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# ON FUNCTIONAL INEQUALITIES ASSOCIATED WITH JORDAN–VON NEUMANN TYPE FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper, it is shown that if $f$ satisfies the following functional inequality

$$(0.1)\qquad \|\sum_{i,j=1}^{3} f(x_i, y_j)\| \le \|f(x_1 + x_2 + x_3, y_1 + y_2 + y_3)\|$$

then $f$ is a bi-additive mapping.

We moreover prove that if $f$ satisfies the following functional inequality (0.2)

$$\|2\sum_{j=1}^{3} f(x_j, z) + 2\sum_{j=1}^{3} f(x_j, w) - f(\sum_{j=1}^{3} x_j, z - w)\| \le \|f(\sum_{j=1}^{3} x_j, z + w)\|$$

then $f$ is an additive-quadratic mapping.

## 1. Introduction and preliminaries

Ulam [27] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group $G$ and a metric group $G'$ with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

Hyers [5] considered the case of approximately additive mappings $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

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exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

Th. M. Rassias [24] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

**Theorem 1.1** ([24]). *Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

*for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$. Then the limit*

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

*exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

*for all $x \in E$. If $p < 0$ then inequality* (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$. Also, if for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then $L$ is linear.*

The inequality (1.1) that was introduced for the first time by Th. M. Rassias [24] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as *Hyers–Ulam–Rassias stability* of functional equations. Czerwik [1] proved the generalized Hyers-Ulam stability of the quadratic functional equation (see also [8, 23] for a number of other new results). Several functional equations have been investigated in [9] and [26]. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [12, 13, 21]). During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam-Rassias stability to a number of functional equations mappings (see [6, 7, 10-14, 20, 22, 24]).

Gilányi [3] showed that if $f$ satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.3}$$

then $f$ satisfies the Jordan–von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [17] and [25]. Fechner [2] and Gilányi [4] proved the Hyers–Ulam–Rassias stability of the functional inequality (1.3). Park, Cho, and Han [19] proved the Hyers–Ulam–Rassias stability of functional inequalities associated with Jordan–von Neumann type additive functional equations. Park [18] proved the Hyers–Ulam–Rassias stability of the *additive-quadratic functional equation*

$$f(\sum_{j=1}^{3} x_j, z+w) + f(\sum_{j=1}^{3} x_j, z-w) = 2\sum_{j=1}^{3} f(x_j, z) + 2\sum_{j=1}^{3} f(x_j, w),$$

whose solution is called an *additive-quadratic mapping.*

In Section 2, we prove that if $f$ satisfies the inequality (0.1), then $f$ is a bi-additive mapping.

In Section 3, we prove that if $f$ satisfies the inequality (0.2), then $f$ is an additive-quadratic mapping.

## 2. Functional inequality associated with Jordan–von Neumann type bi-additive functional equation

Throughout this section, let $G$ be an abelian group. Assume that $Y$ is a Banach space with norm $\|\cdot\|_Y$.

**Theorem 2.1.** *Let $f : G \times G \to Y$ be a mapping such that*

$$\| \sum_{i,j=1}^{3} f(x_i, y_j)\|_Y \le \|f(x_1 + x_2 + x_3, y_1 + y_2 + y_3)\|_Y \tag{2.1}$$

*for all $x_1, x_2, x_3, y_1, y_2, y_3 \in G$. Then $f$ is bi-additive.*

*Proof.* Letting $x_1 = x_2 = x_3 = y_1 = y_2 = y_3 = 0$ in (2.1), we get

$$\|9f(0,0)\|_Y \le \|f(0,0)\|_Y.$$

So $f(0,0) = 0$.

Letting $x_2 = x_3 = y_1 = y_2 = y_3 = 0$ in (2.1), we get

$$\|3f(x_1, 0)\|_Y \le \|f(x_1, 0)\|_Y \quad \text{for all } x_1 \in G.$$

Thus $f(x_1, 0) = 0$ for all $x_1 \in G$.

Letting $x_1 = x_2 = x_3 = y_2 = y_3 = 0$ in (2.1), we get

$$\|3f(0, y_1)\|_Y \le \|f(0, y_1)\|_Y$$

for all $y_1 \in G$. Thus $f(0, y_1) = 0$ for all $y_1 \in G$.

Letting $x_2 = -x_1$ and $x_3 = y_2 = y_3 = 0$ in (2.1), we get

$$\|f(x_1, y_1) + f(-x_1, y_1)\|_Y \le \|f(0, y_1)\|_Y = 0 \quad \text{for all } x_1, y_1 \in G.$$

Hence

$$f(-x_1, y_1) = -f(x_1, y_1) \quad \text{for all } x_1, y_1 \in G. \tag{2.2}$$

Letting $x_3 = -x_1 - x_2$ and $y_2 = y_3 = 0$ in (2.1), we get

$$\|f(x_1, y_1) + f(x_2, y_1) + f(-x_1 - x_2, y_1)\|_Y \le \|f(0, y_1)\|_Y = 0$$

for all $x_1, x_2, y_1 \in G$. By (2.2),

$$f(x_1 + x_2, y_1) = -f(-x_1 - x_2, y_1) = f(x_1, y_1) + f(x_2, y_1)$$

for all $x_1, x_2, y_1 \in G$. Hence the mapping $f : G \times G \to Y$ is additive in the first variable.

Similarly, one can show that the mapping $f : G \times G \to Y$ is additive in the second variable.

Therefore, the mapping $f : G \times G \to Y$ is a bi-additive mapping. □

## 3. Functional inequality associated with Jordan–von Neumann type additive-quadratic functional equation

Throughout this section, let $G$ be an abelian group. Assume that $H$ is a Hilbert space with norm $\|\cdot\|_H$.

**Theorem 3.1.** *Let $f : G \times G \to H$ be a mapping such that*

$$\|2\sum_{j=1}^{3} f(x_j, z)+2\sum_{j=1}^{3} f(x_j, w)-f(\sum_{j=1}^{3} x_j, z-w)\|_H \le \|f(\sum_{j=1}^{3} x_j, z+w)\|_H \tag{3.1}$$

*for all $x_1, x_2, x_3, z, w \in G$. Then $f$ is additive-quadratic.*

*Proof.* Letting $x_1 = x_2 = x_3 = z = w = 0$ in (3.1), we get

$$\|11f(0,0)\|_H \le \|f(0,0)\|_H.$$

So $f(0,0) = 0$.

Letting $x_1 = x_2 = x_3 = w = 0$ in (3.1), we get

$$\|5f(0,z)\|_H \le \|f(0,z)\|_H.$$

So $f(0,z) = 0$ for all $z \in G$.

Letting $x_2 = x_3 = z = w = 0$ in (3.1), we get

$$\|3f(x_1,0)\|_H \le \|f(x_1,0)\|_H.$$

So $f(x_1, 0) = 0$ for all $x_1 \in G$.

Letting $x_2 = -x_1$ and $x_3 = w = 0$ in (3.1), we get

$$\|2f(x_1, z) + 2f(-x_1, z)\|_H \le \|f(0, z)\|_H = 0 \quad \text{for all } x_1, z \in G.$$

Hence

$$f(-x_1, z) = -f(x_1, z) \quad \text{for all } x_1, z \in G. \tag{3.2}$$

Letting $x_3 = -x_1 - x_2$ and $w = 0$ in (3.1), we get

$$\|2f(x_1, z) + 2f(x_2, z) + 2f(-x_1 - x_2, z)\|_H \le \|f(0, z)\|_H = 0$$

for all $x_1, x_2, z \in G$. By (3.2),

$$f(x_1 + x_2, z) = -f(-x_1 - x_2, z) = f(x_1, z) + f(x_2, z)$$

for all $x_1, x_2, z \in G$. Hence the mapping $f : G \times G \to H$ is additive in the first variable.

Letting $x_2 = x_3 = 0$ in (3.1), we get

$$\|2f(x_1, z) + 2f(x_1, w) - f(x_1, z - w)\|_H \le \|f(x_1, z + w)\|_H.$$

for all $x_1, z, w \in G$. By Satz 1 of [3], the mapping $f : G \times G \to H$ is quadratic in the second variable.

Therefore, the mapping $f : G \times G \to H$ is an additive-quadratic mapping. $\square$

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