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A FILTERING FOR DISCRETE MARKET SYSTEM WITH UNKNOWN PARAMETERS

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ABSTRACT. The problem of recursive filtering for discrete market model with unknown parameters is considered. In this paper, we develop an effective filtering algorithm for discrete market systems with unknown parameters and the error covariance equation determining the accuracy of the proposed algorithm is derived.

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1. Introduction

As a model for the evolution of prices(price dynamics) of basic securities on a financial market we consider a system of two discrete stochastic differential equations describing a nonrisky asset B and a risky asset S.

$$\Delta B_n = rB_{n-1}$$

with an interest rate $r > 0$. For the convience, we assume $B_0 = 1$. We assume the stock price $S = (S_n)$ to be evolving according to the law

$$\Delta S_n = \rho_n S_{n-1}$$

where ρ is a "chaotic" sequence. In addition, it is assumed here that the stochastic base $(\Omega, \mathfrak{F}, P)$ is discrete : Ω consists of finitely many elements, $|\Omega| < \infty$, and $\mathfrak{F} = \mathfrak{F}_N$ for some $N \in Z_+$. An investment strategy or portfolio is defined to be a stochastic sequence (β_n, γ_n) whose elements β_n and γ_n are interpreted as the amounts of the respective assets B and S. The value of the portfolio is defined to be the stochastic sequence $X = (X_n)$ with $X_n = \beta_n B_n + \gamma_n S_n$.([3])

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In accordance with this model, the discrete market system is described by the difference equations

$$S_k = (1+\rho_k)S_{k-1},\ \ i = 1,2,\cdots,N$$

$$X_k = \gamma_k S_k + (1+r)^k \beta_k$$

where k is discrete time, $S_k \in \mathbf{R}^n$ is the state vector, $X_k \in \mathbf{R}^n$ is the observation vector. Assume we cannot observe $\{S_0, S_1, S_2, \cdots\}$ and we can only another sequence

$$\breve{S}_n = S_n + \varepsilon_n$$

where ε_n is arbitrary random variables(for example, noise process). Without loss of generality, we assume $\varepsilon_k \sim N(0, R_k)$. We need estimate S on the base $(\breve{S}_1, \breve{S}_2, \cdots)$

In this paper, we develop an effective discrete filtering algorithm for financial market systems with unknown parameters ρ and the error covariance equation determining the accuracy of the proposed algorithm is derived.

2. Main results

Denote the optimal mean square estimate of the state S_k based on observations $\breve{S}$ by $\hat{S}_{k|k} = E(S_k \mid \breve{S}_k)$, and $P_{k|k}$ is its covariance, $P_{k|k} = E[(\hat{S}_{k|k} - S_k)(\hat{S}_{k|k} - S_k)^T \mid \breve{S}_k]$.

We begin with :

Theorem 1. *Consider the market model*

$$\begin{aligned} S_k &= (1+\rho_k)S_{k-1},\ \ i = 1,2,\cdots,N \\ X_k &= \gamma_k S_k + (1+r)^k \beta_k. \end{aligned}$$

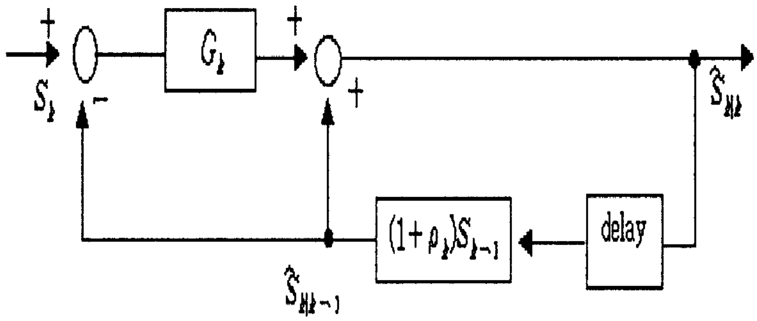
Then we have filtering algorithm

$$\begin{aligned} &\hat{S}_{k|k} = \hat{S}_{k|k-1} + G_k\Big[\breve{S}_k - \hat{S}_{k|k-1}\Big], \\ &\hat{S}_{k|k-1} = (1+\rho_{k-1})\hat{S}_{k-1|k-1}, \\ &P_{k|k-1} = (1+\rho_{k-1})P_{k-1|k-1}(1+\rho_{k-1})^T, \\ &G_k = P_{k|k-1}\Big[P_{k|k-1} + R_k\Big]^{-1}, \\ &P_{k|k} = [I - G_k]P_{k|k-1},\ \ 1 = 1,2,\cdots,N \end{aligned}$$

where I is $n \times n$ identity matrix, superscript T denotes the vector (matrix) transpose.

Proof. Applying the market model to extended Kalman filter, we have result. See Anderson and Moore([1]), Chui and Chen([2]). □

This algorithm may be realized as shown in following figure.



We denote the estimate of the state S_k based on observations $\breve{S}$ by $\hat{S}_{k|k}(\rho_k)$. In Theorem 1, $P_{k|k} = P_{k|k}(\rho_k)$ is the filtering error covariance. Note that the conditional covariance is independent of observations and is, therefore, the unconditional covariance:

$$P_{k|k}(\rho_k) = E\Big[\tilde{S}_{k|k}(\rho_k)\tilde{S}_{k|k}(\rho_k)^T \mid \breve{S}\Big] = E\Big[\tilde{S}_{k|k}(\rho_k)\tilde{S}_{k|k}(\rho_k)^T\Big]$$

$$\tilde{S}_{k|k}(\rho_k) = \hat{S}_{k|k}(\rho_k) - S_k$$

Thus, from Theorem 1, we have N filtering estimates $\hat{S}_{1|1}(\rho_1), \dots, \hat{S}_{N|N}(\rho_N)$. Let f be non-negative payment or reward function. Then the new suboptimal estimate $\hat{S}^*$ of the state S_k is constructed from the estimates $\hat{S}_{1|1}(\rho_1), \dots, \hat{S}_{N|N}(\rho_N)$ by the following formula:

$$\hat{S}^* = \sum_{i=1}^{N} c_{(i)}\ \hat{S}_{i|i}(\rho_i), \quad \sum_{i=1}^{N} c_i = I_n \tag{1}$$

where $c_{(1)}, \dots, c_{(N)}$ are the weighting coefficients determined by the mean square criterion

$$E\left\|\sum_{i=1}^{N} c_{(i)}\ \hat{S}_{i|i}(\rho_i) - f_i\right\|^2 \longrightarrow min_{c_{(i)}} \tag{2}$$

We now meet:

Theorem 2. *Optimal values of the coefficients $c_{(i)}, i = 1, \dots, N$ of the optimization problem (2) are determined by the equations*

$$\sum_{i=1}^{N-1} c_{(i)}\Big[P_{(ij)} - P_{(iN)}\Big] + c_{(N)}\Big[P_{(Nj)} - P_{(NN)}\Big] = 0, \tag{3}$$

$$j = 1, \dots, N-1, \quad c_{(1)} + \dots + c_{(N)} = I_n$$

where $P_{(ij)}$ is cross-covariance of the filtering errors $\tilde{S}_{i|i}(\rho_i)$ and $\tilde{S}_{j|j}(\rho_j)$ at $i \neq j$ and $P_{(ii)}$ is covariance of the error $\tilde{S}_{i|i}(\rho_i)$, i.e.,

$$P_{(ij)} = E\Big[\tilde{S}_{i|i}(\rho_i)\tilde{S}_{j|j}(\rho_j)^T\Big] \quad at\ i \neq j$$

$$P_{(ii)} = P_{i|i}(\rho_i) = E\Big[\tilde{S}_{i|i}(\rho_i)\tilde{S}_{i|i}(\rho_i)^T\Big]$$

Proof. We seek the optimal matrices $c_{(i)}$ minimizing the mean square error, i.e.,

$$\Phi(c_{(1)},\ldots,c_{(N)}) = E\left\|\sum_{i=1}^{N} c_{(i)}\hat{S}_{i|i}(\rho_i) - f_i\right\|^2 = tr(P^*) \quad \longrightarrow \quad min_{c_{(i)}} \tag{4}$$

at the following condition

$$c_{(1)} + \cdots + c_{(N)} = I_n \tag{5}$$

where

$$P^* = \sum_{i,j=1}^{N} c_{(i)}P_{(ij)}[c_{(j)}]^T. \tag{6}$$

Substituting the expression $c_{(N)} = I_n - c_{(1)} - \cdots - c_{(N-1)}$ into (6) we have

$$P^* = \sum_{i,j=1}^{N-1} c_{(i)}P_{(ij)}[c_{(j)}]^T + \sum_{i=1}^{N-1}\Big\{c_{(i)}P_{(iN)} - P_{(Ni)}[c_{(i)}]^T\Big\}$$
$$- \sum_{i,j=1}^{N-1}\Big\{c_{(i)}P_{(iN)}[c_{(j)}]^T + c_{(j)}P_{(Ni)}[c_{(i)}]^T\Big\} + P_{(NN)}$$

$$- \left[\sum_{i=1}^{N-1} c_{(i)}\right] P_{(NN)} - P_{(NN)}\left[\sum_{j=1}^{N-1} c_{(j)}\right]^T + \sum_{i,j=1}^{N-1} c_{(i)}P_{(NN)}[c_{(j)}]^T. \tag{7}$$

So the optimization problem (4)-(6) has the following form:

$$\Phi = \Phi(c_{(1)},\ldots,c_{(N-1)}) = tr(P^*) \quad \longrightarrow \quad min_{c_{(i)}} \tag{8}$$

where the covariance P^* is determined by (7). Next we shall use the following formulae:

$$P_{(ij)} = \Big[P_{(ji)}\Big]^T,\ P_{(ii)} = \Big[P_{(ii)}\Big]^T \quad for\ any\ i,j = 1,\ldots,N \tag{9}$$

$$\Big[\partial/\partial c_{(i)}\Big]\ tr\Big\{c_{(i)}P[c_{(j)}]^T\Big\} = c_{(j)}(P^T + P) \tag{10}$$

$$\Big[\partial/\partial c_{(i)}\Big]\ tr\Big\{c_{(i)}P\Big\} = P^T,\ \Big[\partial/\partial c_{(i)}\Big]\ tr\Big\{P[c_{(i)}]^T\Big\} = P \tag{11}$$

Let's differentiate every summand of the function $\Phi = tr(P^*)$ in (8), (7) with respect to $c_{(i)}$ $(i = 1,\ldots,N-1)$ using (9)-(11) and then set the result to zero.

After simple transformations we have the Equations (3). This completes the proof of the Theorem 2. □

We conclude with :

Corollary 3. *Let the estimate $\hat{S}^*$ satisfy the Eqs. (1). Then the actual filtering error covariance*

$$P^* = E\left(\Delta S_{k|k} \Delta S_{k|k}^T\right), \quad \Delta S_{k|k} = \hat{S}^* - S_k \tag{12}$$

determines by the formula

$$P^* = \sum_{i,j=1}^{N} c_{(i)} P_{(ij)} [c_{(j)}]^T \tag{13}$$

where the matrices $P_{(ij)}$ are determined by the Equations (5).

Proof. Substituting (1) into (12) and taking into account $S_k = \left[\sum_{k=1}^{N} c_{(k)}\right] S_k$, we have (13). □

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