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ON WEAK ARMENDARIZ IDEALS

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ABSTRACT. We introduce weak Armendariz ideals which are a generalization of ideals have the weakly insertion of factors property (or simply weakly IFP) and investigate their properties. Moreover, we prove that, if I is a weak Armendariz ideal of R, then $I[x]$ is a weak Armendariz ideal of $R[x]$. As a consequence, we show that, R is weak Armendariz if and only if $R[x]$ is a weak Armendariz ring. Also we obtain a generalization of [8] and [9].

1. Introduction

Throughout this paper R denotes an associative ring with identity. A ring R is called *semicommutative* if for any $a, b \in R$, $ab = 0$ implies $aRb = 0$. Rege and Chhawchharia [11] introduced the notion of an Armendariz ring. A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. The name "Armendariz ring" was chosen because Armendariz [2, Lemma 1] had noted that a *reduced* ring (i.e., $a^2 = 0$ implies $a = 0$) satisfies this condition. Some properties of Armendariz rings have been studied in Rege and Chhawchharia [11], Armendariz [2], Anderson and Camillo [1], and Kim and Lee [6]. Zhongkui Liu and Renyu Zhao [9] studied a generalization of Armendariz rings, which is called weak Armendariz rings. A ring R is called *weak Armendariz* if whenever $f(x) = a_0 + a_1x + \cdots + a_nx^n, g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$, with $a_i, b_j \in R$ satisfy $f(x)g(x) = 0$, then a_ib_j is a nilpotent element of R for each i, j. They have shown that, if R is a semicommutative ring, then the ring $R[x]$ and the ring $\frac{R[x]}{(x^n)}$, where (x^n) is the ideal generated by x^n, and n is a positive integer, are weak Armendariz. They also give the following question: Let R be a weak Armendariz. Is $R[x]$ weak Armendariz?

We call an ideal I *weak Armendariz*, if whenever $f(x) = a_0+a_1x+\cdots+a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$, satisfy $f(x)g(x) \in I[x]$, then for each i, j,

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there exists a positive integer n_{ij} such that $(a_ib_j)^{n_{ij}} \in I$. Clearly if ideal $I = 0$ is weak Armendariz, then R is a weak Armendariz ring.

Recall from [10] that a one-sided ideal I of a ring R has *the insertion of factors property* (or simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. (H. E. Bell in 1973 introduced this notion for $I = 0$). Observe that every *completely semiprime* ideal (i.e., $a^2 \in I$ implies $a \in I$) of R has the IFP [10, Lemma 3.2(a)]. If $I = 0$ has the IFP, then we say R has the IFP (or R is semicommutative). Li Liang et al. [8] introduced weakly semicommutative rings. A ring R is called *weakly semicommutative*, if for any $a, b \in R$, $ab = 0$ implies arb is a nilpotent element for each $r \in R$.

We say a one-sided ideal I of R has the *weakly IFP* if for each $a, b, r \in R, ab \in I$ implies $(arb)^n \in I$ for some non-negative integer n. Clearly, if ideal $I = 0$ has the weakly IFP, then R is a weakly semicommutative ring.

In this paper we show that if an ideal I has the weakly IFP, then I is weak Armendariz, thus weak Armendariz ideals are a generalization of ideals which has the weakly IFP. Also, for any positive integer n, we study relationship between ideals of R which are weak Armendariz with some ideals of the ring

$$R_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \text{ for all } i, j \right\},$$

the n-by-n upper triangular matrix ring over R and the ring $\frac{R[x]}{(x^n)}$, where (x^n) is the ideal generated by x^n. As a consequence, if R is weak Armendariz, then for any positive integer n, the n-by-n upper triangular matrix ring, the ring $\frac{R[x]}{(x^n)}$ and the ring $R_n(R)$ are weak Armendariz. Also we show that, if I is an ideal of R, then I is weak Armendariz if and only if $I[x]$ is a weak Armendariz ideal of $R[x]$. As a consequence, we show that, R is weak Armendariz if and only if $R[x]$ is weak Armendariz, thus we give an affirmative answer to a question of Liu et al. [9, p. 2614].

For a ring R, we denote by $nil(R)$ the set of all nilpotent elements of R and by $T_n(R)$ the n-by-n upper triangular matrix ring over R.

2. On weak Armendariz ideals

For an ideal I of R put

$$\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some non-negative integer } n \geq 0\}.$$

Definition 2.1. An ideal I of a ring R is said to be *weak Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) \in I[x]$, then $a_ib_j \in \sqrt{I}$ for all i, j.

Clearly, if $I = 0$ is weak Armendariz, then R is a weak Armendariz ring.

It is well-known that for a ring R and any positive integer $n \geq 2$,

$$\frac{R[x]}{(x^n)} \cong \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in R,\ i = 0, 1, \ldots, n-1 \right\},$$

where (x^n) is the ideal of $R[x]$ generated by x^n.

Lemma 2.2. *Let R be a ring and $n \geq 2$ a positive integer. Let $I_0, I_1, \ldots, I_{n-1}$ are ideals of R, such that $I_i \subseteq I_{i+1}$, $i = 0, 1, \ldots, n-2$. Then*

$$J = \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in I_i,\ i = 0, 1, \ldots, n \right\}$$

is an ideal of $\frac{R[x]}{(x^n)}$.

Proof. It is straightforward. □

In Propositions 2.3, 2.6, and Theorem 2.4, I_0 and J are ideals that mentioned in Lemma 2.2.

Proposition 2.3. *Let*

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} \in \frac{R[x]}{(x^n)},$$

such that $a_0^k \in I_0$ for some integer k. Then $A^{nk} \in J$.

Proof. We proceed by induction on n. Let $n = 2$. For a positive integer k, $A^k = \begin{pmatrix} a^k & b_1 \\ 0 & a^k \end{pmatrix}$ and that $A^{2k} = \begin{pmatrix} a_0^{2k} & a_0^k b_1 + b_1 a_0^k \\ 0 & a_0^{2k} \end{pmatrix}$. Hence $A^{2k} \in J$, since $a_0^{2k}, a_0^k b_1 + b_1 a_0^k \in I_0$. Now, let

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ 0 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} \in \frac{R[x]}{(x^n)}$$

such that $a_0^k \in I_0$ for some integer k. Consider

$$A^k = \begin{pmatrix} a_0^k & c_1 & \cdots & c_{n-1} \\ 0 & a_0^k & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0^k \end{pmatrix} \quad \text{and} \quad A^{(n-1)k} = \begin{pmatrix} a_0^{(n-1)k} & b_1 & \cdots & b_{n-1} \\ 0 & a_0^{(n-1)k} & \cdots & b_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0^{(n-1)k} \end{pmatrix}.$$

By the induction hypothesis all b_i's, except b_{n-1}, are in I_0. Let $x = a_0^k b_1 + c_1 b_{n-2} + \cdots + c_{n-1} a_0^{(n-1)k}$. Hence

$$A^{nk} = \begin{pmatrix} a_0^{nk} & y_1 & \cdots & x \\ 0 & a_0^{nk} & \cdots & y_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0^{nk} \end{pmatrix} \in J,$$

since a_0^{nk}, x and all y_i's are in I_0. □

Theorem 2.4. *I_0 is weak Armendariz, if and only if J is weak Armendariz.*

Proof. (⇒) Let $f(y) = A_0 + A_1 y + \cdots + A_m y^m$, $g(y) = B_0 + B_1 y + \cdots + B_t y^t \in \frac{R[x]}{(x^n)}[y]$, such that $f(y)g(y) \in J[y]$. Let

$$A_i = \begin{pmatrix} a_{0i} & a_{1i} & \cdots & a_{n-1i} \\ 0 & a_{0i} & \cdots & a_{n-2i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{0i} \end{pmatrix}, \quad B_j = \begin{pmatrix} b_{0j} & b_{1j} & \cdots & b_{n-1j} \\ 0 & b_{0j} & \cdots & b_{n-2j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{0j} \end{pmatrix}$$

for $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, t$. Let $f_0 = a_{00} + a_{01} y + \cdots + a_{0m} y^m$ and $g_0 = b_{00} + b_{01} y + \cdots + b_{0t} y^t$. Then $f_0 g_0 \in I_0[y]$. Since I_0 is weak Armendariz, there exists $k > 0$, such that $(a_{0i} b_{0j})^k \in I_0$ for each i, j. Then $(A_i B_j)^{nk} \in J$ for all i, j, by Proposition 2.3. Therefore J is weak Armendariz.

(⇐) It is clear. □

Li Liang et al. [8, Theorem 3.9] showed that, if R is a semicommutative ring, then the ring $\frac{R[x]}{(x^n)}$, for each positive integer n, is weak Armendariz. The following result is a generalization of Li Liang et al.'s result.

Corollary 2.5. *Let R be a ring. Then R is weak Armendariz if and only if the ring $\frac{R[x]}{(x^n)}$, for each positive integer n, is weak Armendariz.*

Proposition 2.6. *I_0 has the weakly IFP if and only if J has the weakly IFP.*

Proof. It follows from Proposition 2.3. □

Clearly, if an ideal I has the IFP, then it has the weakly IFP. By the following example, we show that the converse is not true.

Example 2.7. Let

$$J = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a_i \in 2p\mathbb{Z} \right\}$$

be an ideal of $\frac{\mathbb{Z}[x]}{(x^4)}$ where $2 \neq p$ is a prime number and $\mathbb{Z}$ is the set of integers. Then

$$\begin{pmatrix} 0 & p & 1 & 0 \\ 0 & 0 & p & 1 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & p & 1 & 0 \\ 0 & 0 & p & 1 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2p & 2p \\ 0 & 0 & 0 & 2p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in J,$$

but

$$\begin{pmatrix} 0 & p & 1 & 0 \\ 0 & 0 & p & 1 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & p & 1 & 0 \\ 0 & 0 & p & 1 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 3p^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin J.$$

Hence J has not the IFP, but J has the weakly IFP, by Proposition 2.5.

By a similar way as used in Example 2.7, we can construct numerous ideals of $\frac{\mathbb{Z}[x]}{(x^n)}$ such that have the weakly IFP, but have't the IFP for $n \geq 2$.

Theorem 2.8. *Let I be an ideal of R. Then I is weak Armendariz if and only if $I[x]$ is weak Armendariz.*

Proof. ($\Rightarrow$) Let $f(y) = f_0 + f_1y + \cdots + f_my^m$, $g(y) = g_0 + g_1y + \cdots + g_ty^t \in R[x][y]$ are such that $f(y)g(y) \in I[x][y]$. Let $n = \max\{\deg(f_i), \deg(g_j) | i = 0, 1, \ldots, m, j = 0, 1, \ldots, t\}$. Then we can assume $f_i = a_{i0} + a_{i1}x + \cdots + a_{in}x^n$, $g_j = b_{j0} + b_{j1}x + \cdots + b_{jn}x^n$ for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, t$. Let

$$J = \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ 0 & a_0 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in I, \ i = 0, 1, \ldots, n \right\}.$$

By Lemma 2.2, J is an ideal of $\frac{R[x]}{(x^{n+1})}$. Let

$$A_i = \begin{pmatrix} a_{i0} & a_{i1} & \cdots & a_{in} \\ 0 & a_{i0} & \cdots & a_{in-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{i0} \end{pmatrix}, \quad B_j = \begin{pmatrix} b_{j0} & b_{j1} & \cdots & b_{jn} \\ 0 & b_{j0} & \cdots & b_{jn-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{j0} \end{pmatrix}$$

for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, t$. Since $f(y)g(y) \in I[x][y]$, we have $(A_0 + A_1y + \cdots + A_my^m)(B_0 + B_1y + \cdots + B_ty^t) \in J[y]$. By Theorem 2.4, J is weak Armendariz. Hence there exists $k > 0$ such that $(A_iB_j)^k \in J$ for each i, j. Thus $(f_ig_j)^k \in I[x]$ for each i, j. Therefore $I[x]$ is weak Armendariz.

($\Leftarrow$) It is clear. □

Now we give an affirmative answer to a question of Liu et al. [9, p. 2614].

Corollary 2.9. *Let R be a ring. Then R is weak Armendariz if and only if $R[x]$ is weak Armendariz.*

Lemma 2.10. *Let I be an ideal of R and has the IFP. Then $\sqrt{I}$ is an ideal of R and has the IFP.*

Proof. Let $a, b \in \sqrt{I}$. Then $a^n, b^m \in I$ for some integer $m, n \geq 0$. Hence $(a+b)^{m+n+1} = \sum (a^{i_1}b^{j_1}) \cdots (a^{i_{m+n+1}}b^{j_{m+n+1}})$, where $i_k + j_k = 1, 0 \leq i_k \leq 1$, $0 \leq j_k \leq 1$. It can be easily checked that a more that n or b more that m appear in $(a^{i_1}b^{j_1}) \cdots (a^{i_{m+n+1}}b^{j_{m+n+1}})$. Since $a^n, b^m \in I$ and I has the IFP, we have $(a^{i_1}b^{j_1}) \cdots (a^{i_{m+n+1}}b^{j_{m+n+1}}) \in I$. Therefore $(a+b)^{m+n+1} \in I$ and $a + b \in \sqrt{I}$.

Now suppose that $a^m \in I$ and $r \in R$. Then $(ra)^m$, $(ar)^n \in I$, since I has the IFP. Thus $\sqrt{I}$ is an ideal of R. Clearly $\sqrt{I}$ has the IFP. □

Proposition 2.11. *Let I be an ideal of R and has the IFP. Then I and $\sqrt{I}$ are weak Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^m a_i x^i$, $g(x) = \sum_{j=0}^t b_j x^j \in R[x]$ such that $f(x)g(x) \in I[x]$. Then $a_m b_t \in I$. Since $a_m b_{t-1} + a_{m-1} b_t \in I \subseteq \sqrt{I}$, we have $a_m b_{t-1} b_t + a_{m-1} b_t^2 \in I$. Hence $a_{m-1} b_t^2 \in I$, since $a_m b_{t-1} b_n \in I$. Thus $a_{m-1} b_t \in \sqrt{I}$, by Lemma 2.10. Since $\sqrt{I}$ is an ideal of R, hence $a_m b_{t-1} \in \sqrt{I}$. Coefficient of x^{m+t-2} in $f(x)g(x)$ is $a_m b_{t-2} + a_{m-1} b_{t-1} + a_{m-2} b_t$. Then $a_m b_{t-2} + a_{m-1} b_{t-1} + a_{m-2} b_t \in I$, and so $a_m b_{t-2} b_t + a_{m-1} b_{t-1} b_t + a_{m-2} b_t^2 \in \sqrt{I}$. Since $a_m b_{t-2} b_t, a_{m-1} b_{t-1} b_t \in \sqrt{I}$, hence $a_{m-2} b_t^2 \in \sqrt{I}$, and by Lemma 2.10, $a_{m-2} b_t \in \sqrt{I}$. By a similar way as above, we can show that $a_{m-1} b_{t-1}, a_m b_{t-2} \in \sqrt{I}$. Continuing this process, we can prove $a_i b_j \in \sqrt{I}$ for each i, j. Therefore I is weak Armendariz.

Since I has the IFP, hence by Lemma 2.10, $\sqrt{I}$ is an ideal of R and has the IFP. Thus $\sqrt{I}$ is weak Armendariz. □

Corollary 2.12 ([9, Corollary 3.4]). *Semicommutative rings are weak Armendariz.*

Lemma 2.13. *Let I, I_{ij} are ideals of R such that $I \subseteq I_{ij} \subseteq I_{is}$ for $1 \leq i < j \leq s \leq n$, and $I_{pq} \subseteq I_{\ell q}$ for $q = 3, \ldots, n$, $2 \leq \ell \leq p \leq n$. Then*

$$J = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a \in I,\ a_{ij} \in I_{ij} \right\}$$

is an ideal of $R_n(R)$.

Proof. It is straightforward. □

In Propositions 2.14, 2.17, and Theorem 2.15, I and J are ideals that mentioned in Lemma 2.13.

Proposition 2.14. *Let*

$$A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n(R)$$

such that $a^k \in I$ for some integer k. Then $A^{nk} \in J$.

Proof. We proceed by induction on n. Let $n = 2$. For a positive integer k, $A^k = \begin{pmatrix} a^k & b_{12} \\ 0 & a^k \end{pmatrix}$ and that $A^{2k} = \begin{pmatrix} a^{2k} & a^k b_{12} + b_{12} a^k \\ 0 & a^{2k} \end{pmatrix}$. Hence $A^{2k} \in J$, since a^{2k}, $a^k b_{12} + b_{12} a^k \in I$. Now, let

$$A = \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in R_n(R)$$

such that $a^k \in I$ for some integer k. Consider

$$A^k = \begin{pmatrix} a^k & c_{12} & \cdots & c_{1n} \\ 0 & a^k & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^k \end{pmatrix} \quad \text{and} \quad A^{(n-1)k} = \begin{pmatrix} a^{(n-1)k} & b_{12} & \cdots & b_{1n} \\ 0 & a^{(n-1)k} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{(n-1)k} \end{pmatrix}.$$

By the induction hypothesis all b_{ij}'s, except b_{1n}, are in I. Let $x = a^k b_{1n} + c_{12} b_{2n} + \cdots + c_{1n} a^{(n-1)k}$. Hence

$$A^{nk} = \begin{pmatrix} a^{nk} & y_{12} & \cdots & x \\ 0 & a^{nk} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^{nk} \end{pmatrix} \in J,$$

since a^{nk}, x and all y_{ij}'s are in I. □

Theorem 2.15. *I is weak Armendariz if and only if J is weak Armendariz.*

Proof. $(\Rightarrow)$ Let $f(x) = A_0 + A_1 x + \cdots + A_m x^m$, $g(x) = B_0 + B_1 x + \cdots + B_t x^t \in R_n(R)$, such that $f(x)g(x) \in J[x]$. Let

$$A_i = \begin{pmatrix} a^i & a^i_{12} & \cdots & a^i_{1n} \\ 0 & a^i & \cdots & a^i_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^i \end{pmatrix}$$

for $i = 0, 1, \ldots, m$ and

$$B_j = \begin{pmatrix} b^j & b^j_{12} & \cdots & b^j_{1n} \\ 0 & b^j & \cdots & b^j_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^j \end{pmatrix}$$

for $j = 0, 1, \ldots, t$. Let $f_0 = a^0 + a^1x + \cdots + a^mx^m$ and $g_0 = b^0 + b^1x + \cdots + b^tx^t$. Then $f_0g_0 \in I[x]$. Since I is weak Armendariz, there exists $k > 0$ such that $(a^ib^j)^k \in I$ for each i, j. Then $(A_iB_j)^{nk} \in J$ for all i, j, by Proposition 2.14. Therefore J is weak Armendariz.

($\Leftarrow$) It is clear. □

Corollary 2.16 ([9, Example 2.4]). *A ring R is weak Armendariz if and only if, for any positive integer n, $R_n(R)$ is weak Armendariz.*

Proof. It follows from Theorem 2.15. □

Proposition 2.17. *I has the weakly IFP if and only if J has the weakly IFP.*

Proof. It follows from Proposition 2.14. □

Corollary 2.18. *A ring R is weakly semicommutative if and only if, for any positive integer n, the ring $R_n(R)$ is a weakly semicommutative ring.*

Clearly, if an ideal I has the IFP, then it has the weakly IFP. The following is an another example to show that the converse is not true.

Example 2.19. Let

$$J = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a_{ij} \in 2p\mathbb{Z} \right\}$$

be an ideal of $R_4(\mathbb{Z})$ where $2 \neq p$ is a prime number and $\mathbb{Z}$ is the set of integers. Then

$$\begin{pmatrix} 0 & p & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in J,$$

but

$$\begin{pmatrix} 0 & p & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 3p^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin J.$$

Hence J has not the IFP, but J has the weakly IFP, by Proposition 2.17.

By a similar way as used in Example 2.19, we can construct numerous ideals of $R_n(R)$ such that have the weakly IFP, but haven't the IFP for each $n \geq 2$.

Proposition 2.20. *Let R be a ring and I, J be ideals of R. If $I \subseteq \sqrt{J}$ and $\frac{I+J}{I}$ is weak Armendariz, then J is weak Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{t} b_j x^j \in R[x]$ such that $f(x)g(x) \in J[x]$. Then $(\sum_{i=0}^{m} \overline{a}_i x^i)(\sum_{j=0}^{t} \overline{b}_j x^j) \in \frac{I+J}{I}[x]$. Thus $(\overline{a}_i \overline{b}_j)^{n_{ij}} \in \frac{I+J}{I}$ for some positive integer n_{ij}. Hence $(a_i b_j)^{n_{ij}} \in I+J$, and so $(a_i b_j)^{n_{ij}} \in J$, since $I \subseteq \sqrt{J}$. Therefore J is weak Armendariz. □

Corollary 2.21 ([9, Proposition 2.9]). *Let R be a ring and I an ideal of R such that $\frac{R}{I}$ is weak Armendariz. If $I \subseteq nil(R)$, then R is weak Armendariz.*

Lemma 2.22. *Let I_{ij} be ideals of R such that $I_{ij} \subseteq I_{is}$ for $1 \leq i \leq j \leq s \leq n$ and $I_{pq} \subseteq I_{\ell q}$ for $q = 2, \ldots, n$, $1 \leq \ell \leq p \leq n$. Then*

$$J = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \mid a_{ij} \in I_{ij},\ 1 \leq i,j \leq n \right\}$$

is an ideal of $T_n(R)$.

Proof. It is straightforward. □

In Propositions 2.23, 2.26, and Theorem 2.24, I_{ij}'s are ideals that mentioned in Lemma 2.22. By a similar way as used in the proof of Proposition 2.3 and Theorem 2.4, one can prove Proposition 2.23 and Theorem 2.24.

Proposition 2.23. *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$$

such that $a_{ii}^k \in I_{ii}$ for some positive integer k and $i = 1, \ldots, n$. Then $(A^{2k+1})^{n-1} \in J$.

Theorem 2.24. *J is weak Armendariz if and only if all I_{ii} are weak Armendariz for $i = 1, \ldots, n$.*

Corollary 2.25 ([9, Proposition 2.2]). *A ring R is weak Armendariz if and only if $T_n(R)$, for any positive integer n, is weak Armendariz.*

Proposition 2.26. *J has the weakly IFP if and only if all I_{ii} has the weakly IFP for $i = 1, \ldots, n$.*

Proof. It follows from Proposition 2.23. □

Corollary 2.27 ([8, Claim 2.1]). *A ring R is a weakly semicommutative ring if and only if, for any n, the n-by-n upper triangular matrix ring $T_n(R)$ is a weakly semicommutative ring.*

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