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WEAKLY PRIME LEFT IDEALS IN NEAR-SUBTRACTION SEMIGROUPS

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ABSTRACT. In this paper we introduce the notion of weakly prime left ideals in near-subtraction semigroups. Equivalent conditions for a left ideal to be weakly prime are obtained. We have also shown that if $(M,\ L)$ is a weak m^*-system and if P is a left ideal which is maximal with respect to containing L and not meeting M, then P is weakly prime.

1. Introduction

Schein [7] considered systems of the form $(\phi; \circ, \backslash)$ where ϕ is a set of functions closed under the composition " $\circ$ " of functions (and hence $(\phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence$(\phi; \backslash)$ is a subtraction algebra in the sense of [1]). Zelinka [8] discussed a problem proposed by Schein concerning the structure of multiplication in a subtraction semigroup. Eun Hwan Roh, Kyung Ho Kim, and Jong Geol Lee [6] obtained significant results in subtraction semigroups. The notion of near-subtraction semigroup was studied in [2]. We introduce the notion of weakly prime left ideals in near-subtraction semigroup which is a generalization of prime left ideals and give some characterizations of weakly prime left ideals. In this process the concept of weak m^*-system has been introduced which plays the same part as the m-system plays for prime ideals in ring and near-ring theory.

2. Preliminaries

Definition 2.1. A nonempty set X together with a binary operation " $-$ " is said to be a subtraction algebra if it satisfies the following identities:

(1) $x - (y - x) = x$.
(2) $x - (x - y) = y - (y - x)$.
(3) $(x - y) - z = (x - z) - y$ for every $x,\ y,\ z \in X$.

The subtraction determines an order relation on X : $a \leq b \Leftrightarrow a - b = 0$, where $a - a = 0$ is an element that does not depend on the choice of $a \in X$.

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Example 2.2. Let A be any nonempty set. Then $(P(A), \backslash)$ is a subtraction algebra, where "$P(A)$" denotes the power set of A and "$\backslash$" denotes the set theoretic subtraction.

In a subtraction algebra the following holds:

(1) $x - 0 = x$ and $0 - x = 0$.
(2) $(x - y) - x = 0$.
(3) $(x - y) - y = x - y$.
(4) $(x - y) - (y - x) = x - y$.

Following [4], we have the following definition of subtraction semigroup.

Definition 2.3. A nonempty set X together with two binary operations "$-$" and "." is said to be a subtraction semigroup if it satisfies the following:

(1) $(X; -)$ is a subtraction algebra.
(2) $(X; \cdot)$ is a semigroup.
(3) $x(y - z) = xy - xz$ and $(x - y)z = xz - yz$ for every $x,\ y,\ z \in X$.

Note that it is clear that $0x = 0$ and $x0 = 0$ for every $x \in X$.

Example 2.4. Let Γ be a subtraction algebra. Then the set $M_h(\Gamma)$ of all homomorphisms of Γ into Γ is a subtraction semigroup under point wise subtraction and composition of mappings.

Definition 2.5. Let $(X, -, \cdot)$ be a subtraction semigroup. A nonempty subset I of X is called a left (right) ideal if $x - y \in I$, for every $x \in I,\ y \in X$ and $XI \subseteq I$ $(IX \subseteq I)$. If I is both a left and right ideal then I is an ideal, denoted by $I \trianglelefteq X$, where $AB = \{ab \mid a \in A,\ b \in B\}$ for any nonempty subsets $A,\ B$ of X.

Remark 2.6. Let X be a subtraction algebra and $I \subseteq X$. Then the following are equivalent:

(i) $(\forall x \in I,\ y \in X)\ x - y \in I$.
(ii) $x \leq y$ and $y \in I \Rightarrow x \in I$.

Proof. (i)$\Rightarrow$ (ii) Let $y \in I$ and $x \in X$ such that $x \leq y$. Hence $x - y$=0. Then $x = x - (x - y) = y - (y - x) \in I$ by (i).

(ii)$\Rightarrow$ (i) Let $x \in I$ and $y \in X$. Since $(x - y) - x = 0,\ (x - y) \leq x$. Hence by (ii) $x - y \in I$. □

Definition 2.7 ([2]). A nonempty set X together with two binary operations "$-$" and "." is said to be a near-subtraction semigroup if

(1) $(X; -)$ is a subtraction algebra,
(2) $(X; \cdot)$ is a semigroup and
(3) $(x - y)z = xz - yz$ for every $x,\ y,\ z \in X$.

Example 2.8. Let Γ be a subtraction algebra. Then the set $M(\Gamma)$ of all mappings of Γ into Γ is a near-subtraction semigroup under point wise subtraction and composition of mappings. $M(\Gamma)$ is not a subtraction semigroup, because if $f_\delta : \Gamma \to \Gamma$ is given by $f_\delta(\gamma) = \delta$ for all $\gamma \in \Gamma$ (f_δ is a constant map), then for any $g,\ h \in M(\Gamma)$, $f_\delta = f_\delta \circ (g-h) \neq f_\delta \circ g - f_\delta \circ h = 0$.

Example 2.9. Let $\Gamma = \{0,1\}$ in which " $-$ " is defined by

$-$	0	1
0	0	0
1	1	0

Then Γ is a subtraction algebra. Now $M(\Gamma) = \{0,\ a,\ b,\ 1\}$ where 0, a, b, 1 are all functions from Γ to Γ. $M(\Gamma)$ is a near-subtraction semigroup under point wise subtraction and composition and we have

$-$	0	a	b	1
0	0	0	0	0
a	a	0	1	b
b	b	0	0	b
1	1	0	1	0

$\cdot$	0	a	b	1
0	0	0	0	0
a	a	a	a	a
b	a	0	1	b
1	0	a	b	1

Definition 2.10 ([2]). A near-subtraction semigroup X is said to be zero-symmetric if $x0 = 0$ for every $x \in X$.

Example 2.11. Let Γ be a subtraction algebra. Then $M_0(\Gamma) = \{f : \Gamma \to \Gamma | f(0) = 0\}$ is a zero-symmetric near-subtraction semigroup under pointwise subtraction and composition of mappings.

Now we introduce the notion of a ideal which is different from that of [2].

Definition 2.12. Let $(X, -, \cdot)$ be a near-subtraction semigroup. A nonempty subset I of X such that $x - y \in I$ for every $x \in I,\ y \in X$ is called

(1) a left ideal if $xi - x(x' - i) \in I$ for all $x,\ x' \in X$ and $i \in I$ denoted by $I \trianglelefteq_l X$.
(2) a right ideal if $IX \subseteq I$ denoted by $I \trianglelefteq_r X$.
(3) an ideal if I is both a left and right ideal denoted by $I \trianglelefteq X$.

Note:

(1) Suppose if X is a subtraction semigroup and I is a left ideal of X, then for $i \in I$ and $x,\ x' \in X$, we have $xi - x(x'-i) = xi - (xx' - xi) = xi \in I$ by Property 1 of subtraction algebra. Thus we have $XI \subseteq I$.
(2) If X is a zero-symmetric near-subtraction semigroup and I is a left ideal of X, then for $i \in I$ and $x \in X$, we have $xi - x(0-i) = xi - 0 = xi \in I$.

Remark 2.13. Let X be a zero-symmetric near-subtraction semigroup. Let I be a subset of X such that $x - y \in I$ for every $x \in I,\ y \in X$. Then the following are equivalent:

(i) $XI \subseteq I$.
(ii) $xi - x(x' - i) \in I$ for all $x,\ x' \in X$ and $i \in I$.

Proof. (i)⇒(ii) Let $i \in I,\ x,\ x' \in X$. By (i) $xi \in I$. Since I is a subset of X such that $x - y \in I$ for every $x \in I,\ y \in X$, we have $xi - x(x' - i) \in I$ for all $x,\ x' \in X$ and $i \in I$.

(ii)⇒(i) Since X is zero-symmetric and I is a left ideal of X by Note (2), we have $XI \subseteq I$. □

Definition 2.14. Let X be a near-subtraction semigroup. For $S \subseteq X$, $< S >$ denotes the ideal of X generated by S which is the intersection of all ideals of X containing S and hence is the smallest ideal of X containing S.

$< S)$ denotes the left ideal of X generated by S which is the intersection of all left ideals of X containing S and hence is the smallest left ideal of X containing S.

3. Weakly prime left ideals

Unless stated otherwise throughout this paper X stands for a zero-symmetric near-subtraction semigroup.

Definition 3.1. A left ideal P of X is said to be prime if $L_1L_2 \subseteq P$ implies $L_1 \subseteq P$ or $L_2 \subseteq P$ for any left ideals $L_1,\ L_2$ of X.

Definition 3.2. A left ideal P of X is said to be weakly prime if $L_1L_2 \subseteq P$ implies $L_1 = P$ or $L_2 = P$ for any left ideals $L_1,\ L_2$ of X containing P.

A prime left ideal is always weakly prime. But the converse need not be true as the following example shows.

Example 3.3. Consider the following near-subtraction semigroup.

$-$	0	a	b	c
0	0	0	0	0
a	a	0	c	b
b	b	0	0	b
c	c	0	c	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Here the left ideal $\{0,\ c\}$ is weakly prime but not prime since $\{0,\ b\}$ is a left ideal such that $\{0,\ b\}\{0,\ b\} \subseteq \{0,\ c\}$.

Lemma 3.4. *Let X be a near-subtraction semigroup. Then*

$$\text{for } A,\ B \trianglelefteq_l X,\ (A : B) = \{x \in X | xB \subseteq A\} \text{ is an ideal of } X.$$

Proof. Let $i \in (A : B)$ and $x \in X$. Then for $b \in B$, $(i-x)b = ib - xb \in A$, since A is a left ideal and $ib \in A$, $\forall b \in B$. Hence $(i - x)B \subseteq A$. Thus $i - x \in (A : B)$ for all $i \in (A : B)$ and $x \in X$. Now let $i \in (A : B),\ x,\ x' \in X$, and $b \in B$. Then $iB \subseteq A$ and hence $ib = a$ for some $a \in A$. Now $(xi - x(x' - i))b = xib - x(x'b - ib) = xa - x(x' - a) \in A$ since A is a left ideal. Thus $xi - x(x' - i) \in (A : B)$ for every $i \in (A : B),\ x,\ x' \in X$. Now let $i \in (A : B),\ x \in X$ and $b \in B$. Then $ixb = ib'$ for some $b' \in B$ as X is zero symmetric and B is a left ideal. Since $iB \subseteq A,\ ixb \in A$. Hence $ix \in (A : B)$ for every $i \in (A : B),\ x \in X$. Thus $(A : B)X \subseteq (A : B)$. □

Theorem 3.5. *The following are equivalent for a left ideal P of X.*

(i) *P is weakly prime.*

(ii) $\forall\, L_1,\ L_2 \trianglelefteq_l X : (P \cup L_1)(P \cup L_2) \subseteq P$ *implies* $L_1 \subseteq P$ *or* $L_2 \subseteq P$.

(iii) $\forall\, L_1,\ L_2 \trianglelefteq_l X : L_1 \supseteq P$ *and* $L_1L_2 \subseteq P$ *imply* $L_1 = P$ *or* $L_2 \subseteq P$.

(iv) $\forall\, L_1,\ L_2 \trianglelefteq_l X : (P \cup L_1)L_2 \subseteq P$ *implies* $L_1 \subseteq P$ *or* $L_2 \subseteq P$.

(v) $\forall a,\ b \in X : (a \cup P)(< b) \cup P) \subseteq P$ *implies* $a \in P$ *or* $b \in P$, *where* $< b)$ *denotes the left ideal of* X *generated by* b.

Proof. (i)$\Rightarrow$ (ii). Since $(P \cup L_1)$ and $(P \cup L_2)$ are left ideals containing P, $L_1 \subseteq P$ or $L_2 \subseteq P$.

(ii)$\Rightarrow$ (iii). Let $L_1 \supseteq P$ and $L_1L_2 \subseteq P$. Clearly $(P \cup L_1)(P \cup L_2) \subseteq P$. Hence $L_1 = P$ or $L_2 \subseteq P$.

(iii)$\Rightarrow$ (iv). It is obvious.

(iv)$\Rightarrow$ (v). Let $(a \cup P)(< b) \cup P) \subseteq P$. Then $a(< b) \cup P) \subseteq P$ and $P(< b) \cup P) \subseteq P$. Now $< a) \subseteq (P :< b) \cup P)$ and hence $< a)(< b) \cup P) \cup P(< b) \cup P) \subseteq P$. Thus $(< a) \cup P)(< b) \cup P) \subseteq P$ and hence by (iv) $a \in P$ or $b \in P$.

(v)$\Rightarrow$ (i). Let $L_1,\ L_2$ be left ideals of X containing P such that $L_1L_2 \subseteq P$. If $L_1 \neq P$, choose $a \in L_1 \backslash P$. Clearly $a \cup P \subseteq L_1$. Now for any $b \in L_2$, $< b) \cup P \subseteq L_2$. Hence $(a \cup P)(< b) \cup P) \subseteq L_1L_2 \subseteq P$. Hence by (v) $L_2 \subseteq P$. □

Theorem 3.6. *For a left ideal P of X which is not two sided the following are equivalent.*

(i) *P is weakly prime.*

(ii) *$PL \subseteq P$ for a left ideal L implies $L \subseteq P$.*

Proof. (i)$\Rightarrow$ (ii). Let $PL \subseteq P$ for some left ideal L of X. Then $P \subseteq (P : L)$. Hence $< P > \subseteq (P : L)$ so that $< P > L \subseteq P$. By Theorem 3.5 (iii), $< P >= P$ or $L \subseteq P$. Since P is not a two sided ideal, $< P > \neq P$. Hence $L \subseteq P$.

(ii)$\Rightarrow$ (i). Let $L_1,\ L_2$ be left ideals of X such that $(P \cup L_1)L_2 \subseteq P$. Then $PL_2 \cup L_1L_2 \subseteq P$, which implies $PL_2 \subseteq P$. Hence $L_2 \subseteq P$. By Theorem 3.5 (iv), P is weakly prime. □

Definition 3.7. A non empty subset M of X is said to be an m^*-system if for any $m_1,\ m_2 \in M$, there exists $m_1' \in < m_1 >$ and $m_2' \in < m_2)$ such that $m_1'm_2' \in M$.

Clearly every m^*-system is an m-system.

Definition 3.8. A weak m^*-system in X is a pair $(M,\ L)$ where L is a left ideal in X and M is a non empty subset of X such that $L \cap M = \phi$ and $((m \cup L)(< n) \cup L)) \cap M \neq \phi$ for all $m,\ n \in M$.

A weak m^*-system need not be an m^*-system. In Example 3.3, let $M = \{a,\ b\}$ and $L = \{0,\ c\}$, then $(M,\ L)$ is a weak m^*-system but not an m^*-system since for $b,\ a \in M$, $< b >< a) \cap M = \phi$.

Theorem 3.9. *A left ideal P is weakly prime if and only if $(X \backslash P,\ P)$ is a weak m^*-system.*

Proof. Assume that P is a weakly prime ideal. Let $X\backslash P = M$ and $m,\ n \in M$. We claim that $(m \cup P)(< n) \cup P) \nsubseteq P$. Suppose not. Then by Theorem 3.5 (v), we have $m \in P$ or $n \in P$, which is a contradiction to $M \cap P = \phi$. Hence $((m \cup P)(< n) \cup P)) \cap M \neq \phi$. Thus $(M,\ P)$ is a weak m^*-system.

Conversely let $(X\backslash P,\ P)$ be a weak m^*-system. Let $L_1,\ L_2$ be left ideals of X containing P such that $L_1L_2 \subseteq P$. Suppose $L_1 \neq P$. Choose $a \in L_1\backslash P$. We claim that $L_2 = P$. Suppose not. Then choose $b \in L_2\backslash P$. Now $a,\ b \in X\backslash P$. Since $a \in L_1$ and $P \subseteq L_1$, we have $a \cup P \subseteq L_1$. Similarly $< b) \cup P \subseteq L_2$. Hence $(a \cup P)(< b) \cup P) \subseteq L_1L_2 \subseteq P$. Thus $(a \cup P)(< b) \cup P) \cap X\backslash P = \phi$, which is a contradiction to $(X\backslash P,\ P)$ is a weak m^*-system. Hence $L_2 = P$. □

Theorem 3.10. *Let $(M,\ L)$ be a weak m^*-system. If P is a left ideal which is maximal with respect to containing L and not meeting M, then P is weakly prime.*

Proof. Suppose there exist left ideals $L_1,\ L_2$ of X properly containing P such that $L_1L_2 \subseteq P$. Since $L_1,\ L_2$ are left ideals properly containing P, by the maximality of P we have $L_1 \cap M \neq \phi$ and $L_2 \cap M \neq \phi$. Let $m_1 \in L_1 \cap M$ and $m_2 \in L_2 \cap M$. Since $P \subseteq L_1$ and $L \subseteq P$ we have $L \subseteq L_1$. Hence $m_1 \cup L \subseteq L_1$. Again since $P \subseteq L_2$ and $L \subseteq P$ we have $L \subseteq L_2$. Hence $< m_2) \cup L \subseteq L_2$. Now $(m_1 \cup L)(< m_2) \cup L) \cap M = \phi$ which is a contradiction to the fact that $(M,\ L)$ is a weak m^*-system. Hence $L_1L_2 \nsubseteq P$. Thus P is weakly prime. □

Lemma 3.11. *For any left ideal L of X, $B(L) = \{y \in L | yX \subseteq L\}$ is the largest two-sided ideal of X contained in L.*

Proof. Let $x \in B(L)$ and $z \in X$. Then $(x - z)r = xr - zr \in L$ for every $r \in X$. Hence $x - z \in B(L)$. Let $r,\ r' \in X$ and $i \in B(L)$. Then $(ri - r(r' - i))x = rix - r(r'x - ix) \in L$ as $ix \in L$ and L is a left ideal. Hence $B(L)$ is a left ideal. Clearly $B(L)X \subseteq B(L)$. Suppose that there is a two-sided ideal A which is contained in L such that $B(L) \subseteq A$. Now for $y \in A,\ yX \subseteq A \subseteq L$. Hence $B(L) = A$. □

Theorem 3.12. *A left ideal P is prime if and only if there is an m^*-system M such that P is a maximal left ideal not meeting M and $B(P)$ is the maximal two-sided ideal not meeting M.*

Proof. Suppose P is a prime left ideal of X. Let $M = X\backslash P$ and $x_1,\ x_2 \in M$. Since P is prime we have $< x_1) < x_2) \nsubseteq P$. Hence there exist $x_1' \in < x_1) \subseteq < x_1 >$ and $x_2' \in < x_2)$ such that $x_1'x_2' \in M$. Hence M is an m^*-system. Now we claim that P is a maximal left ideal not meeting M. If there is a left ideal L such that $L \cap M = \phi$, then $L \subseteq X\backslash M$ so that $L \subseteq P$. Hence P is the maximal left ideal not meeting M. Suppose A is any two-sided ideal not meeting M, then $A \subseteq X\backslash M$. Hence $A \subseteq P$. But $B(P)$ is the largest two-sided ideal contained in P. Hence $A \subseteq B(P)$. Thus $B(P)$ is the maximal two-sided ideal not meeting M.

Conversely, let M be any m^*-system. Let P be a maximal left ideal not meeting M and let $B(P)$ be a maximal two-sided ideal not meeting M. We show that P is prime. Suppose that there exist left ideals L_1 and L_2 in X such that $L_1L_2 \subseteq P$ with $L_1 \nsubseteq P$, $L_2 \nsubseteq P$. Now $L_1L_2 \subseteq P$ implies $L_1 \subseteq (P : L_2)$. Since $(P : L_2)$ is an ideal, $< L_1 > \subseteq (P : L_2)$ and thus $< L_1 > L_2 \subseteq P$. Hence $(< L_1 > \cup B(P))(P \cup L_2) \subseteq P$. Since $L_1 \nsubseteq P$ and $L_1 \subseteq < L_1 >$, we have $< L_1 > \nsubseteq P$. Now since $B(P)$ is the largest two-sided ideal contained in P and since $< L_1 > \nsubseteq P$, $< L_1 > \cup B(P)$ is a two-sided ideal properly containing $B(P)$. Hence by the maximality of $B(P)$, there is an $m_1 \in (< L_1 > \cup B(P)) \cap M$. Also since $L_2 \nsubseteq P$, $P \cup L_2$ is a left ideal properly containing P. Hence by the maximality of P there is an $m_2 \in (P \cup L_2) \cap M$. Since m_1, $m_2 \in M$ and M is an m^*-system, there exist $m_1' \in < m_1 >$ and $m_2' \in < m_2)$ such that $m_1'm_2' \in M$. But $m_1' \in < m_1 > \subseteq < L_1 > \cup B(P)$ and $m_2' \in < m_2) \subseteq P \cup L_2$. Hence $m_1'm_2' \in (< L_1 > \cup B(P))(P \cup L_2) \subseteq P$. Thus $m_1'm_2' \in P \cap M$, a contradiction. Therefore $L_1L_2 \nsubseteq P$. Hence P is a prime ideal. □

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