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PERTURBATIONS OF HIGHER TERNARY DERIVATIONS IN BANACH TERNARY ALGEBRAS

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Abstract. We investigate approximately higher ternary derivations in Banach ternary algebras via the Cauchy functional equation

$$f(\lambda_1 x + \lambda_2 y + \lambda_3 z) = \lambda_1 f(x) + \lambda_2 f(y) + \lambda_3 f(z).$$

1. Introduction and preliminaries

A ternary algebra $\mathcal{A}$ is a real or complex linear space, endowed with a linear mapping, the so-called a ternary product $(x, y, z) \to [x, y, z]$ of $\mathcal{A}\times\mathcal{A}\times\mathcal{A}$ into $\mathcal{A}$ such that $[[x, y, z,], w, v] = [x, [y, z, w], v] = [x, y, [z, w, v]]$ for all $v, w, x, y, z \in \mathcal{A}$.

If $(\mathcal{A}, \odot)$ is a usual (binary) algebra, then $[x, y, z] := (x \odot y) \odot z$ makes $\mathcal{A}$ into a ternary algebra. Hence the ternary algebra is a natural generalization of the binary case. In particular, if a ternary algebra $(\mathcal{A}, [\ ])$ has a unit, i.e., an element $e \in \mathcal{A}$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in \mathcal{A}$, then $\mathcal{A}$ with the binary product $x \odot y := [x, e, y]$, is a usual algebra. By a normed ternary algebra we mean a ternary algebra with a norm $\|\cdot\|$ such that $\|[x, y, z]\| \le \|x\|\|y\|\|z\|$ for all $x, y, z \in \mathcal{A}$. A Banach ternary algebra is a normed ternary algebra such that the normed linear space with norm $\|\cdot\|$ is complete.

Ternary algebraic operations were considered in the XIX-th century by several mathematicians such as A. Cayley [5] who first introduced in 1840 the notion of "cubic matrices" and a generalization of the determinant, called the "hyperdeterminant", then were found again and generalized by M. Kapranov, I. M. Gelfand and A. Zelevinskii in 1990 [15]. The simplest example of this (non-commutative and non-associative) ternary algebra is given by the following composition rule:

$$\{a, b, c\}_{i,j,k} = \sum_{l,m,n=1}^{N} a_{nil} b_{ljm} c_{mkn}, \qquad i, j, k = 1, 2, \ldots, N.$$

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As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called "Nambu mechanics" which has been proposed by Y. Nambu [19] in 1973, is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the "anyons"), supersymmetric theories, Yang-Baxter equation, etc, (cf. [1, 16, 28]).

Throughout this paper, we assume that $\mathcal{A}$ and $\mathcal{B}$ are real or complex ternary algebras. For the sake of convenience, we use the same symbol $[\ ]$ (resp. $\|\cdot\|$) in order to represent the ternary products (resp. norms) on ternary algebras $\mathcal{A}$ and $\mathcal{B}$.

A linear mapping $h : \mathcal{A} \to \mathcal{B}$ is said to be a *ternary homomorphism* if $h([x,y,z]) = [h(x),h(y),h(z)]$ holds for all $x,y,z \in \mathcal{A}$. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a *ternary derivation* if $d([x,y,z]) = [d(x),y,z] + [x,d(y),z] + [x,y,d(z)]$ holds for all $x,y,z \in \mathcal{A}$ (see [20]).

Let $\mathbb{N}$ be the set of natural numbers. For $m \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$, a sequence $H = \{h_0, h_1, \ldots, h_m\}$ (resp. $H = \{h_0, h_1, \ldots, h_n, \ldots\}$) of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a *higher ternary derivation* of rank m (resp. infinite rank) from $\mathcal{A}$ into $\mathcal{B}$ if

$$h_n([x,y,z]) = \sum_{i+j+k=n} [h_i(x), h_j(y), h_k(z)]$$

holds for each $n \in \{0,1,\ldots,m\}$ (resp. $n \in \mathbb{N}_0$) and all $x,y,z \in \mathcal{A}$ (cf, see [11, 25]). The higher ternary derivation H from $\mathcal{A}$ into $\mathcal{B}$ is said to be *onto* if $h_0 : \mathcal{A} \to \mathcal{B}$ is onto. The higher ternary derivation H on $\mathcal{A}$ is called be *strong* if h_0 is an identity mapping on $\mathcal{A}$. Of course, a higher ternary derivation of rank 0 from $\mathcal{A}$ into $\mathcal{B}$ (resp. a strong higher ternary derivation of rank 1 on $\mathcal{A}$) is a ternary homomorphism (resp. a ternary derivation). So a higher ternary derivation is a generalization of both a ternary homomorphism and a ternary derivation.

Here let us consider an approximately higher ternary derivation which is not an exactly higher ternary derivation in Banach ternary algebras.

The following remark is a slight modification of an example given by P. Šemrl [26] which is due to B. E. Johnson [12] (see also [17, Example 1.1]).

Remark 1.1. Let X be a compact Hausdorff space and let $(\mathcal{A}, [\ ])$ be the Banach ternary algebra of complex-valued continuous functions on X under the usual addition of complex-valued continuous functions, the ternary operation $[\rho_1, \rho_2, \rho_3] = \rho_1 * \rho_2 * \rho_3$ and the supremum norm $\|\cdot\|_\infty$, where $*$ denotes the usual multiplication of complex-valued continuous functions. Assume that $\tau : \mathcal{A} \to \mathcal{A}$ is a continuous ternary homomorphism. We define $f : \mathcal{A} \to \mathcal{A}$ by

$$f(x)(a) = \begin{cases} \tau(x)(a) \log|\tau(x)(a)| & \text{if } \tau(x)(a) \neq 0, \\ 0 & \text{if } \tau(x)(a) = 0 \end{cases}$$

for all $x \in \mathcal{A}$ and all $a \in X$. It is easy to see that $f([x,y,z]) = [f(x), \tau(y), \tau(z)] + [\tau(x), f(y), \tau(z)] + [\tau(x), \tau(y), f(z)]$ for all $x,y,z \in \mathcal{A}$. Let $h_0 = \tau$, $h_n = 0$, $1 \leq$

$n \leq m-1$ and $h_m = f$. Then we see that the sequence $H = \{h_0, h_1, \ldots, h_m\}$ satisfies the relation

$$h_n([x,y,z]) = \sum_{i+j+k=n} [h_i(x), h_j(y), h_k(z)]$$

for all $x, y, z \in \mathcal{A}$. Observe that for all $u, v, w \in \mathbb{C} \setminus \{0\}$ with $u+v+w \neq 0$, where $\mathbb{C}$ is a complex field,

$$\big|(u+v+w)\log|u+v+w| - u\log|u| - v\log|v| - w\log|w|\big| \leq 2(|u|+|v|+|w|).$$

Indeed, fix $u, v, w \in \mathbb{C} \setminus \{0\}$, $u+v+w \neq 0$ arbitrarily. Since $\log(1+x) \leq x$ for all $x \geq 0$, we get

$$\begin{aligned}
&\big|(u+v+w)\log|u+v+w| - u\log|u| - v\log|v| - w\log|w|\big| \\
&\leq |u|\left|\log\frac{|u+v+w|}{|u|}\right| + |v|\left|\log\frac{|u+v+w|}{|v|}\right| + |w|\left|\log\frac{|u+v+w|}{|w|}\right| \\
&\leq |u|\log\left(1+\frac{|v+w|}{|u|}\right) + |v|\log\left(1+\frac{|u+w|}{|v|}\right) + |w|\log\left(1+\frac{|u+v|}{|w|}\right) \\
&\leq |u|\frac{|v+w|}{|u|} + |v|\frac{|u+w|}{|v|} + |w|\frac{|u+v|}{|w|} \\
&= |v+w| + |u+w| + |u+v| \\
&\leq 2(|u|+|v|+|w|).
\end{aligned}$$

This yields

$$\|h_n(x+y+z) - h_n(x) - h_n(y) - h_n(z)\|_\infty \leq 2\|\tau\|(\|x\|_\infty + \|y\|_\infty + \|z\|_\infty)$$

for each $n = 0, 1, \ldots, m$ and all $x, y, z \in \mathcal{A}$. Hence H is not an exactly higher ternary derivation on $\mathcal{A}$ since h_n is not exactly linear for each $n \in \mathbb{N}_0$. That is, we may regard H as an approximately higher ternary derivation of rank m on $\mathcal{A}$.

In 1940, S. M. Ulam [27] gave a talk concerning approximate mappings before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems: "*Under what condition does there exists a homomorphism near an approximate homomorphism?*" In 1941, D. H. Hyers [9] answered affirmatively the question of Ulam for Banach spaces, which states that if $\varepsilon > 0$ and $f : X \to Y$ is a mapping with X a normed space, Y a Banach space such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$. This stability phenomenon is called the *Hyers-Ulam stability* of the additive functional equation $g(x+y) = g(x) + g(y)$.

A generalized version of the theorem of Hyers for approximately additive mappings was given by Th. M. Rassias [24] in 1978 by considering the case when the above inequality is unbounded: if there exist $\theta \geq 0$ and $0 \leq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$$

for all $x \in X$. From this fact, several authors say that the additive functional equation $g(x+y) = g(x) + g(y)$ has the *Hyers-Ulam-Rassias stability* property. Since then, a great deal of work of Rassias type has been done by a number of authors (cf. [13, 21, 22, 23] and reference therein).

In 1949, D. G. Bourgin [4] proved the following result, which is sometimes called the superstability of ring homomorphisms: suppose that A and B are Banach algebras with unit. If $f : A \to B$ is a surjective mapping such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon,$$
$$\|f(xy) - f(x)f(y)\| \leq \delta$$

for some $\varepsilon \geq 0$, $\delta \geq 0$ and all $x, y \in A$, then f is a ring homomorphism.

Recently, R. Badora [3] and T. Miura *et al.* [17] proved the Hyers-Ulam stability, the Isac and Rassias-type stability [10], the Hyers-Ulam-Rassias stability and the Bourgin-type superstability of ring derivations on Banach algebras.

On the other hand, C. Park [20] and M. S. Moslehian [18] have contributed works on the stability problem of ternary homomorphisms and ternary derivations.

Our purpose in this note is to show the existence of an exact higher ternary derivation near to an approximately higher ternary derivation by investigating the Hyers-Ulam stability for higher ternary derivations in Banach ternary algebras. Furthermore, we are going to examine the Isac and Rassias-type stability [10] and the Bourgin-type superstability for higher ternary derivations in Banach ternary algebras.

2. Main results

By a similar to in [2], we first obtain the Hyers-Ulam stability result.

Theorem 2.1. *Let $\mathcal{A}$ be a normed ternary algebra and $\mathcal{B}$ a Banach ternary algebra. Suppose that $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for some $\delta \geq 0$, $\varepsilon \geq 0$ and each $n \in \mathbb{N}_0$,*

$$\|f_n(\lambda_1 x + \lambda_2 y + \lambda_3 z) - \lambda_1 f_n(x) - \lambda_2 f_n(y) - \lambda_3 f_n(z)\| \leq \varepsilon \tag{2.1}$$

and

$$\left\| f_n([x,y,z]) - \sum_{i+j+k=n} [f_i(x), f_j(y), f_k(z)] \right\| \leq \delta \tag{2.2}$$

hold for all $x, y, z \in \mathcal{A}$ and all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$. Then there exists a unique higher ternary derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n \in \mathbb{N}_0$,

$$\|f_n(x) - h_n(x)\| \leq \frac{\varepsilon}{2} \tag{2.3}$$

holds for all $x \in \mathcal{A}$. Moreover, we have

$$\sum_{i+j+k=n} [h_i(x), h_j(y), \{h_k(z) - f_k(z)\}] = 0 \tag{2.4}$$

for each $n \in \mathbb{N}_0$ and all $x, y, z \in \mathcal{A}$.

Proof. Putting $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $x = y = z$ in (2.1) implies

$$\left\|\frac{1}{3} f_n(3x) - f_n(x)\right\| \leq \frac{\varepsilon}{3} \tag{2.5}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. So, given $l, m \in \mathbb{N}$, it follows from the Hyers' direct method that

$$\left\|\frac{1}{3^l} f_n(3^l x) - f_n(x)\right\| \leq \frac{\varepsilon}{2}\left(1 - \frac{1}{3^l}\right)$$

from which we infer that

$$\left\|\frac{1}{3^{l+m}} f_n(3^{l+m} x) - \frac{1}{3^m} f_n(3^m x)\right\| \leq \frac{\varepsilon}{2 \cdot 3^m}. \tag{2.6}$$

Thus, by the Cauchy criterion, the limit

$$h_n(x) := \lim_{l \to \infty} \frac{1}{3^l} f_n(3^l x) \tag{2.7}$$

exists for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Let $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in (2.1) and then let us replace x by $3^l x$, y by $3^l y$ and z by $3^l z$. By dividing the result by 3^l and taking $l \to \infty$, we see that for each $n \in \mathbb{N}_0$, h_n satisfies the functional equation $f(x+y+z) - f(x) - f(y) - f(z) = 0$ for all $x, y, z \in \mathcal{A}$. Note that the functional equation $f(x + y + z) - f(x) - f(y) - f(z) = 0$ is equivalent to the Cauchy additive functional equation $f(x + y) - f(x) - f(y) = 0$. Hence h_n is additive for each $n \in \mathbb{N}_0$. Putting $l = 0$ in (2.6) and taking $m \to \infty$, we obtain (2.3).

Setting $x = y = z$ in (2.1) yields

$$\|f_n((\lambda_1 + \lambda_2 + \lambda_3)x) - (\lambda_1 + \lambda_2 + \lambda_3) f_n(x)\| \leq \varepsilon$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Thus we see that

$$3^{-l}\|f_n(3^l(\lambda_1 + \lambda_2 + \lambda_3)x) - (\lambda_1 + \lambda_2 + \lambda_3) f_n(3^l x)\| \to 0$$

as $l \to \infty$ which implies that for each $n \in \mathbb{N}_0$,

$$\begin{aligned} h_n((\lambda_1+\lambda_2+\lambda_3)x) &= \lim_{l\to\infty} \frac{f_n(3^l(\lambda_1+\lambda_2+\lambda_3)x)}{3^l} \\ &= \lim_{l\to\infty} \frac{(\lambda_1+\lambda_2+\lambda_3)f_n(3^l x)}{3^l} \\ &= (\lambda_1+\lambda_2+\lambda_3)h_n(x) \end{aligned}$$

for all $x \in \mathcal{A}$ and all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{U}$.

Clearly, $h_n(0x) = 0 = 0h_n(x)$ for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Now, let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$), and let $M \in \mathbb{N}$ be an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [14, Theorem 1], there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{U}$ such that $3\frac{\lambda}{M} = \lambda_1 + \lambda_2 + \lambda_3$. By the additivity of each h_n, $n \in \mathbb{N}_0$, we get $h_n(\frac{1}{3}x) = \frac{1}{3}h_n(x)$ for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$.
Therefore

$$\begin{aligned} h_n(\lambda x) &= h_n\left(\frac{M}{3}\cdot 3\cdot \frac{\lambda}{M}x\right) = Mh_n\left(\frac{1}{3}\cdot 3\cdot\frac{\lambda}{M}x\right) = \frac{M}{3}h_n((\lambda_1+\lambda_2+\lambda_3)x) \\ &= \frac{M}{3}(\lambda_1+\lambda_2+\lambda_3)h_n(x) = \frac{M}{3}\cdot 3\cdot\frac{\lambda}{M}h_n(x) = \lambda h_n(x) \end{aligned}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$, so that h_n is $\mathbb{C}$-linear for each $n \in \mathbb{N}_0$.

Next, we need to show that the sequence $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ satisfies the identity

$$h_n([x,y,z]) = \sum_{i+j+k=n} [h_i(x), h_j(y), h_k(z)]$$

for each $n \in \mathbb{N}_0$ and all $x, y \in \mathcal{A}$. The inequality (2.2) implies that the function $\Delta_n : \mathcal{A}\times\mathcal{A}\times\mathcal{A} \to \mathcal{B}$ defined by

$$\Delta_n(x,y,z) = f_n([x,y,z]) - \sum_{i+j+k=n} [f_i(x), f_j(y), f_k(z)] \tag{2.8}$$

for each $n \in \mathbb{N}_0$ and all $x, y, z \in \mathcal{A}$, is bounded. Hence we see that

$$\lim_{l\to\infty} \frac{\Delta_n(3^l x, y, z)}{3^l} = 0 \tag{2.9}$$

for each $n \in \mathbb{N}_0$ and all $x, y \in \mathcal{A}$. Now, using (2.7), (2.8) and (2.9), we have

$$\begin{aligned} h_n([x,y,z]) &= \lim_{l\to\infty}\frac{f_n(3^l[x,y,z])}{3^l} = \lim_{l\to\infty}\frac{f_n([3^l x, y, z])}{3^l} \\ &= \lim_{l\to\infty}\frac{\sum_{i+j+k=n}[f_i(3^l x), f_j(y), f_k(z)] + \Delta_n(3^l x, y, z)}{3^l} \\ &= \lim_{l\to\infty}\sum_{i+j+k=n}\left[\frac{1}{3^l}f_i(3^l x), f_j(y), f_k(z)\right] + \lim_{l\to\infty}\frac{\Delta_n(3^l x, y, z)}{3^l} \end{aligned}$$

$$= \sum_{i+j+k=n} \left\{ \lim_{l\to\infty} \left[\frac{1}{3^l} f_i(3^l x), f_j(y), f_k(z) \right] \right\}$$
$$= \sum_{i+j+k=n} [h_i(x), f_j(y), f_k(z)].$$

That is, we obtain that

$$h_n([x,y,z]) = \sum_{i+j+k=n} [h_i(x), f_j(y), f_k(z)] \tag{2.10}$$

for each $n \in \mathbb{N}_0$ and all $x, y, z \in \mathcal{A}$. Let $l \in \mathbb{N}$ be fixed. Then, applying (2.10) and the additivity of each h_n, $n \in \mathbb{N}_0$, we get

$$\sum_{i+j+k=n} [h_i(x), f_j(3^l y), f_k(z)] = h_n([x, 3^l y, z]) = h_n([3^l x, y, z])$$
$$= \sum_{i+j+k=n} [h_i(3^l x), f_j(y), f_k(z)]$$
$$= 3^l \sum_{i+j+k=n} [h_i(x), f_j(y), f_k(z)].$$

Hence we have

$$\sum_{i+j+k=n} [h_i(x), f_j(y), f_k(z)] = \sum_{i+j+k=n} \left[h_i(x), \frac{1}{3^l} f_j(3^l y), f_k(z) \right] \tag{2.11}$$

for each $n \in \mathbb{N}_0$ and all $x, y, z \in \mathcal{A}$. Letting $l \to \infty$ in (2.11), it follows that

$$\sum_{i+j+k=n} [h_i(x), f_j(y) f_k(z)] = \sum_{i+j+k=n} [h_i(x), h_j(y), f_k(z)] \tag{2.12}$$

for each $n \in \mathbb{N}_0$ and all $x, y \in \mathcal{A}$. Therefore, we obtain that

$$h_n([x,y,z]) = \sum_{i+j+k=n} [h_i(x), h_j(y), f_k(z)] \tag{2.13}$$

for each $n \in \mathbb{N}_0$ and all $x, y, z \in \mathcal{A}$. Again, using (2.13) and the additivity of each h_n, $n \in \mathbb{N}_0$, we get

$$\sum_{i+j+k=n} [h_i(x), h_j(y), f_k(3^l z)] = h_n([x, y, 3^l z]) = h_n([3^l x, y, z])$$
$$= \sum_{i+j+k=n} [h_i(3^l x), h_j(y), f_k(z)]$$
$$= 3^l \sum_{i+j+k=n} [h_i(x), h_j(y), f_k(z)].$$

So we have

$$\sum_{i+j+k=n} [h_i(x), h_j(y), f_k(z)] = \sum_{i+j+k=n} \left[h_i(x), h_j(y), \frac{1}{3^l} f_k(3^l z) \right] \tag{2.14}$$

for each $n \in \mathbb{N}_0$ and all $x, y, z \in \mathcal{A}$. Taking $l \to \infty$ in (2.14), we have

$$\sum_{i+j+k=n} [h_i(x), h_j(y), f_k(z)] = \sum_{i+j+k=n} [h_i(x), h_j(y), h_k(z)] \tag{2.15}$$

for each $n \in \mathbb{N}_0$ and all $x, y \in \mathcal{A}$ which implies (2.4). Combining (2.15) with (2.13), it follows that $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ satisfies the relation

$$h_n([x, y, z]) = \sum_{i+j+k=n} [h_i(x), h_j(y), h_k(z)]$$

for each $n \in \mathbb{N}_0$ and all $x, y, z \in \mathcal{A}$. Thus H is a higher ternary derivation from $\mathcal{A}$ into $\mathcal{B}$.

To show the uniqueness property of H, assume that $H^* = \{h_0^*, h_1^*, \ldots, h_n^*, \ldots\}$ is another higher ternary derivation from $\mathcal{A}$ into $\mathcal{B}$ satisfying

$$\|f_n(x) - h_n^*(x)\| \leq \frac{\varepsilon}{2}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Let $l \in \mathbb{N}$. Since h_n and h_n^* are additive, we deduce that

$$3^l \|h_n(x) - h_n^*(x)\| = \|h_n(3^l x) - h_n^*(3^l x)\| \leq \varepsilon,$$

so that

$$\|h_n(x) - h_n^*(x)\| \leq \frac{\varepsilon}{3^l}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Setting $l \to \infty$, we find that

$$h_n(x) = h_n^*(x)$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. This completes the proof of the theorem. □

Let $\mathbb{R}^+$ be the set of positive real numbers. G. Isac and Th. M. Rassias [10] generalized the Hyers theorem by introducing a mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ subject to the conditions

$$\lim_{t \to \infty} \frac{\psi(t)}{t} = 0, \tag{2.16}$$

$$\psi(ts) \leq \psi(t)\psi(s) \quad \text{for all } t, s \in \mathbb{R}^+, \tag{2.17}$$

$$\psi(t) < t \quad \text{for all } t > 1. \tag{2.18}$$

Here we obtain the Isac and Rassias-type stability result for higher ternary derivations which is a generalization of Theorem 2.1.

Theorem 2.2. *Let $\mathcal{A}$ be a normed ternary algebra, $\mathcal{B}$ a Banach ternary algebra and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a mapping with properties* (2.16), (2.17) *and* (2.18). *In addition, let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping satisfying the condition*

$$\lim_{t \to \infty} \frac{\varphi(t)}{t} = 0. \tag{2.19}$$

Suppose that $F = \{f_0, f_1, \ldots, f_n, \ldots\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for some $\varepsilon \geq 0$ and each $n \in \mathbb{N}_0$,

$$\begin{aligned} &\|f_n(\lambda_1 x + \lambda_2 y + \lambda_3 z) - \lambda_1 f_n(x) - \lambda_2 f_n(y) - \lambda_3 f_n(z)\| \\ \leq\ & \varepsilon\{\psi(\|x\|) + \psi(\|y\|) + \psi(\|z\|)\} \end{aligned} \tag{2.20}$$

and

$$\left\| f_n([x,y,z]) - \sum_{i+j+k=n} [f_i(x), f_j(y), f_k(z)] \right\| \leq \varphi(\|x\|\ \|y\|\ \|z\|) \tag{2.21}$$

hold for all $x, y, z \in \mathcal{A}$ and all $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$. Then there exist a unique higher ternary derivation $H = \{h_0, h_1, \ldots, h_n, \ldots\}$ of any rank from $\mathcal{A}$ into $\mathcal{B}$ and a constant $c \in \mathbb{R}$ such that for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$,

$$\|f_n(x) - h_n(x)\| \leq c\varepsilon\psi(\|x\|).$$

Moreover, the relation (2.4) is fulfilled.

Proof. Putting $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $x = y = z$ in (2.20) yields

$$\left\| \frac{1}{3} f_n(3x) - f_n(x) \right\| \leq \varepsilon\psi(\|x\|)$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Hence, by induction, it is easy to infer that

$$\left\| \frac{1}{3^l} f_n(3^l x) - f_n(x) \right\| \leq \varepsilon\psi(\|x\|) \sum_{m=0}^{l-1} \left(\frac{\psi(3)}{3} \right)^m \tag{2.22}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. From (2.18), we have $\frac{\psi(3)}{3} < 1$; thus the series $\sum_{m=0}^{\infty} \left(\frac{\psi(3)}{3}\right)^m$ converges to the sum $c = \frac{3}{3-\psi(3)}$. It follows that

$$\left\| \frac{1}{3^l} f_n(3^l x) - f_n(x) \right\| \leq c\varepsilon\psi(\|x\|). \tag{2.23}$$

From (2.23) with x replaced by $3^m x$, the result divided by 3^m, and the successive use of (2.17), we find that

$$\left\| \frac{1}{3^{l+m}} f_n(3^{l+m} x) - \frac{1}{3^m} f_n(3^m x) \right\| \leq c\varepsilon \left(\frac{\psi(3)}{3} \right)^m \psi(\|x\|) \tag{2.24}$$

for any $l, m \in \mathbb{N}$ and all $x \in \mathcal{A}$. The Cauchy criterion implies that the limit $h_n(x) = \lim_{l\to\infty} \frac{1}{3^l} f_n(3^l x)$ exists for all $x \in \mathcal{A}$. Let $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in (2.20) and then let us replace x by $3^l x$, y by $3^l y$ and z by $3^l z$. Let us divide the result by 3^l and utilize (2.17). Now, if we take $l \to \infty$, then we see that for each $n \in \mathbb{N}_0$, h_n satisfies the functional equation $f(x+y+z) - f(x) - f(y) - f(z) = 0$ for all $x, y, z \in \mathcal{A}$. So, h_n is additive for each $n \in \mathbb{N}_0$. Setting $x = y = z$ in (2.20) yields

$$\|f_n((\lambda_1 + \lambda_2 + \lambda_3)x) - (\lambda_1 + \lambda_2 + \lambda_3) f_n(x)\| \leq 3\varepsilon\psi(\|x\|) \tag{2.25}$$

for each $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. Then, from (2.16) and (2.25), we deduce that

$$\frac{1}{3^l}\|f_n(3^l(\lambda_1+\lambda_2+\lambda_3)x)-(\lambda_1+\lambda_2+\lambda_3)f_n(3^l x)\| \le 3\varepsilon\frac{\psi(3^l)}{3^l}\psi(\|x\|)\to 0$$

as $l\to\infty$. This implies that for each $n \in \mathbb{N}_0$,

$$\begin{aligned}h_n((\lambda_1+\lambda_2+\lambda_3)x) &= \lim_{l\to\infty}\frac{f_n(3^l(\lambda_1+\lambda_2+\lambda_3)x)}{3^l}\\ &= \lim_{l\to\infty}\frac{(\lambda_1+\lambda_2+\lambda_3)f_n(3^l x)}{3^l}\\ &= (\lambda_1+\lambda_2+\lambda_3)h_n(x)\end{aligned}$$

for all $x \in \mathcal{A}$ and all $\lambda_1,\lambda_2,\lambda_3 \in \mathbb{U}$. The method to show that h_n is $\mathbb{C}$-linear for each $n \in \mathbb{N}_0$ is the same as in the proof of Theorem 2.1.

Let Δ_n be the map defined by (2.8) for each $n \in \mathbb{N}_0$. Then, employing (2.17), (2.19) and (2.21), it follows that the map Δ_n satisfies the condition (2.9) and the further part of the proof is the same as in the proof of Theorem 2.1. Thus, we conclude that $H=\{h_0,h_1,\ldots,h_n,\ldots\}$ is a unique higher ternary derivation of any rank on $\mathcal{A}$ and the relation

$$\sum_{i+j+k=n}[h_i(x),h_j(y),\{h_k(z)-f_k(z)\}]=0$$

holds for each $n \in \mathbb{N}_0$ and all $x,y,z \in \mathcal{A}$ which completes the proof. □

Remark 2.3. The typical example of the mapping ψ fulfilling (2.16), (2.17) and (2.18) is given by $\psi(t)=t^p$, where $p<1$. The example of the mapping φ satisfying (2.19) is $\varphi(t)=t^q$, where $q<1$. If we intend to consider the case of $p,q>1$, then we adopt the method given by Z. Gajda in [6] to obtain the Isac and Rassias-type stability result for the mapping $\psi:\mathbb{R}^+\to\mathbb{R}^+$ fulfilling the conditions

$$\lim_{t\to\infty}\frac{\psi(t)}{t}=0, \tag{2.26}$$

$$\psi(ts)\le\psi(t)\psi(s)\quad\text{for all } t,\ s\in\mathbb{R}^+, \tag{2.27}$$

$$\psi(t)<t\quad\text{for all } t\in(0,1). \tag{2.28}$$

In the proof of Theorem 2.1, if we replace (2.7) by

$$h_n(x)=\lim_{l\to\infty}3^l f_n\Big(\frac{1}{3^l}x\Big)$$

and (2.9) by

$$\lim_{l\to\infty}3^l\Delta_n\Big(\frac{1}{3^l}x,y,z\Big)=0,$$

then Theorem 2.2 is still true under the conditions (2.26), (2.27) and (2.28).

As consequences of Theorem 2.1, we get the following Bourgin-type superstability.

Corollary 2.4. *Let $\mathcal{A}$ be a Banach ternary algebra with unit e and $\mathcal{B}$ a Banach ternary algebra with unit e^*. Suppose that $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ satisfying* (2.1) *and* (2.2), *where f_0 is onto and $f_0(e) = e^*$. Then $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a higher ternary derivation of any rank from $\mathcal{A}$ onto $\mathcal{B}$.*

Proof. Note that we may regard $\mathcal{A}$ and $\mathcal{B}$ as the usual Banach algebras with the binary operations $x \odot y := [x, e, y]$, $x, y \in \mathcal{A}$ and $a \diamond b := [a, e^*, b]$, $a, b \in \mathcal{B}$, respectively. In (2.1) with $n = 0$, putting $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $y = 0$ yields the inequality

$$\|f_0(x+z) - f_0(x) - f_0(z)\| \le \varepsilon + \|f_0(0)\| \tag{2.29}$$

for all $x, z \in \mathcal{A}$. On the other hand, the inequality (2.2) with $n = 0$ can be written by

$$\|f_0([x, e, z]) - [f_0(x), f_0(e), f_0(z)]\| \le \delta$$

which is equivalent to

$$\|f_0(x \odot z) - f_0(x) \diamond f_0(z)\| \le \delta \tag{2.30}$$

for all $x, z \in \mathcal{A}$. In view of (2.29), (2.30), the Bourgin's theorem [4] and (2.7) in the proof of Theorem 2.1, we see that f_0 is a usual homomorphism from $\mathcal{A}$ onto $\mathcal{B}$ and $f_0 = h_0$. By induction, we want to lead the conclusion. If $n = 1$, then it follows from (2.4) that $f_1(z) = h_1(z)$ holds for all $z \in \mathcal{A}$ since $h_0(e) = e^*$. Let us assume that $f_m(x) = h_m(x)$ is valid for all $x \in \mathcal{A}$ and all $m < n$. Then (2.4) implies that $[h_0(x), h_0(y), \{f_n(z) - h_n(z)\}] = 0$ holds for all $x, y, z \in \mathcal{A}$. Since $h_0(e) = e^*$, we have $f_n(z) = h_n(z)$ for all $z \in \mathcal{A}$. Therefore, we conclude that $f_n(x) = h_n(x)$ for all $n \in \mathbb{N}_0$ and all $x \in \mathcal{A}$. So, Theorem 2.1 tells us that $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a higher ternary derivation of any rank from $\mathcal{A}$ onto $\mathcal{B}$. The proof of the theorem is complete. □

Corollary 2.5. *Let $\mathcal{A}$ be a Banach ternary algebra with unit. Suppose that $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a sequence of mappings on $\mathcal{A}$ satisfying* (2.1) *and* (2.2), *where f_0 is an identity mapping on $\mathcal{A}$. Then $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a strong higher ternary derivation of any rank on $\mathcal{A}$.*

Proof. From (2.7), we have

$$h_0(x) = \lim_{l \to \infty} \frac{1}{3^l} f_0(3^l x) = x$$

for all $x \in \mathcal{A}$ and so $h_0 (= f_0)$ is an identity mapping on $\mathcal{A}$. By the same method as in the proof of Corollary 2.4 using the induction and the relation (2.4), we get

$$[x, y, \{f_n(z) - h_n(z)\}] = 0 \tag{2.31}$$

for all $n \in \mathbb{N}$ and all $x, y, z \in \mathcal{A}$. Since $\mathcal{A}$ contains the unit, it follows from (2.31) that $f_n(z) = h_n(z)$ holds for all $n \in \mathbb{N}_0$ and all $z \in \mathcal{A}$. Thus, from Theorem 2.1, we see that $F = \{f_0, f_1, \dots, f_n, \dots\}$ is a strong higher derivation of any rank on $\mathcal{A}$. This completes the proof. □

Remark 2.6. As in Theorem 2.2 and Remark 2.3, we can generalize our results by substituting another functions satisfying appropriate conditions (see, for instance, [7]) for the bounds ε and δ of the inequalities corresponding to the functional equations

$$f_n(\lambda_1 x + \lambda_2 y + \lambda_3 z) = \lambda_1 f_n(x) + \lambda_2 f_n(y) + \lambda_3 f_n(z),$$
$$f_n([x,y,z]) = \sum_{i+j+k=n} [f_i(x), f_j(y), f_k(z)].$$

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