Commun. Korean Math. Soc. **23** (2008), No. 3, pp. 343–348

TETRAGONAL MODULAR CURVES $X_1(M,N)$

Daeyeol Jeon

Abstract. In this work, we determine all the modular curves $X_1(M,N)$ which are tetragonal.

0. Introduction

A smooth projective curve X defined over an algebraically closed field k is called *d-gonal* if it admits a map $\phi : X \to \mathbb{P}^1$ over k of degree d. If the genus $g \geq 2$ and $d = 2$, then X is called *hyperelliptic*. We will say that X is *tetragonal* for $d = 4$.

For positive integers $M|N$, consider the congruence subgroup $\Gamma_1(M,N)$ of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma_1(M,N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mathrm{mod}\ N,\ M \mid b \right\}.$$

Then the modular curve $X_1(M,N)$ corresponding to $\Gamma_1(M,N)$ is related to moduli problems of elliptic curves containing a subgroup isomorphic to

$$\mathbb{Z}/M\mathbb{Z} \bigoplus \mathbb{Z}/N\mathbb{Z}.$$

The author, Kim, and Park [9] get the determination of all the tetragonal modular curves $X_1(1,N)$ and $X_1(2,N)$ which play a central role in determining the structure occurs infinitely often as the torsion of the elliptic curves over quartic number fields.

In this paper, we finish determining all the tetragonal modular curves $X_1(M, N)$. It gives useful information for the torsion subgroups of the elliptic curves over the number fields of higher order.

For X with the genus $g(X) \geq 2$ if one has a map $\phi : X \to C$ of degree 2 onto an elliptic curve C (respectively a hyperelliptic curve C), then X is called *bielliptic* (respectively *bihyperelliptic*).

Our main result is as follows:

Theorem 0.1. *The following are equivalent:*

(a) $X_1(M,N)$ *is tetragonal.*

Received March 21, 2008.
2000 *Mathematics Subject Classification.* 11G18, 11G30.
Key words and phrases. modular curve, tetragonal.

(b) *$X_1(M,N)$ is rational, elliptic, hyperelliptic, bielliptic or bihyperelliptic. Explicitly these (M,N) are:*

- *rational*: $(1, N = 1-10, 12), (2, N = 2, 4, 6, 8), (3, N = 3, 6), (4, 4), (5, 5)$;
- *elliptic*: $(1, N = 11, 14, 15), (2, N = 10, 12), (3, 9), (4, 8)$;
- *hyperelliptic*: $(1, N = 13, 16, 18)$;
- *bielliptic*: $(1, N = 17, 20-22, 24), (2, N = 14, 16), (3, 12), (4, 12), (5, 10), (7, 7), (8, 8)$;
- *bihyperelliptic*: $(2, 18)$.

Remark 0.2. Since a rational curve has maps of degree 2 and 4 to a rational curve, rational and hyperelliptic curve are tetragonal. Since an elliptic admits a map of degree 2 to a rational curve, elliptic and bielliptic curves are also tetragonal. From the definition of a bihyperelliptic curve, we know that it is tetragonal.

1. Preliminaries

1.1. Modular curves $X_\Delta(N)$

Let Δ be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^*$ which contains -1. Let $X_\Delta(N)$ be the modular curve defined over $\mathbb{Q}$ associated to the congruence subgroup

$$\Gamma_\Delta(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \,|\, a \in \Delta, N \mid c \right\}.$$

Note that for $\Delta = \{\pm 1\}$ this is just $X_1(1,N)$. From now on we denote $X_1(1,N)$ by $X_1(N)$.

Conjugating the group $\Gamma_1(M,N)$ with the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & M \end{smallmatrix}\right)$ we obtain a birational map from $X_1(M,N)$ to $X_\Delta(MN)$ with

$$\Delta = \{\pm 1, \pm(N+1), \pm(2N+1), \ldots, \pm((M-1)N+1)\}.$$

For $d|N$, let π_d be the natural projection from $(\mathbb{Z}/N\mathbb{Z})^*$ to $(\mathbb{Z}/\{d, N/d\}\mathbb{Z})^*$, where $\{d, N/d\}$ is the least common multiple of d and N/d.

Theorem 1.1 ([8]). *The genus of the modular curve $X_\Delta(N)$ is given by*

$$g(X_\Delta(N)) = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2},$$

where

$$\begin{aligned}
\mu &= N \cdot \prod_{\substack{p|N \\ prime}} \left(1 + \frac{1}{p}\right) \cdot \frac{\varphi(N)}{|\Delta|} \\
\nu_2 &= \left|\{(b \mod N) \in \Delta \,|\, b^2 + 1 \equiv 0 \mod N\}\right| \cdot \frac{\varphi(N)}{|\Delta|} \\
\nu_3 &= \left|\{(b \mod N) \in \Delta \,|\, b^2 - b + 1 \equiv 0 \mod N\}\right| \cdot \frac{\varphi(N)}{|\Delta|} \\
\nu_\infty &= \sum_{\substack{d|N \\ d>0}} \frac{\varphi(d) \cdot \varphi(\frac{N}{d})}{|\pi_d(\Delta)|}.
\end{aligned}$$

By using the above genus formula and the birational map from $X_1(M, N)$ to $X_\Delta(MN)$, one can calculate the genus of $X_1(M, N)$.

1.2. Abramovich's bound

Let a smooth projective curve X be d-gonal where d is smallest for such d. Then d is called the *gonality* of X and is denoted by $\mathrm{Gon}(X)$. The best general lower bound for the gonality of a modular curve seems to be the one that is obtained in the following way.

Let λ_1 be the smallest positive eigenvalue of the Laplacian operator on the Hilbert space $L^2(X_\Gamma)$ where X_Γ is the modular curve corresponding to a congruence subgroup Γ of $\Gamma(1)$. Let D_Γ be the index of $\pm\Gamma$ in $\Gamma(1)$ and d_Γ the gonality of X_Γ. Abramovich [1] shows the following inequality:

$$\lambda_1 D_\Gamma \leq 24 d_\Gamma.$$

Using the best known lower bound for λ_1, due to Henry Kim and Peter Sarnak, as reported on page 18 of [2], i.e., $\lambda_1 > 0.238$, we get the following result.

Theorem 1.2. *Let X_Γ be the modular curve corresponding to a congruence subgroup Γ of index $D_\Gamma := [\Gamma(1) : \pm\Gamma]$ and let d_Γ be the gonality of X_Γ. Then*

$$D_\Gamma < \frac{12000}{119} d_\Gamma.$$

In the following, we call the inequality in Theorem 1.2 Abramovich's bound.

1.3. Clifford index

For a line bundle $L \in \mathrm{Pic}X$, the *Clifford index* of L is the integer

$$\mathrm{Cliff}(L) := \deg(L) - 2(h^0(X, L) - 1)$$

and the *Clifford index of* X itself is defined as

$$\mathrm{Cliff}(X) := \min\{\mathrm{Cliff}(L) \mid h^0(X, L) \geq 2, h^1(X, L) \geq 2\}.$$

It is well-known that $\mathrm{Cliff}(X) + 2 \leq \mathrm{Gon}(X) \leq \mathrm{Cliff}(X) + 3$ [3].

1.4. Property $\mathbf{N_p}$

If X is a non-hyperelliptic curve, then the canonical line bundle ω_X defines an embedding $X \hookrightarrow \mathbb{P}H^0(X, \omega_X) = \mathbb{P}^{g-1}$. Consider the minimal free resolution

$$0 \to F_{g-2} \to \cdots \to F_2 \to F_1 \to S \to S_X \to 0$$

of the homogeneous coordinate ring $S_X = S/I_X$ as an S-module where $S = \mathbb{C}[X_0, X_1, \ldots, X_{g-1}]$ and $F_i = \oplus_{j\in\mathbb{Z}} S(-i-j)^{\beta_{i,j}}$. We call $\beta_{i,j}$ the *graded Betti-numbers.* Due to Green and Lazarsfeld[5], $X \hookrightarrow \mathbb{P}^{g-1}$ is said to satisfy *property* N_p if the resolution is of the form

$$\cdots \to S^{\beta_{p,1}}(-p-1) \to \cdots \to S^{\beta_{2,1}}(-3) \to S^{\beta_{1,1}}(-2) \to S \to S_X \to 0.$$

Therefore property N_1 holds if and only if the homogeneous ideal is generated by quadrics, and property N_p holds for $p \geq 2$ if and only if it has property N_1

and the k^{th} syzygies among the quadrics are generated by linear syzygies for all $1 \le k \le p-1$. Now we recall the following:

Theorem 1.3 (M. Green and R. Lazarsfeld, Appendix in [4]). *Let X be a smooth non-hyperelliptic curve of genus $g \ge 3$. Then the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ fails to satisfy property N_p for $p \ge$ Cliff(X).*

Thus if the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$ satisfies property N_p, then Cliff(X) $\ge p+1$ and Gon$(X) \ge p+3$.

2. Tetragonal curves

As we mentioned in Remark 0.2, all the rational, elliptic, hyperelliptic, bielliptic and bihyperelliptic curves are tetragonal. Using Theorem 1.1 one can get the following proposition.

Proposition 2.1. *$X_1(M,N)$ has genus 0 or 1 if and only if (M,N) is one of the 13 following ordered pairs:*
- *genus 0 :* $(1, N = 1-10, 12), (2, N = 2, 4, 6, 8), (3, N = 3, 6), (4, 4), (5, 5)$;
- *genus 1 :* $(1, N = 11, 14, 15), (2, N = 10, 12), (3, 9), (4, 8)$;

Mestre prove that the only hyperelliptic modular curves $X_1(N)$ are $X_1(13)$, $X_1(16)$, $X_1(18)$. Ishii and Momose [7] asserted that there exist no hyperelliptic modular curves $X_\Delta(N)$ with $\{\pm1\} \subsetneq \Delta \subsetneq (\mathbb{Z}/N\mathbb{Z})^*$. But the author and Kim [8] find one and only hyperelliptic modular curve $X_\Delta(21)$ with $\Delta = \{\pm1, \pm8\}$, and correct their result. Since $X_\Delta(21)$ is not birational to any $X_1(M,N)$, we can conclude that the only hyperelliptic modular curves $X_1(M,N)$ are $X_1(13)$, $X_1(16)$, $X_1(18)$.

The author and Kim determine all the bielliptic modular curves $X_1(M,N)$ as follows:

Theorem 2.2. *The curve $X_1(M,N)$ is bielliptic only for the following (M,N) :*

$$(1,13),\ (1,16),\ (1,17),\ (1,18),\ (1,20),\ (1,21),\ (1,22),\ (1,24),$$
$$(2,14),\ (2,16),\ (3,12),\ (4,12),\ (5,10),\ (7,7),\ (8,8).$$

Since there is a natural map $X_1(2,18) \to X_1(18)$ of degree 2 and $X_1(18)$ is hyperelliptic, $X_1(2,18)$ is bihyperelliptic.

In the next section, we prove that the modular curves $X_1(M,N)$ as mentioned above are the only tetragonal modular curves.

3. Non-tetragonal curves

Suppose $X_1(M,N)$ is tetragonal. Note that $X_1(M,N)$ is birational to $X_\Delta(MN)$. Applying Abramovich's bound to $X_\Delta(MN)$, we have the following:

Lemma 3.1. *$X_1(M,N)$ is not tetragonal if $M > 10$ or $N > MN_M$ with $1 \le M \le 10$, where*

$N_1 = 32,\ N_2 = 12,\ N_3 = 6,\ N_4 = 4,\ N_5 = 2,\ N_6 = 2,\ N_7 = 1,$
$N_8 = 1,\ N_9 = 1,\ N_{10} = 1.$

The author, Kim, and Park [9] prove that the modular curves $X_1(N)$, $X_1(2,\ 2N')$ are tetragonal only for $N = 1 - 18, 20, 21, 22, 24$ and $N' = 1 - 9$. Therefore we can cut out $X_1(1,N)$, $X_1(2,2N')$ with $N = 19, 23, 25 - 32$ and $N' = 10, 11, 12$.

The remaining cases we don't determine are $X_1(M,N)$ for following (M,N):

$$(3,15), (3,18), (4,16), (6,12), (9,9), (10,10).$$

To treat the above cases we need to compute the graded Betti numbers of the canonical embedding of $X_\Delta(MN)$. We use the computer programs "Maple" and "Singular". First we calculate the homogeneous ideal of the canonical embedding of $X_\Delta(MN)$ by using Maple.

Note that, for such (M,N), $X_\Delta(MN)$ is not hyperelliptic. Thus $X_\Delta(MN)$ can be identified with the canonical curve which is the image of the canonical embedding

$$X_\Delta(MN) \ni P \mapsto (f_1(P) : \cdots : f_g(P)) \in \mathbb{P}^{g-1},$$

where $\{f_1, \ldots, f_g\}$ is a basis of the space of cusp forms of weight 2 on $X_\Delta(MN)$. One can get such a basis and their Fourier coefficients from [10]. Then to obtain the minimal generating system of the homogeneous ideal $I(X_\Delta(MN))$, we have only to compute the relations of the $f_i f_j$ $(1 \leq i, j \leq g)$ by Petri's theorem. Since there are $(g-2)(g-3)/2$ linear relations among the $f_i f_j$, we get quadric generators $Q_k(x_1, \ldots, x_g)$ with $1 \leq k \leq (g-2)(g-3)/2$ by assigning x_i to f_i (for details see [6]).

Now we compute the Betti numbers by using Singular. In fact when the genus of $X_\Delta(MN)$ is big then Singular doesn't produce Betti numbers well. Note that since the canonical embedding is always projectively Cohen-Macauly, the Betti numbers of the canonical curve are equal to those of the hyperplane section, which allows us to get Betti numbers easier.

We exhibit the so-called Betti-table of the canonical embedding for our cases in Table 1. All the cases satisfy property N_p for $p \geq 2$. Thus $\mathrm{Gon}(X_1(M,N)) \geq 5$ by § 1.4 for $(M,N) = (3,15), (3,18), (4,16), (6,12), (9,9), (10,10)$.

Table 1: The Graded Betti-numbers for the canonical embedding.

genus	$X_1(M,N)$	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$	genus	$X_1(M,N)$	$\beta_{1,2}$	$\beta_{2,2}$	$\beta_{3,2}$
		$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$			$\beta_{1,1}$	$\beta_{2,1}$	$\beta_{3,1}$
9	$X_1(3,15)$	0	0	0		$X_1(9,9)$	0	0	20
		21	64	70			28	105	162
	$X_1(6,12)$	0	0	0	13	$X_1(4,16)$	0	0	0
		21	64	70			55	320	891
10	$X_1(3,18)$	0	0	0		$X_1(10,10)$	0	0	0
		28	105	162			55	320	891

References

[1] D. Abramovich, *A linear lower bound on the gonality of modular curves,* Int. Math. Res. Not. **1996** (1996), no. 20, 1005–1011.
[2] M. H. Baker, E. González-Jiménez, J. González, and B. Poonen, *Finiteness results for modular curves of genus at least* 2, Amer. J. Math. **127** (2005), no. 6, 1325–1387.
[3] M. Coppens and G. Martens, *Secant spaces and Clifford's theorem*, Compositio Math. **78** (1991), 193–212.
[4] M. Green, *Koszul cohomology and the geometry of projective varieties I*, J. Differ. Geom. **19** (1984), 125–171.
[5] M. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Compositio Math. **67** (1988), 301–314.
[6] Y. Hasegawa and M. Shimura, *Trigonal modular curves,* Acta Arith. **88** (1999), 129–140.
[7] N. Ishii and F. Momose, *Hyperelliptic modular curves,* Tsukuba J. Math. **15** (1991), 413–423.
[8] D. Jeon and C. H. Kim, *On the arithmetic of certain modular curves,* Acta Arith. **130** (2007), no. 2, 181–193.
[9] D. Jeon, C. H. Kim, and E. Park, *On the torsion of elliptic curves over quartic number fields,* J. London Math. Soc. (2) **74** (2006), no. 1, 1–12.
[10] William A. Stein, *http://modular.fas.harvard.edu.*

Department of Mathematics education
Kongju National University
Chungnam 314-701, Korea
E-mail address: dyjeon@kongju.ac.kr