_n - O_{n+1}$ for each $n \in \mathbb{N}$. Since $\{x_\alpha : \alpha \in I\} \subset \cup\{U_n : n \in \mathbb{N}\}$, there exists $n \in \mathbb{N}$ such that $\{x_\alpha \in U_n : \alpha \in I\}$ is uncountable. Since $\{x_\alpha : a \in I\}$ converges to x, there is $\beta \in I$ such that if $\alpha > \beta$, then $x_\alpha \in O_{n+1}$. Note that $|I| = \omega_1$ under CH. Hence $\{x_\alpha : \alpha \leq \beta\}$ is countable and $\{x_\alpha \in U_n : \alpha \in I\} \subset \{x_\alpha : \alpha \leq \beta\}$. This is a contradiction.

3. Function theorems on AP and WAP spaces

The purpose of the present section is to show to what extent a continuous mapping satisfying certain conditions transfers properties of the domain spaces to the range spaces.

Recall that a compact, closed, and continuous map is called a *perfect map.* A continuous function $f : X \to Y$ is called *bi-quotient* if for each point $y \in Y$ and each open collection $\mathcal{U}$ in X which covers $f^{-1}(y)$, there is a finite subcollection $\mathcal{U}'$ of $\mathcal{U}$ such that $\bigcup\{f(U) : U \in \mathcal{U}'\}$ is a neighborhood of y.

A continuous function $f : X \to Y$ is called *pseudo-open* if for each point $y \in Y$ and every neighborhood U of $f^{-1}(y)$ we have $y \in Int(f(U))$. We have the following basic diagram:

$$\begin{array}{ccccccc} \text{open} & \Rightarrow & \text{bi-quotient} & \Rightarrow & \text{pseudo-open} & \Rightarrow & \text{quotient} \\ & & \Uparrow & & \Uparrow & & \\ & & \text{perfect} & \Rightarrow & \text{closed} & & \end{array}$$

It is well-known that the image of a Fréchet-Urysohn space under a continuous pseudo-open map is a Fréchet-Urysohn space. It is also well-known that every continuous closed (open) function is pseudo-open.

In [10], Tkachuk and Yaschenko have shown that any closed continuous image of an AP(WAP)-space is an AP(WAP)-space, but a quotient image of an AP-space is not necessarily an AP-space. They also asked the following question (Problem 4.6): Is an open image of an AP-space an AP-space? How about open images of WAP-spaces?

We give a positive answer to the question above. Before doing this, we give a characterization of a pseudo-open map.

Lemma 3.1 ([1]). *Let $f : X \to Y$ be a continuous map of X onto Y. Then the following conditions are equivalent:*

(a) *for each $Y' \subset Y$ the restriction f to $X' = f^{-1}(Y')$, the inverse image of Y', is a quotient map of X' onto Y';*

(b) *for each $y \in Y$ and every open set U in X containing $f^{-1}(y)$, the interior $Int(f(U))$ of the image of U contains y (that is, f is pseudo-open);*

(c) *whenever $B \subset Y$ and $y \in Y$ satisfies $y \in \overline{B}$, we have $f^{-1}(y) \cap \overline{f^{-1}(B)} \neq \emptyset$.*

Theorem 3.2. (a) *Any pseudo-open continuous image of an* AP*-space is an* AP*-space.*

(b) *Any pseudo-open continuous image of an* WAP*-space is an* WAP*-space.*

Proof. (a) Let X be an AP-space and $f : X \to Y$ be a pseudo-open continuous onto map. Suppose $B \subset Y$ and $y \in \overline{B} - B$. Let $A = f^{-1}(B)$. Then $\overline{A} \cap f^{-1}(y) \neq \emptyset$ by Lemma 3.1. Fix $x \in \overline{A} \cap f^{-1}(y)$. Then $x \in \overline{A} - A$. Since X is AP, there exists an almost closed $F \subset A$ with $x \in \overline{F}$ for some $x \in f^{-1}(y)$. It is easy to check that $D = f(F)$ is an almost closed subset of B and $y \in \overline{D}$. Hence Y is AP.

(b) A proof is similar to the proof of (a). □

Since every continuous closed (open) function is pseudo-open, we have the following corollaries.

Corollary 3.3 ([10]). (a) *Any closed continuous image of an* AP*-space is an* AP*-space.*

(b) *Any closed continuous image of an* WAP*-space is an* WAP*-space.*

Corollary 3.4. (a) *Any open continuous image of an* AP*-space is an* AP*-space.*
(b) *Any open continuous image of an* WAP*-space is an* WAP*-space.*

Remark 3.5. The above corollary answers the question posed by V. V. Tkachuk and I. V. Yaschenko [10].

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