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ON COMPATIBLE MAPPINGS OF TYPE (I) AND (II) IN INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. In this paper, we give some new definitions of compatible mappings in intuitionistic fuzzy metric spaces and we prove a common fixed point theorem for four mappings under the condition of compatible mappings of type (I) and of type (II) in complete intuitionistic fuzzy metric spaces.

1. Introduction

The notion of fuzzy sets was introduced by Zadeh [23] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. For example, Deng [3], Erceg [5], Fang [6], George and Veeramani [7], Kaleva and Seikkala [11], Kramosil and Michalek [12] have introduced the concept of fuzzy metric spaces in different ways. They also showed that every metric induces a fuzzy metric. Miheţ [14] obtained some new results of modifying the notion of convergence in fuzzy metric spaces. Turkoglu et al. [22] gave definitions of compatible mappings of type (I), (II) and some examples for various compatible mappings in fuzzy metric spaces. After than, we prove common fixed point theorem for four mappings satisfying some conditions in fuzzy metric spaces.

Park [16] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t-conorm as a generalization of fuzzy metric space due to George and Veeramani [7] and introduced the notion of Cauchy sequences in an intuitionistic fuzzy metric space and proved the Baire's theorem and finding a necessary and sufficient condition for an intuitionistic fuzzy metric space to be complete and shown that every separable intuitionistic fuzzy metric space is second countable and that every subspace of an intuitionistic fuzzy metric space is separable and proved the uniform limit theorem for intuitionistic fuzzy metric spaces. Alaca et al. [1] using the idea of intuitionistic fuzzy sets, they defined the notion

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of intuitionistic fuzzy metric space as Park [16] with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [12]. Further, they introduced the notion of Cauchy sequences in an intuitionistic fuzzy metric spaces and proved the well-known fixed point theorems of Banach [2] and Edelstein [4] extended to intuitionistic fuzzy metric spaces with the help of Grabiec [8]. Turkoglu et al. [20] gave generalization of Jungck's common fixed point theorem [10] to intuitionistic fuzzy metric spaces. They first formulate the definition of weakly commuting and R-weakly commuting mappings in intuitionistic fuzzy metric spaces and proved the intuitionistic fuzzy version of Pant's theorem [15]. Turkoglu et al. [21] introduced the concept of compatible maps and compatible maps of types (α) and (β) in intuitionistic fuzzy metric spaces and gave some relations between the concepts of compatible maps and compatible maps of types (α) and (β). Many authors studied the concept of intuitionistic fuzzy metric space and its applications [9, 17, 18].

The purpose of this paper, we give concepts of A-compatible and B-compatible and later afterwards introduce definitions of compatible mappings of type (I), (II) and some examples for various compatible mappings in intuitionistic fuzzy metric spaces. After than, we prove common fixed point theorem for four mappings satisfying some conditions in intuitionistic fuzzy metric spaces and give an example to validate our main theorem.

2. Intuitionistic fuzzy metric spaces

Definition 1 ([19]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2 ([19]). A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\Diamond$ is satisfying the following conditions:

(i) $\Diamond$ is commutative and associative;
(ii) $\Diamond$ is continuous;
(iii) $a \Diamond 0 = a$ for all $a \in [0,1]$;
(iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

The following definition was introduced by Alaca et al. [1].

Definition 3 ([1]). A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;

(ii) $M(x,y,0) = 0$ for all $x, y \in X$;
(iii) $M(x,y,t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(iv) $M(x,y,t) = M(y,x,t)$ for all $x, y \in X$ and $t > 0$;
(v) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$ for all $x, y, z \in X$ and $s, t > 0$;
(vi) for all $x, y \in X$, $M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous;
(vii) $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x, y \in X$ and $t > 0$;
(viii) $N(x,y,0) = 1$ for all $x, y \in X$;
(ix) $N(x,y,t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;
(x) $N(x,y,t) = N(y,x,t)$ for all $x, y \in X$ and $t > 0$;
(xi) $N(x,y,t) \Diamond N(y,z,s) \geq N(x,z,t+s)$ for all $x, y, z \in X$ and $s, t > 0$;
(xii) for all $x, y \in X$, $N(x,y,.) : [0,\infty) \to [0,1]$ is right continuous;
(xiii) $\lim_{t\to\infty} N(x,y,t) = 0$ for all x, y in X.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 1. Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are associated ([13]), i.e., $x \Diamond y = 1 - ((1-x) * (1-y))$ for all $x, y \in X$.

Remark 2. In intuitionistic fuzzy metric space X, $M(x,y,.)$ is non-decreasing and $N(x,y,.)$ is non-increasing for all $x, y \in X$.

Definition 4 ([1]). Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1, \quad \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0.$$

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \lim_{n\to\infty} N(x_n, x, t) = 0.$$

Since $*$ and $\Diamond$ are continuous, the limit is uniquely determined from (v) and (xi), respectively.

Definition 5 ([1]). An intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 6 ([1]). An intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be compact if every sequence in X contains a convergent subsequence.

Lemma 1. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and $\{y_n\}$ be a sequence in X. If there exists a number $k \in (0,1)$ such that*

(2.1) $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t), \; N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$

for all $t > 0$ and $n = 1, 2, \ldots$, then $\{y_n\}$ is a Cauchy sequence in X.

Proof. By the simple induction with the condition (2.1) with the help of Alaca et al. [1], we have, for all $t > 0$ and $n = 1, 2, \dots,$

(2.2)
$$M(y_{n+1}, y_{n+2}, t) \geq M\left(y_1, y_2, \frac{t}{k^n}\right), \; N(y_{n+1}, y_{n+2}, t) \leq N\left(y_1, y_2, \frac{t}{k^n}\right).$$

Thus, by (2.2) and Definition 3 ((v) and (xi)), for any positive integer p and real number $t > 0$, we have

$$\begin{aligned} M(y_n, y_{n+p}, t) &\geq M(y_n, y_{n+1}, \frac{t}{p}) \overset{p-\text{times}}{* \cdots *} M(y_{n+p-1}, y_{n+p}, \frac{t}{p}) \\ &\geq M(y_1, y_2, \frac{t}{pk^{n-1}}) \overset{p-\text{times}}{* \cdots *} M(y_1, y_2, \frac{t}{pk^{n+p-2}}) \end{aligned}$$

and

$$\begin{aligned} N(y_n, y_{n+p}, t) &\leq N(y_n, y_{n+1}, \frac{t}{p}) \overset{p-\text{times}}{\Diamond \cdots \Diamond} N(y_{n+p-1}, y_{n+p}, \frac{t}{p}) \\ &\leq N(y_1, y_2, \frac{t}{pk^{n-1}}) \overset{p-\text{times}}{\Diamond \cdots \Diamond} N(y_1, y_2, \frac{t}{pk^{n+p-2}}). \end{aligned}$$

Therefore, by Definition 3 ((vii) and (xiii)), we have

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 \overset{p-\text{times}}{* \cdots *} 1 \geq 1$$

and

$$\lim_{n\to\infty} N(y_n, y_{n+p}, t) \leq 0 \overset{p-\text{times}}{\Diamond \cdots \Diamond} 0 \leq 0,$$

which implies that $\{y_n\}$ is a Cauchy sequence in X. This completes the proof. □

Lemma 2. *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0, 1)$,*

(2.3) $$M(x, y, kt) \geq M(x, y, t) \text{ and } N(x, y, kt) \leq N(x, y, t)$$

then $x = y$.

Proof. Since $t > 0$ and $k \in (0, 1)$, we get $t > kt$. Using Remark 2 (In intuitionistic fuzzy metric space X, $M(x, y, \cdot)$ is non-decreasing and $N(x, y, \cdot)$ is non-increasing for all $x, y \in X$.), we have,

$$M(x, y, t) \geq M(x, y, kt) \text{ and } N(x, y, t) \leq N(x, y, kt).$$

Using (2.3) and the definition of intuitionistic fuzzy metric, we have,

$$x = y.$$

□

3. Various definitions of compatibility

In this section, we give some definitions of compatible mappings, some properties and some examples in intuitionistic fuzzy metric spaces.

Definition 7 ([21]). Let A and B be maps from an intuitionistic fuzzy metric space (IFM-space) $(X, M, N, *, \Diamond)$ into itself. The maps A and B are said to be compatible if, for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 8 ([21]). Let A and B be maps from an IFM-space $(X, M, N, *, \Diamond)$ into itself. The maps A and B are said to be compatible of type (α) if, for all $t > 0$,

$$\begin{aligned}
\lim_{n\to\infty} M(ABx_n, BBx_n, t) &= 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0, \\
\lim_{n\to\infty} M(BAx_n, AAx_n, t) &= 1 \text{ and } \lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0
\end{aligned}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 9 ([21]). Let A and B be maps from an IFM-space $(X, M, N, *, \Diamond)$ into itself. The maps A and B are said to be compatible of type (β) if, for all $t > 0$,

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Proposition 1. *Let $(X, M, N, *, \Diamond)$ be an IFM-space with $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0, 1]$ and A and B be continuous mappings from X into itself. Then A and B are compatible if and only if they are compatible mappings of type (α).*

Proof. Suppose that A and B are compatible and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Since A and B are continuous, we have

$$\begin{aligned}
\lim_{n\to\infty} AAx_n &= \lim_{n\to\infty} ABx_n = Az, \\
\lim_{n\to\infty} BAx_n &= \lim_{n\to\infty} BBx_n = Bz.
\end{aligned}$$

Further, since A and B are compatible,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

for all $t > 0$. Now, since we have

$$M(ABx_n, BBx_n, t) \geq M(ABx_n, BAx_n, \frac{t}{2}) * M(BAx_n, BBx_n, \frac{t}{2})$$

and

$$N(ABx_n, BBx_n, t) \leq N(ABx_n, BAx_n, \frac{t}{2}) \Diamond N(BAx_n, BBx_n, \frac{t}{2})$$

for all $t > 0$, it follows that

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) \geq 1 * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(ABx_n, BBx_n, t) \leq 0 \Diamond 0 \leq 0$$

which implies that

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0.$$

Similarly, we have also, for all $t > 0$,

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0.$$

Therefore, A and B are compatible of type (α).

Conversely, suppose that A and B are compatible of type (α) and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Since A and B are continuous, we have

$$\begin{aligned}\lim_{n\to\infty} AAx_n &= \lim_{n\to\infty} ABx_n = Az,\\ \lim_{n\to\infty} BAx_n &= \lim_{n\to\infty} BBx_n = Bz.\end{aligned}$$

Further, since A and B are compatible of type (α), we have, for all $t > 0$,

$$\begin{aligned}\lim_{n\to\infty} M(ABx_n, BBx_n, t) &= 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0,\\ \lim_{n\to\infty} M(BAx_n, AAx_n, t) &= 1 \text{ and } \lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0.\end{aligned}$$

Thus, from the inequality

$$M(ABx_n, BAx_n, t) \geq M(ABx_n, BBx_n, \frac{t}{2}) * M(BBx_n, BAx_n, \frac{t}{2})$$

and

$$N(ABx_n, BAx_n, t) \leq N(ABx_n, BBx_n, \frac{t}{2}) \Diamond N(BBx_n, BAx_n, \frac{t}{2})$$

for all $t > 0$, it follows that

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) \geq 1 * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(ABx_n, BAx_n, t) \leq 0 \Diamond 0 \leq 0$$

for all $t > 0$, which implies that

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0.$$

Therefore, A and B are compatible. This completes the proof. □

Proposition 2 ([21]). *Let $(X, M, N, *, \lozenge)$ be an IFM-space with $t * t \geq t$ and $(1-t)\lozenge(1-t) \leq (1-t)$ for all $t \in [0,1]$ and let A and B be continuous maps from X into itself. Then A and B are compatible if and only if they are compatible maps of type (β).*

Proposition 3 ([21]). *Let $(X, M, N, *, \lozenge)$ be an IFM-space with $t * t \geq t$ and $(1-t)\lozenge(1-t) \leq (1-t)$ for all $t \in [0,1]$ and let A and B be continuous maps from X into itself. Then A and B are compatible maps of type (β) if and only if they are compatible maps of type (α).*

We now introduce the following definitions.

Definition 10. Let A and B be mappings from an IFM-space $(X, M, N, *, \lozenge)$ into itself. Then the pair (A, B) is called A-compatible if, for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 11. Let A and B be mappings from an IFM-space $(X, M, N, *, \lozenge)$ into itself. Then the pair (A, B) is called B-compatible if and only if (B, A) is B-compatible.

Definition 12. Let A and B be mappings from an IFM-space $(X, M, N, *, \lozenge)$ into itself. Then the pair (A, B) is said to be compatible of type (I) if, for all $t > 0$,

$$\underline{\lim}_{n\to\infty} M(ABx_n, z, \lambda t) \leq M(Bz, z, t) \text{ and } \overline{\lim}_{n\to\infty} N(ABx_n, z, \lambda t) \geq N(Bz, z, t)$$

whenever $\lambda \in (0, 1]$ and $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 13. Let A and B be mappings from an IFM-space $(X, M, N, *, \lozenge)$ into itself. Then the pair (A, B) is said to be compatible of type (II) if and only if (B, A) is compatible of type (I).

Proposition 4. *Let $(X, M, N, *, \lozenge)$ be an IFM-space and A, B, be mappings from X into itself such that B (resp., A) is continuous. If the pair (A, B) is A-compatible (resp., B-compatible), then it is compatible of type (I) (resp., of type (II)).*

Proof. Suppose that the pair (A, B) is A-compatible and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Since B is continuous, we have

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1 = \lim_{n\to\infty} M(ABx_n, Bz, t)$$

and

$$\lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0 = \lim_{n\to\infty} N(ABx_n, Bz, t).$$

Further,

$$M(Bz, z, t) \geq M(ABx_n, z, \frac{t}{2}) * M(ABx_n, Bz, \frac{t}{2}),$$

$$N(Bz, z, t) \leq N(ABx_n, z, \frac{t}{2}) \Diamond N(ABx_n, Bz, \frac{t}{2})$$

and so

$$\begin{aligned} M(Bz, z, t) &\geq \lim_{n\to\infty} \left(M(ABx_n, z, \frac{t}{2}) * M(ABx_n, Bz, \frac{t}{2}) \right) \\ &= \lim_{n\to\infty} M(ABx_n, z, \frac{t}{2}) \end{aligned}$$

and

$$\begin{aligned} N(Bz, z, t) &\leq \lim_{n\to\infty} \left(N(ABx_n, z, \frac{t}{2}) \Diamond N(ABx_n, Bz, \frac{t}{2}) \right) \\ &= \lim_{n\to\infty} N(ABx_n, z, \frac{t}{2}). \end{aligned}$$

Hence it follows that

$$\underline{\lim}_{n\to\infty} M(ABx_n, z, \frac{t}{2}) \leq M(Bz, z, t) \text{ and } \overline{\lim}_{n\to\infty} N(ABx_n, z, \frac{t}{2}) \geq N(Bz, z, t).$$

This inequality holds for every choice of the sequence $\{x_n\}$ in X with the corresponding $z \in X$ and so the pair (A, B) is compatible of type (I). This completes the proof. □

Proposition 5. *Let $(X, M, N, *, \Diamond)$ be an IFM-space and A, B be mappings from X into itself with B (resp., A) is continuous. If the pair (A, B) is compatible of type (I) (resp., of type (II)) and, for every sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$, it follows that $\lim_{n\to\infty} ABx_n = z$ (resp., $\lim_{n\to\infty} BAx_n = z$), then it is A-compatible (resp., B-compatible).*

Proof. Suppose that the pair (A, B) is compatible of type (I) and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Since B is continuous, we have

$$M(Bz, z, t) \geq \underline{\lim}_{n\to\infty} M(ABx_n, z, \lambda t) \text{ and } N(Bz, z, t) \leq \overline{\lim}_{n\to\infty} N(ABx_n, z, \lambda t)$$

and

$$\begin{aligned}\lim_{n\to\infty} M(BBx_n, z, t) &\geq \underline{\lim}_{n\to\infty} \left(M(Bz, z, \frac{t}{2}) * M(BBx_n, Bz, \frac{t}{2}) \right) \\ &= M(Bz, z, \frac{t}{2}),\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} N(BBx_n, z, t) &\leq \overline{\lim}_{n\to\infty} \left(N(Bz, z, \frac{t}{2}) \Diamond N(BBx_n, Bz, \frac{t}{2}) \right) \\ &= N(Bz, z, \frac{t}{2})\end{aligned}$$

and so

$$\begin{aligned}\lim_{n\to\infty} M(BBx_n, z, t) &\geq \underline{\lim}_{n\to\infty} M(ABx_n, z, \frac{\lambda t}{2}),\ \lim_{n\to\infty} N(BBx_n, z, t) \\ &\leq \overline{\lim}_{n\to\infty} N(ABx_n, z, \frac{\lambda t}{2}).\end{aligned}$$

Now, we have

$$M(ABx_n, BBx_n, t) \geq M(ABx_n, z, \frac{t}{2}) * M(BBx_n, z, \frac{t}{2}),$$

$$N(ABx_n, BBx_n, t) \leq N(ABx_n, z, \frac{t}{2}) \Diamond N(BBx_n, z, \frac{t}{2}).$$

Hence, letting $n \to \infty$,

$$\begin{aligned}\underline{\lim}_{n\to\infty} M(ABx_n, BBx_n, t) &\geq \underline{\lim}_{n\to\infty} \left(M(ABx_n, z, \frac{t}{2}) * M(BBx_n, z, \frac{t}{2}) \right) \\ &\geq \underline{\lim}_{n\to\infty} \left(M(ABx_n, z, \frac{t}{2}) * M(ABx_n, z, \frac{\lambda t}{4}) \right) \\ &= 1,\end{aligned}$$

$$\begin{aligned}\overline{\lim}_{n\to\infty} N(ABx_n, BBx_n, t) &\leq \overline{\lim}_{n\to\infty} \left(N(ABx_n, z, \frac{t}{2}) \Diamond N(BBx_n, z, \frac{t}{2}) \right) \\ &\leq \overline{\lim}_{n\to\infty} \left(N(ABx_n, z, \frac{t}{2}) \Diamond N(ABx_n, z, \frac{\lambda t}{4}) \right) \\ &= 0.\end{aligned}$$

Thus $\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1$ and $\lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0$. This limit always exists and these are 1 and 0 for any sequence $\{x_n\}$ in X with the corresponding $z \in X$. Hence the pair (A, B) is A-compatible. This completes the proof. □

By unifying Propositions 4 and 5, we have the following:

Proposition 6. *Let $(X, M, N, *, \Diamond)$ be an IFM-space and A, B be mappings from X into itself with B (resp., A) is continuous. Suppose that, for every sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$, we have $\lim_{n\to\infty} ABx_n = z$ (resp., $\lim_{n\to\infty} BAx_n = z$). Then the pair (A, B) is compatible of type (I) (resp., of type (II)) if and only if it is A-compatible (resp., B-compatible).*

Now, we give some examples.

Example 1. Let $X = [0, \infty)$ with the metric d defined by $d(x, y) = |x - y|$. For each $t \in (0, \infty)$ and $x, y \in X$, define (M, N) by

$$M(x,y,t) = \begin{cases} \left[\exp\left(\frac{|x-y|}{t}\right)\right]^{-1}, & t > 0, \\ 0, & t = 0, \end{cases}$$

$$N(x,y,t) = \begin{cases} \left[\exp\left(\frac{|x-y|}{t}\right) - 1\right]\left[\exp\left(\frac{|x-y|}{t}\right)\right]^{-1}, & t > 0, \\ 1, & t = 0. \end{cases}$$

Clearly, $(X, M, N, *, \Diamond)$ is an IFM-space, where $*$ and $\Diamond$ are define by $a*b = \min\{a, b\}$ and $a\Diamond b = \max\{a, b\}$ respectively. Let A and B be defined by $Ax = 1$ for all $x \in [0, 1]$, $Ax = 1 + x$ for all $x \in (1, \infty)$ and $Bx = 1 + x$ for all $x \in [0, 1)$, $Bx = 1$ for all $x \in [1, \infty)$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$. By definition of A and B, $z \in \{1\}$ and $\lim_{n\to\infty} x_n = 0$. A and B both are discontinuous at $z = 1$. Therefore, we have

$$\lim_{n\to\infty} M(AAx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BAx_n, t) = 0,$$
$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0.$$

Also, we consider the sequence $\{x_n\}$ in X defined by $x_n = \frac{1}{2n}$, $n = 1, 2, \ldots$. Then we have $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 1$. Further, for $t > 0$, we have

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = \left[\exp\left(\frac{1}{t}\right)\right]^{-1} < 1,$$
$$\lim_{n\to\infty} N(ABx_n, BAx_n, t) = \left[\exp\left(\frac{1}{t}\right) - 1\right]\left[\exp\left(\frac{1}{t}\right)\right]^{-1} > 0,$$

and

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = \left[\exp\left(\frac{1}{t}\right)\right]^{-1} < 1,$$
$$\lim_{n\to\infty} N(ABx_n, BBx_n, t) = \left[\exp\left(\frac{1}{t}\right) - 1\right]\left[\exp\left(\frac{1}{t}\right)\right]^{-1} > 0.$$

Therefore, (A, B) is compatible of type (β) and B-compatible, but they are neither compatible nor A-compatible. Moreover, $z = 1$ is a common fixed point of A and B. Hence the pair (A, B) is compatible of type (I) as well as of type (II).

Example 2. Let $X = [0,1]$ and let $*$ be the continuous t-norm and $\Diamond$ be the continuous t-conorm defined by $a * b = \min\{a, b\}$ and $a\Diamond b = \max\{a, b\}$ respectively, for all $a, b \in [0,1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define (M, N) by

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, & t > 0, \\ 0, & t = 0 \end{cases} \quad \text{and } N(x,y,t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, & t > 0, \\ 1, & t = 0. \end{cases}$$

Clearly, $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space, where $*$ and $\Diamond$ are defined by $a * b = \min\{a, b\}$ and $a\Diamond b = \max\{a, b\}$ respectively. Let A and B be defined by $Ax = 0$ for $\frac{1}{3} < x < \frac{1}{2}$, $Ax = 1$ for $0 \leq x \leq \frac{1}{3}$ and $\frac{1}{2} \leq x \leq 1$ and $Bx = x$ for all $x \in X$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$. By definition of A and B, $z \in \{1\}$ and $\lim_{n\to\infty} x_n = 1$. Therefore, we have

$$\begin{aligned} \lim_{n\to\infty} M(AAx_n, BAx_n, t) &= 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BAx_n, t) = 0, \\ \lim_{n\to\infty} M(ABx_n, BAx_n, t) &= 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0, \\ \lim_{n\to\infty} M(ABx_n, BBx_n, t) &= 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0, \\ \lim_{n\to\infty} M(AAx_n, BBx_n, t) &= 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0. \end{aligned}$$

Thus (A, B) is compatible, A-compatible, B-compatible, compatible of type (α), compatible of type (β). Moreover, $z = 0$ is a fixed point of B and $z = 1$ is a fixed point of A. Hence the pair (A, B) is compatible of type (I) as well as of type (II).

In the previous two examples, the pair (A, B) was compatible of type (I) as well as of type (II). But the following example shows that this need not to be case always and also shows that the conclusion of Proposition 4 need not to be true B is not continuous.

Example 3. Let $X = [0,2]$ with the usual metric. For each $t > 0$ and $x, y \in X$, define (M, N) by

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, & t > 0, \\ 0, & t = 0 \end{cases} \quad \text{and } N(x,y,t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, & t > 0, \\ 1, & t = 0. \end{cases}$$

Clearly, $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space, when $*$ and $\Diamond$ are defined by $a * b = ab$ and $a\Diamond b = \min\{1, a + b\}$ respectively. Let A and B be defined as $Ax = 1$ for all $x \in X$ and $Bx = 1$ for $x \neq 1$, $Bx = 2$ for $x = 1$. Then B is not continuous at $z = 1$. We assert that the pair (A, B) is compatible of type (II), but not of type (I), of type (α), of type (β), A-compatible, B-compatible or compatible.

To see this, we suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$. By definition of A and B, $z \in \{1\}$. Since A and B agree on $X\backslash\{1\}$, we need only consider $x_n \to 1$. Now, $ABx_n = 1$, $BAx_n = 2$, $AAx_n = 1$, $BBx_n = 2$, $A1 = 1$ and $B1 = 2$. Thus, for $t > 0$,

$$\begin{aligned}
\lim_{n\to\infty} M(ABx_n, BAx_n, t) &= \frac{t}{t+1} < 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = \frac{1}{t+1} > 0,\\
\lim_{n\to\infty} M(AAx_n, BAx_n, t) &= \frac{t}{t+1} < 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BAx_n, t) = \frac{1}{t+1} > 0,\\
\lim_{n\to\infty} M(ABx_n, BBx_n, t) &= \frac{t}{t+1} < 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = \frac{1}{t+1} > 0,\\
\lim_{n\to\infty} M(AAx_n, BBx_n, t) &= \frac{t}{t+1} < 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = \frac{1}{t+1} > 0.
\end{aligned}$$

Therefore, the pair (A, B) is none of compatible of type (α), of type (β), A-compatible, B-compatible or compatible. Also, for $t > 0$,

$$M(B1, 1, t) = \frac{t}{t+1} < 1 = \underline{\lim}_{n\to\infty} M(ABx_n, 1, t)$$

and

$$N(B1, 1, t) = \frac{1}{t+1} > 0 = \overline{\lim}_{n\to\infty} N(ABx_n, 1, t),$$

$$M(A1, 1, t) = 1 > \frac{t}{t+1} = \underline{\lim}_{n\to\infty} M(BAx_n, 1, t)$$

and

$$N(A1, 1, t) = 0 < \frac{1}{t+1} = \overline{\lim}_{n\to\infty} N(BAx_n, 1, t).$$

Therefore, the pair (A, B) compatible of type (II), but not of type (I).

Proposition 7. *Let $(X, M, N, *, \Diamond)$ be an IFM-space and A, B be mappings from X into itself. Suppose that the pair (A, B) is compatible of type (I) (resp., of type (II)) and $Az = Bz$ for some $z \in X$. Then, for $t > 0$ and $\lambda \in (0, 1]$, $M(Az, BBz, t) \geq M(Az, ABz, \lambda t)$ and $N(Az, BBz, t) \leq N(Az, ABz, \lambda t)$ (resp., $M(Bz, AAz, t) \geq M(Bz, BAz, \lambda t)$ and $N(Bz, AAz, t) \leq N(Bz, BAz, \lambda t)$).*

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_n = z$ for $n = 1, 2, \ldots$ and $Az = Bz$ for some $z \in X$. Then we have $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$. Suppose the pair (A, B) is compatible of type (I). Then, for $t > 0$ and $\lambda \in (0, 1]$,

$$M(Az, BBz, t) \geq \underline{\lim}_{n\to\infty} M(Az, ABx_n, \lambda t) = M(Az, ABz, \lambda t)$$

and

$$N(Az, BBz, t) \leq \overline{\lim}_{n\to\infty} N(Az, ABx_n, \lambda t) = N(Az, ABz, \lambda t).$$

□

4. Fixed point theorem

In this section, we prove a fixed point theorem for four mappings under the condition of compatible mappings of type (I) and (II) in intuitionistic fuzzy metric spaces.

Theorem 1. *Let $(X, M, N, *, \Diamond)$ be an complete IFM-space with $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$. Let A, B, S and T be mappings from X into itself such that*

$$A(X) \subseteq S(X) \text{ and } B(X) \subseteq T(X) \tag{4.1}$$

there exists a constant $k \in (0,1)$ such that

$$\begin{aligned} M(Ax, By, kt) \geq\ & M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ & * M(Ax, Ty, \alpha t) * M(By, Sx, (2-\alpha)t) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} N(Ax, By, kt) \quad \leq \quad & N(Sx, Ty, t)\Diamond N(Ax, Sx, t)\Diamond N(By, Ty, t) \\ & \Diamond\, N(Ax, Ty, \alpha t)\Diamond N(By, Sx, (2-\alpha)t) \end{aligned}$$

for all $x, y \in X$, $\alpha \in (0,2)$ and $t > 0$. Let A, B, S and T are satisfying conditions any one of the following:

- (C_1) *A is continuous and the pairs (A,S) and (B,T) are compatible of type (II).*
- (C_2) *B is continuous and the pairs (A,S) and (B,T) are compatible of type (II).*
- (C_3) *S is continuous and the pairs (A,S) and (B,T) are compatible of type (I).*
- (C_4) *T is continuous and the pairs (A,S) and (B,T) are compatible of type (I).*

Then A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point of X. By (4.1), we can construct a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n}, \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$$

for $n = 0, 1, \ldots$. Then, by (4.2), for $\alpha = 1 - q$, $q \in (0,1)$, we have

$$\begin{aligned} & M(Ax_{2n}, Bx_{2n+1}, kt) \\ \geq\ & M(Sx_{2n}, Tx_{2n+1}, t) * M(Ax_{2n}, Sx_{2n}, t) \\ & * M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Ax_{2n}, Tx_{2n+1}, (1-q)t) \\ & * M(Bx_{2n+1}, Sx_{2n}, (1+q)t) \end{aligned}$$

and

$$\begin{aligned} & N(Ax_{2n}, Bx_{2n+1}, kt) \\ \le\ & N(Sx_{2n}, Tx_{2n+1}, t) \Diamond N(Ax_{2n}, Sx_{2n}, t) \\ & \Diamond N(Bx_{2n+1}, Tx_{2n+1}, t) \Diamond N(Ax_{2n}, Tx_{2n+1}, (1-q)t) \\ & \Diamond N(Bx_{2n+1}, Sx_{2n}, (1+q)t) \end{aligned}$$

and so

$$\begin{aligned} & M(y_{2n}, y_{2n+1}, kt) \\ \ge\ & M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n-1}, t) \\ & * M(y_{2n+1}, y_{2n}, t) * M(y_{2n}, y_{2n}, (1-q)t) \\ & * M(y_{2n+1}, y_{2n-1}, (1+q)t) \\ \ge\ & M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) \\ & * M(y_{2n+1}, y_{2n}, qt) \end{aligned}$$

and

$$\begin{aligned} & M(y_{2n}, y_{2n+1}, kt) \\ \le\ & N(y_{2n-1}, y_{2n}, t) \Diamond N(y_{2n}, y_{2n-1}, t) \\ & \Diamond N(y_{2n+1}, y_{2n}, t) \Diamond N(y_{2n}, y_{2n}, (1-q)t) \\ & \Diamond N(y_{2n+1}, y_{2n-1}, (1+q)t) \\ \le\ & N(y_{2n-1}, y_{2n}, t) \Diamond N(y_{2n}, y_{2n+1}, t) \\ & \Diamond N(y_{2n+1}, y_{2n}, qt). \end{aligned}$$

Thus it follows that

$$M(y_{2n}, y_{2n+1}, kt) \ge M(y_{2n-1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, qt)$$

and

$$N(y_{2n}, y_{2n+1}, kt) \le N(y_{2n-1}, y_{2n}, t) \Diamond N(y_{2n+1}, y_{2n}, t) \Diamond N(y_{2n+1}, y_{2n}, qt).$$

Since t-norm and t-conorm $*$ and $\Diamond$ are continuous and $M(x, y, \cdot)$ and $N(x, y, \cdot)$ are continuous, letting $q \to 1$, we have

$$M(y_{2n}, y_{2n+1}, kt) \ge M(y_{2n-1}, y_{2n}, t) * M(y_{2n+1}, y_{2n}, t)$$

and

$$N(y_{2n}, y_{2n+1}, kt) \le N(y_{2n-1}, y_{2n}, t) \Diamond N(y_{2n+1}, y_{2n}, t).$$

Similarly, we also have

$$M(y_{2n+1}, y_{2n+2}, kt) \ge M(y_{2n}, y_{2n+1}, t) * M(y_{2n+2}, y_{2n+1}, t)$$

and

$$N(y_{2n+1}, y_{2n+2}, kt) \le N(y_{2n}, y_{2n+1}, t) \Diamond N(y_{2n+2}, y_{2n+1}, t).$$

In general, we have, for $m = 1, 2, \ldots$,

$$M(y_{m+1}, y_{m+2}, kt) \ge M(y_m, y_{m+1}, t) * M(y_{m+1}, y_{m+2}, t)$$

and

$$N(y_{m+1}, y_{m+2}, kt) \leq N(y_m, y_{m+1}, t) \Diamond N(y_{m+1}, y_{m+2}, t).$$

Consequently, it follows that, for $m, p = 1, 2, \ldots,$

$$M(y_{m+1}, y_{m+2}, kt) \geq M(y_m, y_{m+1}, t) * M(y_{m+1}, y_{m+2}, \frac{t}{k^p})$$

and

$$N(y_{m+1}, y_{m+2}, kt) \leq N(y_m, y_{m+1}, t) \Diamond N(y_{m+1}, y_{m+2}, \frac{t}{k^p}).$$

By noting that $M(y_{m+1}, y_{m+2}, \frac{t}{k^p}) \to 1$ and $N(y_{m+1}, y_{m+2}, \frac{t}{k^p}) \to 0$ as $p \to \infty$, we have, for $m = 1, 2, \ldots,$

$$M(y_{m+1}, y_{m+2}, kt) \geq M(y_m, y_{m+1}, t)$$

and

$$N(y_{m+1}, y_{m+2}, kt) \leq N(y_m, y_{m+1}, t).$$

Hence, by Lemma 1, $\{y_n\}$ is a Cauchy sequence in X. Since $(X, M, N, *, \Diamond)$ is complete, it converges to a point z in X. Since $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n+2}\}$ and $\{Tx_{2n+1}\}$ are subsequence of $\{y_n\}$. Therefore, Ax_{2n}, Bx_{2n+1}, Sx_{2n+2}, $Tx_{2n+1} \to z$ as $n \to \infty$.

Now, suppose that the condition (C_4) holds. Then, since the pair (B, T) is compatible of type (I) and T is continuous, we have

$$\begin{aligned} M(Tz, z, t) &\geq \underline{\lim}_{n \to \infty} M(BTx_{2n+1}, z, \lambda t), \\ N(Tz, z, t) &\leq \overline{\lim}_{n \to \infty} N(BTx_{2n+1}, z, \lambda t), \\ TTx_{2n+1} &\to Tz. \end{aligned}$$

Now, for $\alpha = 1$, setting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (4.2), we obtain

$$\begin{aligned} &M(Ax_{2n}, BTx_{2n+1}, kt) \\ &\geq M(Sx_{2n}, TTx_{2n+1}, t) * M(Ax_{2n}, Sx_{2n}, t) \\ &\quad * M(BTx_{2n+1}, TTx_{2n+1}, t) * M(Ax_{2n}, TTx_{2n+1}, t) \\ &\quad * M(BTx_{2n+1}, Sx_{2n}, t) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &N(Ax_{2n}, BTx_{2n+1}, kt) \\ \leq\ &N(Sx_{2n}, TTx_{2n+1}, t) \Diamond N(Ax_{2n}, Sx_{2n}, t) \\ &\Diamond N(BTx_{2n+1}, TTx_{2n+1}, t) \Diamond N(Ax_{2n}, TTx_{2n+1}, t) \\ &\Diamond NM(BTx_{2n+1}, Sx_{2n}, t). \end{aligned}$$

Thus, by letting the limit inferior on both sides of (4.3), we have

$$\begin{aligned} &\varliminf_{n\to\infty} M(z, BTx_{2n+1}, kt) \\ \geq\ & M(z,Tz,t) * M(z,z,t) * \varliminf_{n\to\infty} M(Tz, BTx_{2n+1}, t) \\ & * M(z,Tz,t) * \varliminf_{n\to\infty} M(z, BTx_{2n+1}, t) \end{aligned}$$

and

$$\begin{aligned} &\varlimsup_{n\to\infty} N(z, BTx_{2n+1}, kt) \\ \leq\ & N(z,Tz,t) \Diamond M(z,z,t) \Diamond \varlimsup_{n\to\infty} N(Tz, BTx_{2n+1}, t) \\ & \Diamond N(z,Tz,t) \Diamond \varlimsup_{n\to\infty} N(z, BTx_{2n+1}, t). \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} &\varliminf_{n\to\infty} M(z, BTx_{2n+1}, kt) \\ \geq\ & M(z,Tz,t) * \varliminf_{n\to\infty} M(Tz, BTx_{2n+1}, t) \\ & * \varliminf_{n\to\infty} M(z, BTx_{2n+1}, t) \\ \geq\ & M(z,Tz,t) * M(z,Tz,\frac{t}{2}) * \varliminf_{n\to\infty} M(z, BTx_{2n+1}, \frac{t}{2}) \\ & * \varliminf_{n\to\infty} M(z, BTx_{2n+1}, t) \\ \geq\ & \varliminf_{n\to\infty} M(z, BTx_{2n+1}, \lambda t) * \varliminf_{n\to\infty} M(Tz, BTx_{2n+1}, \frac{\lambda t}{2}) \\ & * \varliminf_{n\to\infty} M(z, BTx_{2n+1}, \frac{t}{2}) * \varliminf_{n\to\infty} M(z, BTx_{2n+1}, t) \end{aligned}$$

and

$$\begin{aligned} &\varlimsup_{n\to\infty} N(z, BTx_{2n+1}, kt) \\ \leq\ & N(z,Tz,t) \Diamond \varlimsup_{n\to\infty} N(Tz, BTx_{2n+1}, t) \\ & \Diamond \varlimsup_{n\to\infty} N(z, BTx_{2n+1}, t) \\ \leq\ & N(z,Tz,t) \Diamond N(z,Tz,\frac{t}{2}) \Diamond \varlimsup_{n\to\infty} N(z, BTx_{2n+1}, \frac{t}{2}) \\ & \Diamond \varlimsup_{n\to\infty} N(z, BTx_{2n+1}, t) \\ \leq\ & \varlimsup_{n\to\infty} N(z, BTx_{2n+1}, \lambda t) \Diamond \varlimsup_{n\to\infty} N(Tz, BTx_{2n+1}, \frac{\lambda t}{2}) \\ & \Diamond \varlimsup_{n\to\infty} N(z, BTx_{2n+1}, \frac{t}{2}) \Diamond \varlimsup_{n\to\infty} N(z, BTx_{2n+1}, t) \end{aligned}$$

and, for $\lambda = 1$,

$$\underline{\lim}_{n\to\infty} M(z, BTx_{2n+1}, kt) \geq \underline{\lim}_{n\to\infty} M(z, BTx_{2n+1}, \frac{t}{2})$$

and

$$\overline{\lim}_{n\to\infty} N(z, BTx_{2n+1}, kt) \leq \overline{\lim}_{n\to\infty} N(z, BTx_{2n+1}, \frac{t}{2}).$$

Thus, by Remark 3, it follows that $\lim_{n\to\infty} BTx_{2n+1} = z$. Now using the compatibility of type (I), we have

$$\begin{aligned} M(Tz, z, t) &\geq \underline{\lim}_{n\to\infty} M(z, BTx_{2n+1}, \lambda t) = 1, \\ N(Tz, z, t) &\leq \overline{\lim}_{n\to\infty} N(z, BTx_{2n+1}, \lambda t) = 0 \end{aligned}$$

and so it follows that $Tz = z$.

Again, replacing x by x_{2n} and y by z in (4.2), for $\alpha = 1$, we have

$$\begin{aligned} M(Ax_{2n}, Bz, kt) \geq\; & M(Sx_{2n}, z, t) * M(Ax_{2n}, Sx_{2n}, t) \\ & * M(Bz, z, t) * M(Ax_{2n}, z, t) * M(Bz, Sx_{2n}, t) \end{aligned}$$

and

$$\begin{aligned} N(Ax_{2n}, Bz, kt) \leq\; & N(Sx_{2n}, z, t) \Diamond N(Ax_{2n}, Sx_{2n}, t) \\ & \Diamond N(Bz, z, t) \Diamond N(Ax_{2n}, z, t) \Diamond N(Bz, Sx_{2n}, t) \end{aligned}$$

and so, letting $n \to \infty$, we have

$$M(Bz, z, kt) \geq M(Bz, z, t) \text{ and } N(Bz, z, kt) \leq N(Bz, z, t),$$

which implies that, by Lemma 2, $Bz = z$. Since $B(X) \subseteq S(X)$, there exists a point $u \in X$ such that $Bz = Su = z$. By (4.2), for $\alpha = 1$, we have

$$\begin{aligned} M(Au, z, kt) \geq\; & M(Su, z, t) * M(Au, Su, t) \\ & * M(z, z, t) * M(Au, z, t) * M(z, Su, t) \end{aligned}$$

and

$$\begin{aligned} N(Au, z, kt) \leq\; & N(Su, z, t) \Diamond N(Au, Su, t) \\ & \Diamond N(z, z, t) \Diamond N(Au, z, t) \Diamond N(z, Su, t) \end{aligned}$$

and so

$$M(Au, z, kt) \geq M(Au, z, t) \text{ and } N(Au, z, kt) \leq N(Au, z, t)$$

which implies that, by Lemma 2, $Au = z$. Since the pair (A, S) is compatible of type (I) and $Au = Su = z$, by Proposition 7, we have

$$M(Au, SSz, t) \geq M(Au, ASz, t) \text{ and } N(Au, SSz, t) \leq N(Au, ASz, t)$$

and so

$$M(z, Sz, t) \geq M(z, Az, t) \text{ and } N(z, Sz, t) \leq N(z, Az, t).$$

Again, by (4.2), for $\alpha = 1$, we have

$$\begin{aligned} M(Az, z, kt) &\geq M(Sz, z, t) * M(Az, Sz, t) \\ &\quad * M(z, z, t) * M(Az, z, t) * M(z, Sz, t) \end{aligned}$$

and

$$\begin{aligned} N(Az, z, kt) &\leq N(Sz, z, t) \Diamond N(Az, Sz, t) \\ &\quad \Diamond N(z, z, t) \Diamond N(Az, z, t) \Diamond N(z, Sz, t). \end{aligned}$$

Thus it follows that

$$\begin{aligned} M(Az, z, kt) &\geq M(Sz, z, t) * M(Az, Sz, t) * M(Az, z, t) \\ &\geq M(Az, z, \frac{t}{2}) \end{aligned}$$

and

$$\begin{aligned} N(Az, z, kt) &\leq N(Sz, z, t) \Diamond N(Az, Sz, t) \Diamond N(Az, z, t) \\ &\leq N(Az, z, \frac{t}{2}) \end{aligned}$$

and so, by Lemma 2, $Az = z$. Therefore, $Az = Bz = Sz = Tz = z$ and z is a common fixed point of A, B, S and T. The uniqueness of a common fixed point can be easily verified by using (4.2).

The other cases (C_1), (C_2) and (C_3) can be disposed from a similar argument as above. □

Now we give an example to support our main theorem.

Example 4. Let $X = \{\frac{1}{n} : n = 1, 2, \ldots\} \cup \{0\}$ with the usual metric and, for all $t > 0$ and $x, y \in X$, define (M, N) by

$$M(x, y, t) = \begin{cases} \frac{t}{t+|x-y|}, & t > 0, \\ 0, & t = 0 \end{cases}$$

and

$$N(x, y, t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, & t > 0, \\ 1, & t = 0. \end{cases}$$

Clearly, $(X, M, N, *, \Diamond)$ is a complete intuitionistic fuzzy metric space, where $*$ and $\Diamond$ are defined by $a * b = \min\{a, b\}$ and $a \Diamond b = \max\{a, b\}$ respectively. Let A, B, S and T be define by $Ax = \frac{x}{4}$, $Sx = \frac{x}{2}$, $Bx = \frac{x}{6}$, $Tx = \frac{x}{3}$ for all $x \in X$. Then we have

$$\begin{aligned} A(X) &= \{\frac{1}{4n} : n = 1, 2, \ldots\} \cup \{0\} \subseteq \{\frac{1}{2n} : n = 1, 2, \ldots\} \cup \{0\} = S(X) \\ B(X) &= \{\frac{1}{6n} : n = 1, 2, \ldots\} \cup \{0\} \subseteq \{\frac{1}{3n} : n = 1, 2, \ldots\} \cup \{0\} = T(X). \end{aligned}$$

Also, the condition (4.2) of Theorem 1 is satisfied and A, B, S and T are continuous. Further, the pairs (A, S) and (B, T) are compatible of type (I) and of type (II) if $\lim_{n\to\infty} x_n = 0$, where $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 0$ for some $0 \in X$. Thus all the conditions of Theorem 1 are satisfied and also 0 is the unique common fixed point of A, B, S and T.

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