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SOME CONVEX PROPERTIES IN BANACH SPACES

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ABSTRACT. In this paper, we study property (B_2) and property (D_2) and their implications.

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1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X. By B_X and S_X, we denote the closed unit ball and the unit sphere of X, respectively. For any subset A of X by $\mathrm{co}(A)(\overline{\mathrm{co}}(A))$ we denote the convex hull (closed convex hull) of A

$(X, \|\cdot\|)$ is said to be uniformly convex (UC) if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $x,\ y \in B_X$ with $\|x - y\| \geq \epsilon$,

$$\left\| \frac{1}{2}(x+y) \right\| \leq 1 - \delta.$$

A Banach space is said to have Banach-Saks property (BS) if any bounded sequence in the space admits a subsequence whose arithmetic means converges in norm. S. Kakutani [5] showed that Uniform convexity implies Banach-Saks property. And T. Nishiura and D. Waterman [8] proved that Banach-Saks property implies reflexivity in Banach spaces.

For a sequence (x_n) in X, we let

$$\mathrm{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\}.$$

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For any subset C, we denote by $\alpha(C)$ its Kuratowski measure of non-compactness, i.e., the infimum of such $\epsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less than ϵ.

For any $x \notin B_X$, the drop determined by x is the set

$$D(x, B_X) = \mathrm{co}\Big(\{x\} \cup B_X\Big)$$

Rolewicz [9] has defined property (β). A Banach space X is said to have property (β) if, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\alpha\Big(D(x, B_X)\backslash B_X\Big) < \epsilon$$

whenever $1 < \|x\| < 1 + \delta$.

The following result is found in [6]. A Banach space X has property (β) if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that for each element $x \in B_X$ and each sequence $(x_n) \in B_X$ with $\mathrm{sep}(x_n) \geq \epsilon$, there is $k \in \mathbb{N}$ such that

$$\left\|\frac{x + x_k}{2}\right\| \leq 1 - \delta.$$

2. Property (B_2) and property (D_2)

We start with the following definition.

Definition 1. A Banach space X have *property* (B_2) if there exists a number $\delta > 0$ such that for $x_1, x_2 \in B_X$,

$$\inf\left\{\left\|\frac{1}{2}(x_1 + x_2)\right\|, \left\|\frac{1}{2}(x_1 - x_2)\right\|\right\} \leq 1 - \delta$$

We can easily see that uniformly convexity implies property (B_2). The converse is not true [4]. A. Brunel and L. Sucheston [1] show that property (B_2) implies Banach-Saks property. The converse is not true [3].

We get the following strict implications.

$$\text{(UC)} \Rightarrow \text{property } \ (B_2) \Rightarrow \text{(BS)} \tag{1}$$

Definition 2. A Banach space X is said to have *property* (D_2) if it is reflexive and there exists a number $0 < \alpha < 1$ such that for a weakly null sequence (x_n) in B_X, there exist $n_1 < n_2$ with

$$\left\|\frac{1}{2}(x_{n_1} - x_{n_2})\right\| < \alpha.$$

We can see that Uniformly convexity implies property (D_2).

Proposition 3. *If X is uniformly convex, then it has property (D_2).*

Proof. Suppose that X is uniformly convex. Then for all $0 < \epsilon < 2$, there exists $0 < \delta(\epsilon) < 1$ such that for $x, y \in B_X$ if $\frac{1}{2}\|x + y\| \geq 1 - \delta(\epsilon)$, $\|x - y\| < \epsilon$.

Since if $\epsilon \uparrow 2$, then $\delta(\epsilon) \uparrow 1$, there exists $0 < \epsilon_0 < 2$ such that $\delta(\epsilon_0) > \frac{1}{2}$. Take $\theta = \max\left\{\frac{3}{2} - \delta(\epsilon_0), \frac{\epsilon_0}{2}\right\}$. Then $0 < \theta < 1$.

We show that for a weakly null sequence (x_n) in B_X, there exists $n_1 < n_2$ such that $\frac{1}{2}\|x_{n_1} - x_{n_2}\| \leq \theta$.

Let (x_n) be a weakly null sequence in B_X. If $\|x_1\| \leq 2(1 - \delta(\epsilon_0))$,

$$\begin{aligned}
\frac{1}{2}\|x_1 - x_2\| &\leq \frac{1}{2}\left(\|x_1\| + \|x_2\|\right) \\
&\leq \frac{1}{2}\left(2(1 - \delta(\epsilon_0)) + 1\right) \\
&= \frac{3}{2} - \delta(\epsilon_0) \leq \theta.
\end{aligned}$$

Suppose that $\|x_1\| > 2(1-\delta(\epsilon_0))$. Then there exists $N \in \mathbb{N}$ such that $\|x_1+x_n\| \geq 2(1 - \delta(\epsilon_0))$. (Indeed, if $\|x_1 + x_n\| < 2(1 - \delta(\epsilon_0))$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}
2(1 - \delta(\epsilon_0)) < \|x_1\| &= \sup_{\|x^*\|=1} \lim_n |x^*(x_1 + x_n)| \\
&\leq \sup_{\|x^*\|=1} \limsup_n \|x^*\|\|x_1 + x_n)\| \\
&= \limsup_n \|x_1 + x_n)\| \\
&\leq 2(1 - \delta(\epsilon_0)).
\end{aligned}$$

We get the contradiction.) Since X is uniformly convex,

$$\|x_1 - x_N\| < \epsilon_0 \leq 2\theta.$$

This completes the proof. □

We consider the converse of Proposition 3. The implication of Proposition 3 is strict.

Example 4. There exists a non-uniformly convex Banach space with property (D_2). Consider $(\mathbb{R}^2, \|\cdot\|_\infty)$. Let $x = (1,1)$ and $y = (1,0)$. Then $\|x\|_\infty = \|y\|_\infty = 1$ and $\|x-y\|_\infty = 1$. But $\frac{1}{2}\|x+y\|_\infty = 1$. This means that $(\mathbb{R}^2, \|\cdot\|_\infty)$ is not uniformly convex. We show that $(\mathbb{R}^2, \|\cdot\|_\infty)$ has property (D_2). Since $(\mathbb{R}^2, \|\cdot\|_\infty)$ is a finite dimensional Banach space, it is reflexive. Take $\alpha = \frac{1}{2}$. Let $x_n = (a_n, b_n)$ be a weakly null sequence in $B_{(\mathbb{R}^2, \|\cdot\|_\infty)}$. Since $(\mathbb{R}^2, \|\cdot\|_\infty)$ is a finite dimensional Banach space, $a_n \to 0$ and $b_n \to 0$. It is easy to show that there exist $n_1 < n_2$ such that $\frac{1}{2}\|x_{n_1} - x_{n_2}\| < \alpha$. This means that $(\mathbb{R}^2, \|\cdot\|_\infty)$ has property (D_2).

A Banach space X is said to have weak Banach-Saks property if every weakly null sequence (x_n) in X admits a subsequence whose arithmetic means converges in norm.

The following definition and theorem are found in [3].

Definition 5. A Banach space X is said to have *alternate signs weak Banach-Saks property* if every weakly null sequence (x_n) in X there exists a subsequence (x'_n) of (x_n) and a sequence (ϵ_n) of $\{\pm 1\}$ such that $(1/n)\sum_{i=1}^{n} \epsilon_i x'_i$ converges in norm.

Theorem 6. *A Banach space has weak Banach-Saks property if and only if it has alternate signs weak Banach-Saks property.*

Banach spaces with property (D_2) have alternate Banach-Saks property.

Theorem 7. *If X has property (D_2), it has alternate signs weak Banach-Saks property (hence weak Banach-Saks property).*

Proof. Suppose that X has property (D_2). Then there exists $0 < \alpha < 1$ such that for all weakly null sequence (x_n) in B_X, there exist $n_1 < n_2$ with

$$\left\|\frac{1}{2}(x_{n_1} - x_{n_2})\right\| < \alpha.$$

Suppose (x_n) is a weakly null sequence in X. Without loss of generality, we may assume that $\|x_n\| \le 1$. Then there exist $n_1 < n_2$ such that

$$\frac{1}{2}\|x_{n_1} - x_{n_2}\| < \alpha.$$

Since $(x_n)_{n > n_2}$ is weakly null and $\|x_n\| \le 1$ for $n > n_2$, there exist $(n_2 <) n_3 < n_4$ such that

$$\frac{1}{2}\|x_{n_3} - x_{n_4}\| < \alpha.$$

Continue this process, we obtain a subsequence (x_{n_m}) for which given any $k \in \mathbb{N}$

$$\frac{1}{2}\|x_{n_{2k-1}} - x_{n_{2k}}\| < \alpha.$$

Now, using Kakutani's result [5], we conclude that there exists a subsequence (x'_n) of (x_n) such that

$$\left\|\frac{1}{n}\sum_{i=1}^{n}(-1)^{i+1}x'_n\right\| \to 0 \qquad \text{as} \to \infty.$$

This means that X has alternate weak Banach-Saks property (hence weak Banach-Saks property). □

Since weak Banach-Saks property is equivalent to Banach-Saks property in reflexive Banach spaces, we get the following.

Corollary 8. *If X has property (D_2), then it has Banach-Saks property.*

We consider the converse of Corollary 8. The implication of Corollary 8 is strict.

Example 9. There exists a Banach space with Banach-Saks property which has no property (D_2). The following is found in [2]. For $x = (x_n) \in l_2$, define sequences x^+ and x^- as follows:

$$(x^+)_n = \sup\{x_n, 0\} \text{ and } (x^-)_n = \sup\{-x_n, 0\}$$

Denote the l_2 norm by $\|\cdot\|_2$. Let $l_{2,1}$ denote the set of elements of l_2 with the norm

$$\|x\|_{2,1} = \|x^+\|_2 + \|x^-\|_2.$$

It is easy to show that $l_{2,1}$ is equivalent to l_2 [2]. Since Banach-Saks property is isomorphic invariant and l_2 has Banach-Saks property, $l_{2,1}$ has Banach-Saks property.

Since $l_{2,1}$ is equivalent to l_2, the sequence (e_n) of usual unit vectors is weakly null in $l_{2,1}$. Since

$$\|e_n - e_m\|_{2,1} = 2 \qquad \text{for } n \neq m,$$

$l_{2,1}$ has no property (D_2).

By Proposition 3, Example 4, Corollary 8 and Example 9, we get the following strict implications.

$$\text{(UC)} \Rightarrow \text{property } (D_2) \Rightarrow \text{(BS)} \tag{2}$$

By (1), (2), it is natural to consider the relation of Property (B_2) and (D_2).

3. The relation of property (B_2) and (D_2)

A Banach space X is said to be weakly orthogonal if every weakly null sequence (x_n) in X satisfies

$$\lim_{n\to\infty}\Big|\|x_n+x\|-\|x_n-x\|\Big|=0 \qquad \text{for all } x\in X$$

Proposition 10. *Let X be a weakly orthogonal Banach space. If X has property (B_2), then it has property (D_2).*

Proof. Suppose that X has property (B_2). Then there exists $\delta>0$ such that for $x,y\in B_X$,

$$\inf\left\{\left\|\frac{1}{2}(x+y)\right\|,\left\|\frac{1}{2}(x-y)\right\|\right\}\le 1-\delta.$$

Take $\alpha=1-\frac{\delta}{2}$. Let (x_n) be a weakly null sequence in B_X. We show that there exists $n_1<n_2$ such that

$$\left\|\frac{1}{2}(x_{n_1}-x_{n_2})\right\|\le\alpha.$$

Since X is weakly orthogonal, there exists $N\in\mathbb{N}$ such that if $n\ge N$

$$\Big|\|x_n+x_1\|-\|x_n-x_1\|\Big|<\delta.$$

Since

$$\|x_N-x_1\|<\inf\{\|x_N+x_1\|,\|x_N-x_1\|\}+\delta,$$

$$\begin{aligned}\left\|\frac{1}{2}(x_N-x_1)\right\|&\le\inf\left\{\left\|\frac{1}{2}(x_N+x_1)\right\|,\left\|\frac{1}{2}(x_N-x_1)\right\|\right\}+\frac{1}{2}\delta\\&\le 1-\delta+\frac{1}{2}\delta=\alpha.\end{aligned}$$

This completes the proof. □

We need the following lemma.

Lemma 11. *Let x_n, $x \in X$. If (x_n) is weakly null and $\|x\| > \alpha$, for some $\alpha \in \mathbb{R}^+$ then there exists a subsequence (x_{n_m}) of (x_n) such that $\|x - x_{n_m}\| \geq \alpha$ for all $m \in \mathbb{N}$.*

Proof. The proof is by contradiction. Assume the assertion were false ; $\|x - x_n\| < \alpha$ except finite n. Then

$$\begin{aligned} \alpha < \|x\| &= \sup_{\|x^*\|=1} |x^*(x)| \\ &= \sup_{\|x^*\|=1} \lim_{n\to\infty} |x^*(x - x_n)| \\ &\leq \sup_{\|x^*\|=1} \limsup_n \|x^*\|\|x - x_n\| \\ &= \limsup_n \|x - x_n\| \leq \alpha. \end{aligned}$$

We get the contradiction. □

Property (β) implies property (D_2).

Theorem 12. *If X has property (β), then it has property (D_2).*

Proof. Suppose that X have property (β). Then there exists $\delta > 0$ such that for $x, x_n \in B_X$ with $\text{sep}(x_n) \geq \frac{1}{2}$,

$$\left\|\frac{x + x_m}{2}\right\| \leq 1 - \delta \qquad \text{for some} \quad m \in \mathbb{N}. \tag{3}$$

Let $\theta = \max\left\{\frac{3}{4}, 1 - \delta\right\}$. Then $0 < \theta < 1$. Let (x_n) be a weakly null sequence in B_X. We show that there exist $n_1 < n_2$ such that

$$\left\|\frac{x_{n_1} - x_{n_2}}{2}\right\| \leq \theta$$

If there exists $N \in \mathbb{N}$ such that $\|x_N\| \leq \dfrac{1}{2}$,

$$\left\|\frac{x_N - x_{N+1}}{2}\right\| \leq \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4} \leq \theta.$$

Suppose that $\|x_n\| > \dfrac{1}{2}$ for all $n \in \mathbb{N}$. Let $x_{n_1} = x_2$. Since $\|x_{n_1}\| > \dfrac{1}{2}$, there exists a subsequence $(x_n^{(1)})$ of $(x_n)_{n > n_1}$ such that

$$\|x_{n_1} - x_n^{(1)}\| \geq \frac{1}{2} \qquad \text{for all } n \in \mathbb{N} \text{ by Lemma 11}$$

Let $x_{n_2} = x_1^{(1)}$. Then $\|x_{n_1} - x_{n_2}\| \geq \frac{1}{2}$. Continue this process, we get a subsequence (x_{n_i}) of (x_n) with $\text{sep}(x_{n_i}) \geq \frac{1}{2}$. By (3), there exists $m \in \mathbb{N}$ such that

$$\left\| \frac{x_1 - x_m}{2} \right\| = \left\| \frac{-x_1 + x_m}{2} \right\| \leq 1 - \delta \leq \theta.$$

This completes the proof. □

We need the following lemma.

Lemma 13 [7]. *Let $(Y, \|\cdot\|)$ be a Banach space with basis $(e_i : i \in I)$ (unconditional if I is noncountable) and such that, for every finite subset J of I,*

$$\text{if } 0 \leq |\alpha_j| \leq \beta_j, j \in J, \text{ then } \left\| \sum_{j\in J} \alpha_j e_j \right\| \leq \left\| \sum_{j \in J} \beta_j e_j \right\|.$$

Let $(X_i, i \in I)$ be a family of finite dimensional Banach space. Let

$$Z := \left\{ (x_i)_{i \in I} \in \prod_{i \in I} X_i : \sum_{i \in I} \|x_i\| e_i \in Y \right\}$$

equipped with the norm $\|(x_i)_{i\in I}\| = \left\| \sum_{i \in I} \|x_i\| e_i \right\|$. Then, if $(Y, \|\cdot\|)$ has property (β), $(Z, \|\cdot\|)$ has property (β), too.

There exist a Banach space with property (D_2) which have no property (B_2).

Example 14. Let

$$Z = \left\{ (x_i) \in \prod_{i=1}^{\infty} \mathbb{R}^i : \sum_{i=1}^{\infty} \|x_i\|_\infty e_i \in l_2, \ x_i \in \mathbb{R}^i \right\}$$

equipped with the norm $\|(x_i)\| = \left\| \sum_{i=1}^{\infty} \|x_i\|_\infty e_i \right\|_2$ where (e_n) is usual unit vector basis of l_2. Then Z has property (β) by Lemma 13. By Theorem 12, it has property (D_2).

We prove that Z has no property (B_2). It suffices to show that for all $n \in \mathbb{N}$ there exist $x^{(1)}$, $x^{(2)} \in Z$ such that $\|x^{(k)}\| = 1$, $k = 1, 2$ and

$$\inf \left\{ \left\| x^{(1)} + x^{(2)} \right\|, \left\| x^{(1)} - x^{(2)} \right\| \right\} = 2.$$

Define

$$x^{(1)} = \left(0, \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), (0,0,0,0), \cdots\right)$$

and

$$x^{(2)} = \left(0, \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), (0,0,0,0), \cdots\right).$$

Then

$$\|x^{(1)}\| = \|x^{(2)}\| = 1 \text{ and } \|x^{(1)} + x^{(2)}\| = \|x^{(1)} - x^{(2)}\| = 2.$$

This implies that Z has no property (B_2). □

Finally, we investigate the question whether property (B_2) implies (D_2) or not.

Example 15. There exists a Banach space with property (B_2) which has no property (D_2). $l_{2,1}$ is uniformly non-square [2]. This means that $l_{2,1}$ has property (B_2). $l_{2,1}$ has no property (D_2), by Example 9

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