

ON THE (n, k) -TH CATALAN NUMBERS

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ABSTRACT. In this paper, we generalize the Catalan number to the (n, k) -th Catalan numbers and find a combinatorial description that the (n, k) -th Catalan numbers is equal to the number of partitions of $n(k-1)+2$ polygon by $(k+1)$ -gon where all vertices of all $(k+1)$ -gons lie on the vertices of $n(k-1)+2$ polygon.

1. Introduction

Let D^2 be the closed unit disc in \mathbb{R}^2 . We label $2n$ points on the boundary of D^2 by $\{1, 2, \dots, 2n\}$ with a counterclockwise orientation. We embed n arcs into D^2 so that the boundaries of intervals map to the labeled points. Such an embedding is called an $(n, 2)$ diagram if it does not have any crossings. We consider the isotopy classes of $(n, 2)$ diagrams relative to the boundary. The number of isotopy classes of $(n, 2)$ diagrams is called the n -th Catalan number [2], denoted by C_n and

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

66 different combinatorial descriptions of n -th Catalan number are given in a famous literature [13, Exercise 6. 19] and currently 96 more are added on his Catalan addendum [14]. This number plays a major role in many different areas of not only mathematics but also natural science [4, 5]. Harary found an implication of n -th Catalan number on the cell growth problem [8]. Cameron [1] found representations for binary and ternary trees which can be used to enumerate Rothe numbers [6].

There have been many attempts to generalize Catalan numbers. We are particularly interested in two of them. Gould developed a generalization of n -th Catalan number as follows [7],

$$A_n(a, b) = \frac{a}{a+bn} \binom{a+bn}{n},$$

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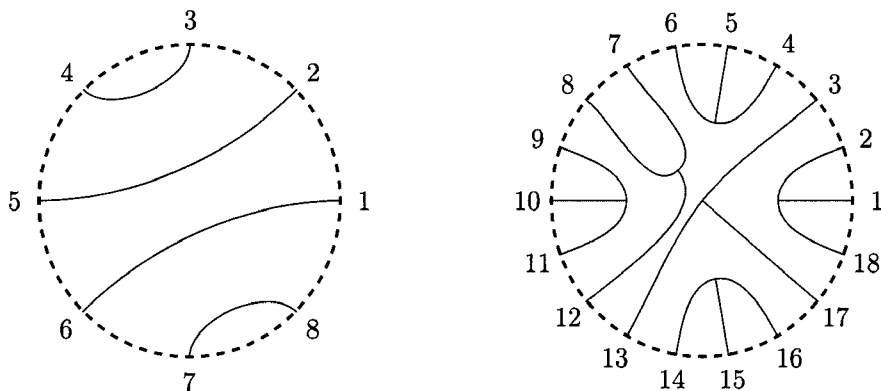


FIGURE 1. A $(4, 2)$ diagram and a $(6, 3)$ diagram.

together with its convolution formula,

$$\sum_{k=0}^n A_k(a, b) A_{n-k}(a, b) = A_n(a + c, b).$$

From Gould's generalization, $A_n(a, b)$, one can obtain C_n by substituting $a = 1$ and $b = 2$. On the other hand, n -th Catalan number is the dimension of the linear skein space of a disc with $2n$ points on its boundary or simply n -th Temperley-Lieb algebras, \mathcal{T}_n . Kuperberg generalized Temperley-Lieb algebras, which corresponds to the invariant subspace of a tensor product of the vector representation of $\mathfrak{sl}(2)$, to web spaces of simple Lie algebras of rank 2, $\mathfrak{sl}(3)$, $\mathfrak{sp}(4)$ and G_2 [12]. It has been extensively studied [9, 10, 11]. In particular, a purpose of the present article is to find a generalization which fits in both directions; Gould's generalization and the web spaces of $\mathfrak{sl}(3)$. The following describes our generalization. For $k \geq 2, n \geq 1$, we label kn points on the boundary of the disc by $\{1, 2, \dots, kn\}$. Then we embed n copies of k -stars with boundaries of k -stars goes the labeled points where a k -star is a neighborhood of a vertex in the complete graph K_{k+1} on $k + 1$ vertices. Then we consider an (n, k) diagram and its isotopy class in a similar fashion. A $(4, 2)$ diagram and a $(6, 3)$ diagram are illustrated in Figure 1. Let $\mathcal{D}(n, k)$ be the set of all isotopy classes of (n, k) -diagrams. The number of elements in $\mathcal{D}(n, k)$ is called the (n, k) -Catalan number.

Theorem 1. *The (n, k) -th Catalan number is $\frac{1}{n} \binom{kn}{n-1}$.*

The first combinatorial description of n -th Catalan number given in [13, Exercise 6. 19 a] is that it is equal to the number of triangulations of $(n + 2)$ polygon P_{n+2} where all vertices of triangles lie on the vertices of P_{n+2} . We also find a similar description for (n, k) -Catalan number in the following theorem.

Theorem 2. *The (n, k) -th Catalan number is equal to the number of partitions of $n(k - 1) + 2$ polygon by $(k + 1)$ -gon where all vertices of all $(k + 1)$ -gon lie on the vertices of $n(k - 1) + 2$ polygon.*

The outline of this paper is as follows. In Section 2, we provide the proof of Theorem 1 and 2. In Section 3 we illustrate how (n, k) -th Catalan number is related with other generalizations and the representation theory of the quantum Lie algebra $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$.

2. Proof of main theorems

2.1. Proof of Theorem 1

Let $I_l = \{1, 2, \dots, l\}$ be $\mathbb{Z}/l\mathbb{Z}$. Let $\mathcal{S}(l, m)$ be the set of all subsets of I_l whose cardinalities are m . We first observe the relation between $\mathcal{S}(nk, n)$ and $\mathcal{D}(n, k)$ by defining a map $\psi : \mathcal{S}(nk, n) \rightarrow \mathcal{D}(n, k)$. Let E be an element in $\mathcal{S}(nk, n)$. To obtain the image $\psi(E)$, an (n, k) diagram, we first array nk points on the boundary of the unit disc D^2 . We assign \uparrow for n points in E , \downarrow for points in $I_{nk} - E$. Let j be the smallest number in E . Now we define a height function $\varphi : I_{nk} \rightarrow \mathbb{Z}$ by $\varphi(i) =$ the number of \uparrow 's minus the number of \downarrow 's from j to i in the clockwise orientation on the boundary of the disc D^2 . Then clearly $\varphi(j) = 1$, $\varphi(j - 1) = -(k - 1)n$ and φ is a step function such that $|\varphi(l) - \varphi(l + 1)| = 1$ for all $l = 1, 2, \dots, kn$ except $j - 1$. Since it starts from 1 at j and ends with $-(k - 1)n$ at $j - 1$, by a simple application of the Pigeon Hole principle, we can see that there exists l such that

$$\varphi(l - 1) < \varphi(l) = \varphi(l + 1) + 1 = \dots = \varphi(l + k - 1) + k - 1,$$

which give us an innermost k -star joining k consecutive points from l . After removing these k consecutive points, inductively we repeat this process from the next innermost k -star. By picking the first point for the points of each k -star from j , we can see that ψ is onto. For example, we will pick $\{3, 4, 7, 9, 14, 18\}$ for $(6, 3)$ diagram in the righthand side of Figure 1 and we will get this $(6, 3)$ diagram from this set of six points if we follow the inductive process.

One can easily notice that there are different sets of n points which make the same (n, k) diagram, i.e., ψ is not one to one, such as $\{4, 7, 9, 13, 14, 18\}$ and $\{1, 3, 4, 7, 9, 14\}$ will give the same $(6, 3)$ diagram in Figure 1. However, it is not clear to find how many different subsets in $\mathcal{S}(nk, n)$ lead us to the same (n, k) diagram. To get the right number of all distinct (n, k) diagrams, we will show the second claim : there exists a surjective map $\vartheta : \mathcal{S}(nk, n - 1) \rightarrow \mathcal{D}(n, k)$, i.e., for a given element F of $\mathcal{S}(nk, n - 1)$, there is a unique way to decide the last point we have to pick so that we get an element E in $\mathcal{S}(nk, n)$ and an (n, k) diagram, $\varphi(E) = \vartheta(F)$. The proof follows from the proof of the first claim because one can see that at the last stage of the induction procedure, we had k points left and the (n, k) diagram was decided already. Therefore, we just pick the smallest label from these k points. For the surjectivity of ϑ , for a given (n, k) diagram D , we knew that there is an innermost k -star from

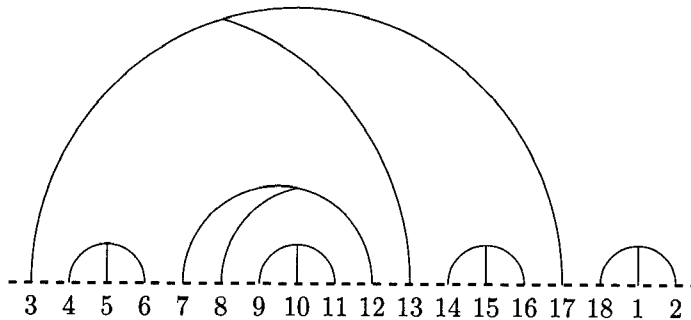


FIGURE 2. The $(6, 3)$ diagram \tilde{D} in the upperhalf plane corresponding to a $(6, 3)$ diagram D in Figure 1.

the first claim. Then to obtain the desired set we first start to read the first label for each k star from the last label of the innermost k -star. Then we will have an ordered set E of n points and furthermore, if we drop the last label to get an ordered set F of $n - 1$ points, the previous process will recover it, i.e., $\varphi(E) = \vartheta(F) = D$.

The last claim for the proof is that ϑ is an n to 1 map, i.e., for a given an (n, k) diagram there are n unique sets of $n - 1$ points in $\mathcal{S}(nk, n - 1)$ such that if we apply the process in the second claim, we recover the original diagram. For a given (n, k) diagram D and each k star in D , we drop it and read the first label in other $(n - 1)$ k -stars. One can easily see that all these $(n - 1)$ points will recover the same (n, k) diagram by the process described in the second claim. Also we can see that if we pick a point other than these $(n - 1)$ points, then we may not recover the k star we have dropped. For example, $\{3, 4, 7, 9, 14\}$, $\{4, 7, 9, 14, 18\}$, $\{4, 7, 9, 7, 18\}$, $\{4, 9, 13, 14, 18\}$, $\{4, 12, 13, 14, 18\}$ and $\{7, 9, 13, 14, 18\}$ are only possible six sets of five elements in $\mathcal{S}(18, 5)$ for the $(6, 3)$ diagram in Figure 1. Therefore, the (n, k) Catalan number is the one n -th of the cardinality of $\mathcal{S}(nk, n - 1)$ which is $\frac{1}{n} \binom{kn}{n-1}$.

2.2. Proof of Theorem 2

The main idea of the proof came from that of the proof for the Catalan number by replacing \uparrow for the first left branch and \downarrow for the other branches [13].

Let D be an (n, k) diagram. We pick a set E of n points as we described in the proof for the surjectivity of φ . We cut open the disc D^2 from a point p which is in between the smallest label in E and its immediate predecessor so that we get a diagram \tilde{D} on the upperhalf plane as in Figure 2. One can notice that all points in E , for \tilde{D} in Figure 2, E can be chosen $\{3, 4, 7, 9, 14, 18\}$, appears in the leftmost leg of each k -star. Then we can make a rooted k -ary tree T_D by the following way. First we start from the k -star contains the smallest

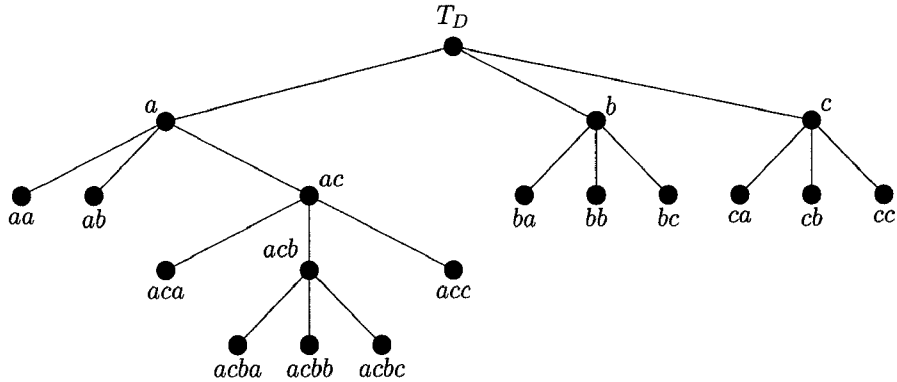


FIGURE 3. The rooted tree T_D corresponding to the $(6, 3)$ diagram D in Figure 1 with words describe the lexicographic order to recover the $(6, 3)$ diagram D .

label in E . We just draw k branches from the root. Since each k -star divides the upper half plane into k disjoint regions and we have the counterclockwise orientation around the *center* of k star, where all legs of k -star join, it gives us a cyclic order of the edges of k star. So we can order the regions from the next region of the unbounded region in counterclockwise orientation. If a region does not contain any other k -stars, that branch stops. Otherwise, we repeat the process from the region in the leftmost k -star, at this stage we ignore the previous k -stars. Inductively, we can complete the rooted tree \tilde{T} . For converse, from the root we assign alphabet letters for children nodes from the left. Inductively at each generation of parents nodes, for each children nodes we append alphabet letters to the word assigned to the parent node from the left. So, it assigns words for every nodes except the root. Then we can array them in line by the lexicographic dictionary order. For the example in Figure 3, we obtain the following words for each labels: 3(a), 4(aa), 5(ab), 6(ac), 7(aca), 8(acb), 9($acba$), 10($acbb$), 11($acbc$), 12(acc), 13(b), 14(ba), 15(bb), 16(bc), 17(c), 18(ca), 1(cb), 2(cc). Then we read n points which is the leftmost branch or words which end with the letter a to recover n points set E . One can see this process in Figure 3.

For the next step, we cover all vertices of valence bigger than one by small quadrilaterals such that they only intersect the tree at the neighborhood of the center vertex and every edges of quadrilaterals centered at the vertex are transversal to only one edge of the neighborhood of the center vertex as depicted in Figure 4. We must have exactly n vertices of valence bigger than one. We make an exception for the root where we allow to have an edge of quadrilateral centered at root does not intersect any part of the tree. Then each edges between two vertices of valence bigger than one, we put a strip which connects

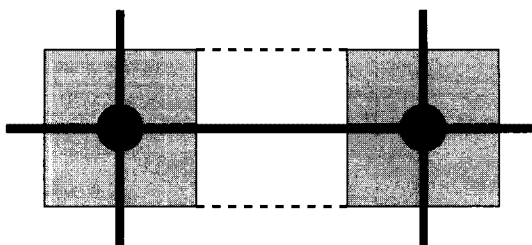


FIGURE 4. Strips for connecting the adjacent quadrilaterals from the rooted tree T_D and dotted lines which will be collapsing to a point to get $n(k-1)+2$ polygon.

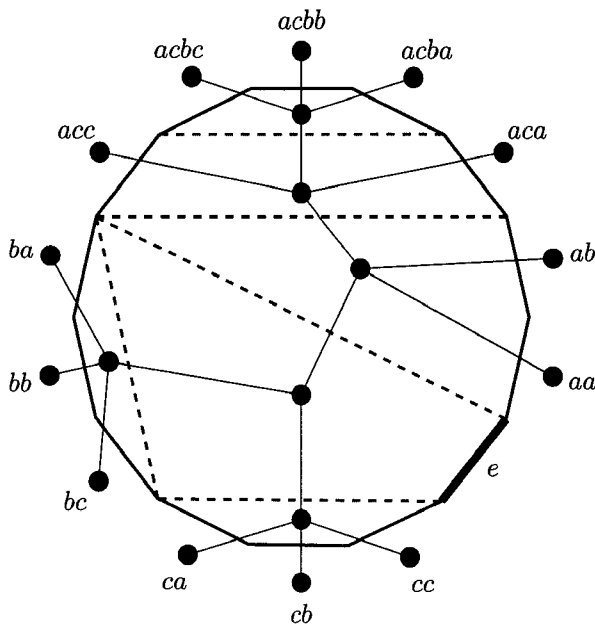


FIGURE 5. The partition of 14-gon by rectangles obtained from the rooted tree T_D in Figure 4.

the edges of quadrilaterals facing each other as in Figure 2. At this stage, we have $n(k+1)$ polygon. Then we collapse these extra strips by shrinking each dotted line to a point as depicted in Figure 5. This process leads us to an $n(k-1)+2$ polygon. Finally, we transform it to a regular $n(k-1)+2$ -gon where an edge of the quadrilateral contains the root which does not intersect the tree

T goes to a fixed edge e . The converse part should be clear. It completes the proof.

3. Discussion and problems

In this section, we discuss the relations between our generalization of the Catalan number and other generalizations and the representation theory of quantum Lie algebra $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{C}))$.

The first easy observation is to see that the (n, k) -th Catalan number can be obtained from $A_n(a, b)$ by substituting $a = 1$ and $b = k$. Consequently, the n -th Catalan number C_n can be obtained from the (n, k) -th Catalan number by substituting $k = 2$. Although, we have not stated separate theorems but the proof of Theorem 2 showed two more combinatorial descriptions of (n, k) -th Catalan number by the rooted k -ary tree and alphabetical words [13, Exercise 6. 19 d].

As we stated before, the n -th Catalan number is the dimension of the invariant space of $V_1^{\otimes 2n}$, where V_1 is the vector representation of $\mathfrak{sl}(2, \mathbb{C})$ because $(n, 2)$ diagrams are precisely geometric realizations of the invariant vectors. Thus, it is natural to ask $(n, 3)$ diagrams may be also geometric realizations of the invariant vectors in the invariant space of $V_1^{\otimes 3n}$, where V_1 is the vector representation of $\mathfrak{sl}(3, \mathbb{C})$. In fact, it is known that there exists a general method to generate all invariant vectors in the invariant space of $V_1^{\otimes 3n}$ [11] and our expectation is right. However, not all invariant vectors are $(n, 3)$ diagrams. Because the full invariant space can be described as

$$\begin{aligned} \text{Inv}(V_1^{\otimes 3n}) = & \bigoplus_{k=1}^{3n-2} \bigoplus_{V_{w_1} \in \mathcal{C}} \dots \bigoplus_{V_{w_k} \in \mathcal{C}} \text{Inv}(V_1^{\otimes a_1} \otimes V_{w_1}) \oplus \text{Inv}(V_{w_1}^* \otimes V_1^{\otimes a_2} \otimes V_{w_2}) \\ & \oplus \dots \oplus \text{Inv}(V_{w_k}^* \otimes V_1^{\otimes a_k}), \end{aligned}$$

where \mathcal{C} be the set of all irreducible representations of $\mathfrak{sl}(3, \mathbb{C})$. But the invariant vectors corresponding to $(n, 3)$ diagrams are the subspace of $\text{Inv}(V_1^{\otimes 3n})$ where V_{w_i} is the trivial representation, \mathbb{C} .

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