J. Appl. Math. & Informatics Vol. **26***(2008), No. 1 - 2, pp. 417 - 425*
Website: http://www.kcam.biz

ON THE FUNCTIONAL CENTRAL LIMIT THEOREMS FOR MARTINGALE DIFFERENCE RANDOM VECTORS

KWANG HEE HAN

ABSTRACT. For stationary m-dimensional martingale difference sequences we prove the random functional central limit theorems and propose an almost sure consistent estimator for the limiting covariance matrix.

AMS Mathematics Subject Classifications : 60F17, 60G10
Key words and phrases : Functional central limit theorem, Martingale difference, Random indices, Mixing in the sense of Renyi.

1. Introduction

Let $\{\mathbb{Z}_t, t = 0, \pm 1, \pm 2, \cdots,\}$ be a sequence of m-dimensional random vectors. We say that $\{\mathbb{Z}_t\}$ is an m-dimensional martingale difference sequence if

$$E(\mathbb{Z}_t|\mathcal{F}_{t-1}) = \mathbf{0} \ a.s. \tag{1.1}$$

where $\mathcal{F}_t$ is the σ-field generated by $\mathbb{Z}_u, u \leq t$. Let W^m denote Wiener measure on $C^m[0,1]$, the space of all continuous functions f defined on $[0,1]$ into $\mathbb{R}^m$ equipped with norms $\|f\|_\infty = \max\limits_{1\leq i\leq m} \sup\limits_{0\leq t\leq 1} |f_i(t)|$ and let Γ_t denote the conditional covariance matrix of $\mathbb{Z}_t$, $E(\mathbb{Z}_t\mathbb{Z}_t^{'}|\mathcal{F}_{t-1}) = \Gamma_t \ a.s.$, such that

$$\frac{1}{n}\sum_{t=1}^{n} \Gamma_t \to^p \Gamma, \tag{1.2}$$

where the prime denotes transpose and Γ is a positive definite(d.f.) nonrandom matrix. Further, let $\mathbb{S}_n = \sum\limits_{t=1}^{n} \mathbb{Z}_t, (n \geq 0)(\mathbb{S}_o = \mathbb{O})$, and define for $n \geq 1$ the stochastic process ξ_n by

$$\xi_n(u) = n^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}\Big[\mathbb{S}_r + (nu - r)\mathbb{Z}_{r+1}\Big], r \leq nu < r + 1, \tag{1.3}$$

Received June 28, 2007.

where $r = 0, 1, \cdots, (n-1)$.

For $m = 1$, under certain conditions, Babu and Ghosh (1976) and Rootzen (1976) show that the process ξ_n defined by (1.3) converges weakly (or in distribution) to W^1, Wiener measure on $C[0,1]$. We will extend their results on functional central limit theorems to the case of m-dimensional $\mathbb{Z}_t$ under slightly weaker conditions. More general results, however, are obtainable. As it will appear in Theorem 3.1 below, the weak convergence of ξ_n to W^m is mixing in the sense of Renyi(see Definition 2.1). Furthermore, the convergence holds if the process ξ_n is randomly indexed.

There are, of course, many ways of defining the functional central limit theorem, because there are alternative ways of norming or scaling the partial sum process ξ_n(see e.g., Hall and Heyde, 1980, p.98). We will have more to say on this point in Remark following the proof of Corollary 3.2.

In this paper we will also construct a consistent estimator for the limiting covariance matrix Γ, under the assumption that the m-dimensional sequence $\{\mathbb{Z}_t\}$ is weakly stationary. Using this estimator we will be able to normalize or scale the partial sum process $\{\xi_n\}$ based on the observed data and obtain the same limiting distribution W^m(see Corollary 3.2 stated below).

Let us now consider a consistent estimate of Γ. First observe that if $\{\mathbb{Z}_t\}$ is weakly stationary, then

$$\Gamma = \Gamma(0) + \sum_{h=1}^{\infty} \left[\Gamma(h) + \Gamma(h)'\right], \tag{1.4}$$

where $\Gamma(h) = E(\mathbb{Z}_{t+h}\mathbb{Z}_t')$ is the covariance matrix of $\{\mathbb{Z}_t\}$ at lag h and

$$nE\left(\bar{\mathbb{Z}}_n\bar{\mathbb{Z}}_n'\right) = \sum_{|h|<n} \left(1 - \frac{|h|}{n}\right)\Gamma(h) \to \Gamma \ as\ n \to \infty, \tag{1.5}$$

where $\bar{\mathbb{Z}}_n = \frac{1}{n}\sum_{t=1}^{n} \mathbb{Z}_t$. Let $\{h_n\}$ be a sequence of positive integers such that

$$h_n \to \infty \text{ and } \frac{h_n}{n^\delta} \to 0 \ as\ n \to \infty \text{ for all } \delta > 0. \tag{1.6}$$

A typical example of such sequence is $h_n = [(\log n)^q]$, for some $q < \infty$. From (1.4), it is clear that Γ involves an infinite number of unknown parameters. In view of (1.4), given n consecutive observations $\mathbb{Z}_1, \cdots, \mathbb{Z}_n$, we propose to estimate Γ by

$$\hat{\Gamma}_n = \hat{\Gamma}_n(0) + \sum_{h=1}^{h_n} \left[\hat{\Gamma}_n(h) + \hat{\Gamma}_n'(h)\right], \tag{1.7}$$

where

$$\hat{\Gamma}_n(h) = n^{-1}\sum_{t=1}^{n-h} (\mathbb{Z}_{t+h} - \bar{\mathbb{Z}}_n)(\mathbb{Z}_t - \bar{\mathbb{Z}}_n)', \ 0 \le h \le h_n. \tag{1.8}$$

Our motivation, behind the estimator $\hat{\Gamma}_n$, is that covariances at large lags, relative to the number of data points, are likely to be negligible compared with those at smaller lags.

2. Preliminary results

First we recall the definition of mixing in the sense of Renyi(1958).

Definition 2.1. A sequence $\{Y_n, n \geq 1\}$ of random elements on the probability space $(\Omega, \mathcal{F}, P)$ with values in a metric space is *Reńyi-mixing with limiting distribution* F(notation $Y_n \Rightarrow F$(mixing)) if $P(\{Y_n \in A\} \cap B) \to F(A)P(B)$, *as* $n \to \infty$, for all F-continuity sets A and all events B with $P(B) > 0$.

Lemma 2.2. *Let* $\{Y_n, n \geq 1\}$ *and* $\{Y_n^{'}, n \geq 1\}$ *be two sequences of random elements on the probability space* $(\Omega, \mathcal{F}, P)$ *with values in a metric space* $(\mathcal{T}, \rho)$. *If* $Y_n \Rightarrow F$ *(mixing) and* $\rho(Y_n Y_n^{'}) \to^p 0$ *as* $n \to \infty$, *then* $Y_n^{'} \Rightarrow F$ *(mixing).*

This is also Lemma 2.6 of Rootsen(1976) and its proof is immediate.

Lemma 2.3. *Let* $\{Y_n, n \geq 1\}$ *and* $\{Y_n^{'}, n \geq 1\}$ *be two sequences of random elements with values in a metric space. Further, let* $g(x, y)$ *be a continuous function of two variables. If* $Y_n \Rightarrow F$ *(mixing) and* $Y_n^{'} \to^p Y^{'}$ *as* $n \to \infty$, *then* $g(Y, Y_n^{'}) \Rightarrow g(F, Y^{'})$*(mixing).*

Proof. See Theorem $1^{'}$ of Aldous of Eagleson(1978). □

Lemma 2.4. *Let* $\{\mathbb{Z}_t, t \geq 1\}$ *be an m-dimensional martingale difference with* $E\mathbb{Z}_t = 0$ *and let* $\{\xi_n\}$ *be as in (1.3) and assume that (1.2) and*

$$\sup_t E\|\mathbb{Z}_t\|^2 < \infty \tag{2.1}$$

hold. Define

$$\xi_{n,p_n}(u) = \begin{cases} \mathbb{O}, \ u_n < p_n, \\ n^{-\frac{1}{2}}\Gamma^{-\frac{1}{2}}(\mathbb{S}_r - \mathbb{S}_{p_n}), \ p_n \leq r < un < u+1. \end{cases}$$

Suppose that p_n *is a sequence of positive integers such that*

$$p_n \to \infty \text{ and } p_n/n \to 0 \text{ as } n \to \infty. \tag{2.2}$$

Then we have

$$\sup_{0 \leq u \leq 1} \|\xi_n(u) - \xi_{n,p_n}(u)\| = o_p(1). \tag{2.3}$$

Proof. Note that by Doob's maximal inequality

$$n^{-1}\max_{1\le k\le n}\left\|\sum_{t=1}^{k}\mathbb{Z}_t\right\|^2 \le K\sup_t E\|\mathbb{Z}_t\|^2 \tag{2.4}$$

for some constant K. To prove this lemma, let us first observe that

$$\sup_{0\le u\le 1}\|\xi_n(u)-\xi_{n,p_n}(u)\| \le 3n^{-\frac{1}{2}}\|\Gamma\|^{-\frac{1}{2}}\max_{1\le i\le p_n}\|\mathbb{S}_i\| \tag{2.5}$$

$$+n^{-\frac{1}{2}}\|\Gamma\|^{-\frac{1}{2}}\max_{1\le i\le n}\|\mathbb{Z}_i\|.$$

By (2.1), (2.2) and (2.4)

$$n^{-1}\max_{1\le i\le p_n}\|\mathbb{S}_i\|^2 = O_p(p_n/n) = o_p(1) \tag{2.6}$$

and by assumption (2.1)

$$n^{-\frac{1}{2}}\max_{1\le i\le n}\|\mathbb{Z}_i\| = o_p(1). \tag{2.7}$$

Combining (2.6) and (2.7) shows that the right-hand side of (2.5) converges in probability to zero and hence the proof is complete.

The following lemmas are useful in the sequel.

Lemma 2.5. *Let $\{Y_i, i\ge 1\}$ be a sequence of random variables, and let $S_n = Y_1+\cdots+Y_n$, $S_{a,n} = Y_{a+1}+\cdots+Y_{a+n}$, $M_{a,n} = \max_{1\le i\le n}|S_{a,i}|$. Suppose that for some $\nu>2$, all $n\ge 1$ and all $a\ge 0$*

$$E|S_{a,n}|^\nu \le A_\nu n^{\nu/2}, \tag{2.8}$$

$$E|M_{a,n}|^\nu \le B_\nu n^{\nu/2}, \tag{2.9}$$

where A_ν and B_ν are positive constants depending only on ν. Then for any $r\in(0,\nu)$ and for all $n\ge 1$,

$$E\left(\sup_{k\ge n}|S_k/k|\right)^r \le c_{r,\nu}n^{-r/2},$$

where

$$c_{r,\nu} = \left[1+\frac{r}{\nu-r}(A_\nu+B_\nu)2^{-\nu}(1-2^{-\frac{\nu}{2}})^{-1}\right].$$

Proof. By (2.8) and (2.9) and Theorem 5.1 of Serfling(1970) for any $x>0$,

$$P\left(\sup_{k\ge n}|S_k/k|>x\right) \le a_\nu x^{-\nu}n^{-\frac{\nu}{2}}, \tag{2.10}$$

where $a_\nu = (A_\nu + B_\nu)2^{-\nu}(1 - 2^{-\frac{\nu}{2}})^{-1}$. Note that

$$\begin{aligned} E\left(\sup_{k\geq n}|S_k/k|\right)^r &\leq \epsilon + E\left[\sup_{k\geq n}|S_k/k|^r \cdot I(\sup_{k\geq n}|S_k/k|^r > \epsilon)\right] \\ &\leq \epsilon + \int_\epsilon^\infty P\left(\sup_{k\geq n}|S_k/k|^r > x\right)dx \\ &= \epsilon + \int_{\epsilon^{1/r}}^\infty ry^{r-1}P\left(\sup_{k\geq n}|S_k/k| > y\right)dy \\ &\leq \epsilon + ra_\nu n^{-\frac{\nu}{2}}\int_{\epsilon^{1/r}}^\infty y^{r-1-\nu}dy \\ &= \epsilon + ra_\nu n^{-\frac{\nu}{2}}\left(\frac{\epsilon^{(r-\nu)/r}}{\nu - r}\right), \end{aligned}$$

where the last inequality follows from (2.10). Letling $\epsilon = n^{-r/2}$ we obtain the desired inequality. □

Lemma 2.6. *Let $\{Y_t, t \geq 1\}$ be a sequence of martingale differences such that $\sup_t E|Y_t|^r < \infty$ for some $r \geq 2$, and let $\{c_t\}$ be a sequence of real numbers. Then there exists a constant B_r such that*

$$E\left|\sum_{t=1}^n c_tY_t\right|^r \leq B_r\left(\sup_t E|Y_t|^r\right)\left(\sum_{t=1}^n c_t^2\right)^{r/2}.$$

Proof. The proof follows easily by an application of Burkholder's inequality and then Minkowski's inequality. □

Lemma 2.7. *Let $\{Y_t, t \geq 1\}$ be a sequence of martingale differences such that $\sup_t E|Y_t|^\nu < \infty$ for some $\nu > 2$. Then for any $r \in (0, \nu)$ and for all $n \geq 1$,*

$$E\left(\sup_{k\geq n}|S_k/k|\right)^r \leq D_{r,\nu}\cdot n^{-r/2},$$

where

$$D_{r,\nu} = \left\{1 + (\nu - r)^{-1}rc_\nu\left[1 + \left(\frac{\nu}{\nu-1}\right)^\nu\right]2^{-\nu}(1 - 2^{-\nu/2})^{-1}\left(\sup_t E|Y_t|^\nu\right)\right\},$$

$$c_\nu = \left(18\nu\tilde{\nu}^{1/2}\right)^\nu, \quad \nu^{-1} + \tilde{\nu}^{-1} = 1.$$

Proof. By Doob's maximal inequality and Lemma 2.6 we have that, for all $a \geq 0$ and all $n \geq 1$,

$$E|M_{a,n}|^\nu \leq (\nu/(\nu-1))^\nu E|S_{a,n}|^\nu \leq c_\nu(\nu/(\nu-1))^\nu\left(\sup_t E|Y_t|^\nu\right)n^{\nu/2}.$$

Application of Lemma 2.5 with $A_\nu = c_\nu\Big(\sup_t E|Y_t|^\nu\Big)$ and $B_\nu = c_\nu(\nu/(\nu-1))^\nu\Big(\sup_t E|Y_t|^\nu\Big)$ now completes the proof. □

3. Main results

Theorem 3.1. *Let $\{\mathbb{Z}_t, t \geq 1\}$ be an m-dimensional martingale difference sequence and $\{\xi_n\}$ be as in (1.3) and assume that (1.2) and (2.1) hold. Further, assume that*

$$\frac{1}{n}\sum_{t=1}^{n} E\Big(\mathbb{Z}_t'\mathbb{Z}_t I(\mathbb{Z}_t'\mathbb{Z}_t > n\epsilon)|\mathcal{F}_{t-1}\Big) \to^p 0 \text{ as } n \to \infty, \tag{3.1}$$

for every $\epsilon > 0$, where $I(\cdot)$ denotes the indicator function. Let $\{N_n, n \geq 1\}$ be a sequence of positive integer valued random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ such that, as $n \to \infty$, $N_n/n \to^p N$ with $P(0 < N < \infty) = 1$. Then, the following hold:

(i) $\xi_n \Rightarrow W^m(mixing)$,

(ii) $\xi_{N_n} \Rightarrow W^m$.

Proof. It follows from the multivariate version of Theorem 1 of Babu and Ghosh(1976) or Theorem 2.4 of Rootzen (1976) or Thorem 2 of Aldous and Eagleson(1978) that $\xi_n \Rightarrow W^m(mixing)$. So, Theorem 3.1 (i) is proved.

To prove Theorem 3.1 (ii), first note that $\xi_{n,p_n} \Rightarrow W^m(mixing)$, which follows directly from application of Theorem 3.1 (i), Lemmas 2.4 and 2.2. Combining this result and Lemma 2.4, we arrive at (17.19) of Billingsley(1968, p147). From this point, the proof of Theorem 3.1 (ii) follows the same lines as that given in his Theorem 17.2 and hence the details are omitted. □

Corollary 3.2. *Let $\{\tilde{\Gamma}_n\}$ be a sequence of $m \times m$ p.d. matrices such that $\tilde{\Gamma}_n \to \Gamma$, a.s. as $n \to \infty$, let $\tilde{\xi}_n$ be the same as ξ_n, defined in (1.3), with Γ replaced by $\tilde{\Gamma}_n$. Then, under the assumptions in Theorem 3.1, we have*

(i) $\tilde{\xi}_n \Rightarrow W^m(mixing)$,

(ii) $\tilde{\xi}_{N_n} \Rightarrow W^m$.

Proof. (i) follows immediately from a joint application of Theorem 3.1 (i) and Lemma 2.3. Because $\tilde{\Gamma}_n \to \Gamma$ *a.s.* and $N_n \to^p \infty$, *as* $n \to \infty$,it follows that $\tilde{\Gamma}_{N_n} \to^p \Gamma$ (cf, Gut, 1988 Theorem 2.2). Hence part (ii) follows from this fact, Theorem 3.1 (ii) and Lemma 2.3. □

Remark. As mentioned in the introduction, there are various ways of defining the partial sum process. Corollary 3.2 shows that we may normalize the process by any sequence which converges a.s. to Γ. In the univariate case, due to the nonhomogeneous nature of the variances, Brown(1971) used

$$\xi_n(u) = s_n^{-1}\left(S_r + \frac{s_n^2 u - s_r^2}{s_{r+1}^2 - s_r^2} Z_{r+1}\right) \text{ if } s_r^2 \le u s_n^2 < s_{r+1}^2,$$

where $s_n^2 = ES_n^2$. A similar partial sum process can also be defined in our case. For instance, let

$$\xi_n(u) := n^{-\frac{1}{2}} \hat{\Gamma}_n^{-\frac{1}{2}}\left(S_r + \frac{s_n^2 u - s_r^2}{s_{r+1}^2 - s_r^2} Z_{r+1}\right) \; if \; s_r^2 \le u s_n^2 < s_{r+1}^2, \tag{R.1}$$

where $s_n^2 = E\left\|\sum_{t=1}^{n} \Gamma_t\right\|^2$ and replace the constant n in (1.2) and (3.1) by s_n^2. Note that since Γ is p.d. and $\hat{\Gamma}_n \to \Gamma$ a.s., we may assume that $\hat{\Gamma}_n$ is also p.d.

An alternative choice of particular interest for the partial sum process is

$$\xi_n(u) = \Gamma_n^{-\frac{1}{2}} S_r, \; s_r^2 \le u s_n^2 < s_{r+1}^2. \tag{R.2}$$

This process belongs to $D^m[0,1]$, the space of all functions on [0,1] into R^m which are right continuous and have left-hand limits and usually equipped with the Skorohod topology and a compatible metric(see, e.g., Billingsley, 1968; Pollard, 1984).

For simplicity and clarity of the exposition, we have chosen the partial sum process ξ_n introduced in (1.3). However, it is easy to see that the conclusions of Theorem 3.1 also hold for the partial sum processes given in (R.1) and (R.2) above with suitable modifications.

The next theorem gives us information concerning the rate of convergence of $\hat{\Gamma}_n$ to Γ, which, in turn, implies that $\hat{\Gamma}_n \to \Gamma$ *a.s.*. Thus, $\hat{\Gamma}_n$ satisfies the condition of Corollary 3.2.

Theorem 3.3. *Let $\hat{\tau_{ij}}$ and τ_{ij} denote the (i,j)-th elements of $\hat{\Gamma}_n$ and Γ respectively. Assume that the m-dimensional martingale difference sequence $\{\mathbb{Z}_t\}$ is weakly stationary with* $\sup_t E\|\mathbb{Z}_t\|^{2r} < \infty$ *for some $r > 2$. Then*

$$\max_{1\le i,j\le m} \|\hat{\tau_{ij}} - \tau_{ij}\|_r = O\left(n^{-1/2} h_n\right), \tag{3.2}$$

where $\|.\|_r$ denotes the r-th norm in the space $L^r(\Omega, \mathcal{F}, P)$, and

$$\hat{\Gamma}_n \to \Gamma \; a.s. \; as \; n \to \infty. \tag{3.3}$$

Proof. Denote the components of $\mathbb{Z}_t$ and $\bar{\mathbb{Z}}_n$ by Z_{tj} and $\bar{Z}_{nj}$, $j = 1, \cdots, m$, and the (i,j)-th entries of $\Gamma(h)$ and $\hat{\Gamma}(h)$ by $r_{ij}(h)$ and $\hat{r_{ij}}(h)$. Clearly,

$$\sup_t E|Z_{tj}|^{2r} < \infty \tag{3.4}$$

and by Minkowski's inequality and Lemma 2.7

$$\left\| n^{-1} \sum_{t=1}^{n} Z_{tj} \right\|_{2r} = O(n^{-1/2}). \tag{3.5}$$

Let

$$\tilde{r_{ij}}(h) = n^{-1} \sum_{t=1}^{n} (Z_{t+hi} Z_{tj}), \tag{3.6}$$

$$\begin{aligned} A_1 &= A_1(n,i,j,h) = -n^{-1} \sum_{t=n-h+1}^{n} (Z_{t+hi} Z_{tj}) \\ A_2 &= A_2(n,i,j,h) = -n^{-1} \left(\bar{Z_{ni}} \sum_{t=1}^{n-h} Z_{tj} + \bar{Z_{nj}} \sum_{t=1}^{n-h} Z_{t+hi} \right) \\ A_3 &= A_3(n,i,j,h) = -n^{-1}(n-h) \bar{Z_{ni}} Z_{nj}. \end{aligned}$$

Then, in view of (1.8), we have

$$\hat{r_{ij}}(h) = \tilde{r_{ij}}(h) + A_1 + A_2 + A_3. \tag{3.7}$$

Using the Schwarz inequalities (3.4) and (3.5), it follows immediately that

$$\max_{1 \le i,j \le m} \max_{1 \le h \le h_n} \sum_{k=1}^{3} \|A_k\|_r = O(n^{-1/2}). \tag{3.8}$$

Note that from (1.4) and (1.7),

$$\begin{aligned} \hat{\tau_{ij}} - \tau_{ij} &= \hat{r_{ij}}(0) - r_{ij}(0) + \sum_{h=1}^{h_n} \left[(\hat{r_{ij}}(h) - r_{ij}(h)) + (\hat{r_{ji}}(h) - r_{ji}(h)) \right] \\ &\quad - \sum_{h=h_n+1}^{\infty} [r_{ij}(h) + r_{ji}(h)]. \end{aligned}$$

Hence (3.2) holds if

$$\max_{1 \le i,j \le m} \max_{1 \le h \le h_n} \|\sqrt{n}\hat{r_{ij}}(h) - r_{ij}(h)\|_r = O(1). \tag{3.9}$$

and by virtue of (3.7) and (3.8) it suffices to show that

$$\max_{1 \le i,j \le m} \max_{1 \le h \le h_n} \|\sqrt{n}(\tilde{r_{ij}}(h) - r_{ij}(h))\|_r = O(1). \tag{3.10}$$

By (1.4) and (3.6) we have

$$\begin{aligned} \sqrt{n}(\tilde{r_{ij}}(h) - r_{ij}(h)) &= \sqrt{n} \left(n^{-1} \sum_{t=1}^{n} (Z_{t+hi} Z_{tj}) - E(Z_{t+hi} Z_{tj}) \right) \\ &= n^{-\frac{1}{2}} \sum_{t=1}^{n} \{ Z_{t+hi} Z_{tj} - E(Z_{t+hi} Z_{tj}) \}. \end{aligned} \tag{3.11}$$

Further more,

$$\sup_{i,j}\left\|n^{-\frac{1}{2}}\sum_{t=1}^{n}\{Z_{t+hi}Z_{tj}-E(Z_{t+hi}Z_{tj})\}\right\|_r=O(1) \tag{3.12}$$

where (3.12) can be obtained from the Marcinkiewicz-Zygmund inequality(cf. Chow and Teicher(1978)). Now, from (3.11), (3.12) and Lemma 2.6 we see that

$$\max_{1\le i,j\le m}\sup_{h\ge 0}\|\sqrt{n}(\tilde{r_{ij}}(h)-r_{ij}(h))\|_r=O(1),$$

which proves (3.10). Now by using (3.2) and Markov inequality we obtain

$$\max_{1\le i,j\le m}P(|\hat{\tau_{ij}}(h)-\tau_{ij}(h)|>\epsilon)=O(n^{-\frac{r}{2}}h_n^r),$$

which implies (3.3). This completes the proof of Theorem 3.3.

Acknowledgements

This work was supported by Howon University Research Fund in 2007.

References

1. Aldous, D.J., Eagleson, G.K.(1978), *On mixing and stability of limit theorems,* Ann. Probability. **6** 325-331
2. Babu, G.J., Ghosh, M.(1976), *A ranldom functional central limit theorem for martingales,* Acta. Math. Scie. Hungar. **27** 301-306
3. Billingsley, P.(1968), *Convergence of Probability Measures,* Wiley, New York
4. Brown, B.M.(1971), *Martingale central limit theorems,* Ann. Math. Statist **42** 59-65
5. Chow, Y.S., and Teicher, H.(1978), *Probability Theory : Independence, Interchangeability, Martingales,* Springer, New York
6. Hall, P., Heyde, C.C.(1980), *Martingale Limit Theory and its Applications,* Academic Press, New York
7. Pollard, D.(1984), *Convergence of Stochastic Processes,* Wiley, New York
8. Renyi. (1958), *On mixing sequences of random variables,* Acta. Math. Acad. Sci. Hungar. **9** 389-393
9. Rootzen, H.(1976), *Fluctuations of sequences which converge in distribution,* Ann. Probability. **4** 456-463
10. Serfling, R. J.(1970), *Convergence properties of S_n under moment restrictions,* Ann. Math. Statist. **41** 1235-1248

Kwang Hee Han received his BS, MS and PH.D from Wonkwang University. Since 1990 he has been teaching at Howon University. His research interests are probability theory and and related fields.

Department of Computer Science, Howon University, Jeonbuk 573-718, Korea
e-mail: khhan@howon.ac.kr