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ON CHARACTERIZATIONS OF THE WEIBULL DISTRIBUTION BY THE UPPER RECORD VALUES

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ABSTRACT. In this paper, we establish detailed characterizations of the Weibull distribution by the independence of the upper record values. We prove that $X \in WEI(\alpha)$, if and only if $\frac{X_{U(n)}}{X_{U(n+1)}+X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$. And we show that $X \in WEI(\alpha)$, if and only if $\frac{X_{U(n+1)}-X_{U(n)}}{X_{U(n+1)}+X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.

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1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Suppose $Y_n = \max\{X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence, if $Y_j > Y_{j-1}$ for $j > 1$. By definition, X_1 is an upper as well as a lower record value.

The indices at which the upper record values occur are given by the record times $\{U(n),\ n \geq 1\}$, where $U(n) = \min\{j | j > U(n-1),\ X_j > X_{U(n-1)},\ n \geq 2\}$ with $U(1) = 1$.

A continuous random variable X is said to have the Weibull distribution with parameter $\alpha > 0$ if it has a cdf $F(x)$ of the form

$$F(x) = \begin{cases} 1 - e^{-x^{\alpha}},\ x > 0,\ \alpha > 0, \\ \\ 0, \text{ otherwise.} \end{cases} \tag{1}$$

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A notation that designates that X has the cdf (1) is $X \in WEI(\alpha)$.

Some characterizations by the independence of the upper record values are known. Ahsanullah(1995) characterized that $F(x) = 1 - e^{-\frac{x}{\sigma}}$, $x > 0, \sigma > 0$, if and only if $X_{U(n)} - X_{U(m)}$ and $X_{U(m)}$, $0 < m < n$ are independent. Moreover Ahsanullah(1995, 2004) characterized if $F(x) = 1 - \left(\frac{\alpha}{x}\right)^{\beta}$, $\alpha, \beta > 0$, then $\frac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$, $0 < m < n$ are independent. Also, Lee and Chang(2005) characterized that $F(x) = 1 - e^{-\frac{x}{\sigma}}$, $x > 0, \sigma > 0$, if and only if $\frac{X_{U(n)}}{X_{U(n+1)}}$ and $X_{U(n)}$, $n \geq 1$ are independent and $F(x) = 1 - x^{-\theta}$, $x > 1, \theta > 0$, if and only if $\frac{X_{U(n+1)}}{X_{U(n)}}$ and $X_{U(n)}$, $n \geq 1$ are independent. Above results are characterized by the simple quotient form of the upper record values. By the expansion form of denominator, we can get the Weibull distribution.

In this paper, we will give characterizations of the Weibull distribution by the independence of the upper record values.

2. Main results

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$ for $x > 0$. Then $F(x) = 1 - e^{-x^{\alpha}}$ for $x > 0$ and $\alpha > 0$, if and only if $\frac{X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.*

Proof. If $F(x) = 1 - e^{-x^{\alpha}}$ for $x > 0$ and $\alpha > 0$, then the joint pdf $f_{n,n+1}(x, y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x, y) = \frac{\alpha^2}{\Gamma(n)} x^{\alpha n - 1} y^{\alpha - 1} e^{-y^{\alpha}}$$

for $0 < x < y$, $\alpha > 0$ and $n \geq 1$.

We consider the functions $V = \frac{X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $W = X_{U(n+1)}$. It follows that $x_{U(n)} = \frac{vw}{(1 - v)}$, $x_{U(n+1)} = w$ and $|J| = \frac{w}{(1 - v)^2}$. Thus we can write the joint pdf $f_{V,W}(v, w)$ of V and W as

$$f_{V,W}(v, w) = \frac{\alpha^2}{\Gamma(n)} \frac{v^{\alpha n - 1}}{(1 - v)^{\alpha n + 1}} w^{\alpha(n+1) - 1} e^{-w^{\alpha}} \tag{2}$$

for $0 < v < \frac{1}{2}$, $w > 0$, $\alpha > 0$ and $n \geq 1$.

The marginal pdf $f_V(v)$ of V is given by

$$\begin{aligned} f_V(v) &= \int_0^\infty f_{V,W}(v,w)\, dw \\ &= \frac{\alpha^2}{\Gamma(n)} \frac{v^{\alpha n-1}}{(1-v)^{\alpha n+1}} \int_0^\infty w^{\alpha(n+1)-1} e^{-w^\alpha}\, dw \\ &= \alpha n \frac{v^{\alpha n-1}}{(1-v)^{\alpha n+1}} \end{aligned} \tag{3}$$

for $0 < v < \frac{1}{2}$, $\alpha > 0$ and $n \geq 1$.

Also, the pdf $f_W(w)$ of W is given by

$$\begin{aligned} f_W(w) &= \frac{1}{\Gamma(n+1)} (R(w))^n f(w) \\ &= \frac{\alpha}{\Gamma(n+1)} w^{\alpha(n+1)-1} e^{-w^\alpha} \end{aligned} \tag{4}$$

for $w > 0$, $\alpha > 0$ and $n \geq 1$.

From (2), (3) and (4), we obtain $f_V(v) f_W(w) = f_{V,W}(v,w)$.

Hence $V = \frac{X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $W = X_{U(n+1)}$ are independent for $n \geq 1$.

Now we will prove the sufficient condition. The joint pdf $f_{n,n+1}(x,y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x,y) = \frac{1}{\Gamma(n)} (R(x))^{n-1} r(x) f(y)$$

for $0 < x < y$ and $n \geq 1$, where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}(R(x)) = \frac{f(x)}{1 - F(x)}$.

Let us use the transformation $V = \frac{X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $W = X_{U(n+1)}$. The Jacobian of the transformation is $|J| = \frac{w}{(1-v)^2}$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{1}{\Gamma(n)} \frac{w}{(1-v)^2} \left(R\left(\frac{vw}{(1-v)} \right) \right)^{n-1} r\left(\frac{vw}{(1-v)} \right) f(w) \tag{5}$$

for $0 < v < \frac{1}{2}$, $w > 0$ and $n \geq 1$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{1}{\Gamma(n+1)} (R(w))^n f(w) \tag{6}$$

for $w > 0$ and $n \geq 1$.

From (5) and (6), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{n\,w\left(R\left(\frac{vw}{(1-v)}\right)\right)^{n-1} r\left(\frac{vw}{(1-v)}\right)}{(1-v)^2\,(R(w))^n}$$

for $0 < v < \frac{1}{2}$, $w > 0$ and $n \geq 1$.

That is,

$$f_V(v) = \frac{\partial}{\partial v}\left(\left(\frac{R\left(\frac{vw}{(1-v)}\right)}{R(w)}\right)^n\right).$$

Since V and $W$are independent, the pdf $f_V(v)$ of V is a function of v only. Thus we must have

$$R\left(\frac{vw}{(1-v)}\right) = R\left(\frac{v}{1-v}\right) R(w) \tag{7}$$

for $0 < \frac{v}{1-v} < 1$ and $w > 0$.

By the functional equations [see Aczel (1996)], the only continuous solution of (7) with the boundary condition $R(0) = 0$ is

$$R(x) = x^\alpha$$

for $x > 0$ and $\alpha > 0$. Thus we have

$$F(x) = 1 - e^{-x^\alpha}$$

for $x > 0$ and $\alpha > 0$. This completes the proof. □

Theorem 2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is an absolutely continuous with pdf $f(x)$ and $F(0) = 0$ and $F(x) < 1$ for $x > 0$. Then $F(x) = 1 - e^{-x^\alpha}$ for $x > 0$ and $\alpha > 0$, if and only if $\frac{X_{U(n+1)} - X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $X_{U(n+1)}$ are independent for $n \geq 1$.*

Proof. In the same manner as Theorem 2.1, we consider the functions $V = \frac{X_{U(n+1)} - X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $W = X_{U(n+1)}$. It follows that $x_{U(n)} = \frac{(1-v)w}{(1+v)}$, $x_{U(n+1)} = w$ and $|J| = \frac{2w}{(1+v)^2}$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{2\alpha^2}{\Gamma(n)} \frac{(1-v)^{\alpha n-1}}{(1+v)^{\alpha n+1}} w^{\alpha(n+1)-1} e^{-w^\alpha} \tag{8}$$

for $0 < v < 1$, $w > 0$, $\alpha > 0$ and $n \geq 1$.

The marginal pdf $f_V(v)$ of V is given by

$$\begin{aligned} f_V(v) &= \int_0^\infty f_{V,W}(v,w)\, dw \\ &= \frac{2\alpha^2}{\Gamma(n)} \frac{(1-v)^{\alpha n-1}}{(1+v)^{\alpha n+1}} \int_0^\infty w^{\alpha(n+1)-1} e^{-w^\alpha}\, dw \\ &= 2\alpha n \frac{(1-v)^{\alpha n-1}}{(1+v)^{\alpha n+1}} \end{aligned} \tag{9}$$

for $0 < v < 1$, $\alpha > 0$ and $n \geq 1$.

Also, the pdf $f_W(w)$ of W is given by

$$\begin{aligned} f_W(w) &= \frac{1}{\Gamma(n+1)} (R(w))^n f(w) \\ &= \frac{\alpha}{\Gamma(n+1)}\, w^{\alpha(n+1)-1} e^{-w^\alpha} \end{aligned} \tag{10}$$

for $w > 0$, $\alpha > 0$ and $n \geq 1$. From (8), (9) and (10), we obtain $f_V(v)f_W(w) = f_{V,W}(v,w)$.

Hence $V = \frac{X_{U(n+1)} - X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $W = X_{U(n+1)}$ are independent for $n \geq 1$.

Now we will prove the sufficient condition. The joint pdf $f_{n,n+1}(x,y)$ of $X_{U(n)}$ and $X_{U(n+1)}$ is

$$f_{n,n+1}(x,y) = \frac{1}{\Gamma(n)} (R(x))^{n-1} r(x) f(y)$$

for $0 < x < y$ and $n \geq 1$, where $R(x) = -ln(1 - F(x))$ and $r(x) = \frac{d}{dx}(R(x)) = \frac{f(x)}{1 - F(x)}$.

Let us use the transformation $V = \frac{X_{U(n+1)} - X_{U(n)}}{X_{U(n+1)} + X_{U(n)}}$ and $W = X_{U(n+1)}$. The Jacobian of the transformation is $|J| = \frac{2w}{(1+v)^2}$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{1}{\Gamma(n)} \frac{2w}{(1+v)^2} \left(R\left(\frac{(1-v)w}{(1+v)} \right) \right)^{n-1} r\left(\frac{(1-v)w}{(1+v)} \right) f(w) \tag{11}$$

for $0 < v < 1$, $w > 0$ and $n \geq 1$.

The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{1}{\Gamma(n+1)} (R(w))^n f(w) \tag{12}$$

for $w > 0$ and $n \geq 1$.

From (11) and (12), we obtain the pdf $f_V(v)$ of V

$$f_V(v) = \frac{2nw\left(R\left(\frac{(1-v)w}{(1+v)}\right)\right)^{n-1} r\left(\frac{(1-v)w}{(1+v)}\right)}{(1+v)^2\,(R(w))^n}$$

for $0 < v < 1$, $w > 0$ and $n \geq 1$.

That is,

$$f_V(v) = \frac{\partial}{\partial v}\left(-\left(\frac{R\left(\frac{(1-v)w}{(1+v)}\right)}{R(w)}\right)^n\right).$$

Since V and $W$are independent, the pdf $f_V(v)$ of V is a function of v only. Thus we must have

$$R\left(\frac{(1-v)w}{(1+v)}\right) = R\left(\frac{1-v}{1+v}\right)R(w) \qquad (13)$$

for $0 < \dfrac{1-v}{1+v} < 1$ and $w > 0$.

By the functional equations [see Aczel (1996)], the only continuous solution of (13) with the boundary condition $R(0) = 0$ is

$$R(x) = x^\alpha$$

for $x > 0$ and $\alpha > 0$. Thus we have

$$F(x) = 1 - e^{-x^\alpha}$$

for $x > 0$ and $\alpha > 0$. This completes the proof. □

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