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NEW OSTROWSKI TYPE INEQUALITIES INVOLVING TWO FUNCTIONS

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ABSTRACT. In this paper, new inequalities of Ostrowski type involving two functions and their derivatives for mapping whose derivations belong to $L^p[a,b]$, $p > 1$ are established .

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1. Introduction

In 1938, A. Ostrowski [4] proved the following interesting integral inequality:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f' : (a,b) \to \mathbb{R}$ is bounded in (a,b), that is, $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(x)| < \infty$. Then for any $x \in [a,b]$, we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty. \qquad (1)$$

The inequality is sharp in the sense that the constant $1/4$ cannot be replaced by a smaller one.

In the past few years inequality (1) has received considerable attention from many researchers and a number of papers have appeared in the literature, which deal with alternative proofs, various generalizations, numerous variants and applications. In [2], Dragomir gave a generalization of Ostrowski integral inequality for mappings whose derivatives belong to $L^p[a,b]$, $p > 1$.

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Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and differentiable on (a,b) and its derivative $f' : (a,b) \to \mathbb{R}$ is bounded in (a,b), that is, $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(x)| < \infty$. Then for any $x \in [a,b]$, we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq L(x)\|f'\|_p. \tag{2}$$

where

$$L(x) = \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}. \tag{3}$$

Recently, B. G. Pachpatte [5] established a new Ostrowski type inequality involving two functions and their derivatives.

Theorem 3. *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b), whose derivatives $f', g' : (a,b) \to \mathbb{R}$ are bounded in (a,b), i.e., $\|f'\|_\infty < \infty$, $\|g'\|_\infty < \infty$. Then*

$$\begin{aligned}&\left| f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(t)dt + f(x)\int_a^b g(t)dt\right]\right| \\ &\leq \frac{1}{2}\left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\{g(x)\|f'\|_\infty + f(x)\|g'\|_\infty\}.\end{aligned}$$

for all $x \in [a,b]$.

Motivated by the result of B. G. Pachpatte and Dragomir, in this paper we establish new inequalities of Ostrowski type involving two functions and their derivatives for mapping whose derivations belong to $L^p[a,b]$, $p > 1$. The analysis used in the proofs is based on an integral identity and provide new estimates on these types of inequalities. Our result in special case yield Theorem 2, the inequalities given by Theorem 2 in [1] and Theorem 3.1 in [6].

2. Main results

Our main result is given in the following theorem.

Theorem 4. *Let $f, g : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and differentiable on (a,b), $f', g' \in L^p[a,b]$ for some $p > 1$ and $1/p + 1/q = 1$. Then*

$$\begin{aligned}&\left| f(x)\left[\frac{1}{b-a}\int_a^b g(t)dt\right] + g(x)\left[\frac{1}{b-a}\int_a^b f(t)dt\right]\right. \\ &\left. -\frac{2}{1-h}\left[\frac{1}{b-a}\int_a^b f(t)dt\right]\left[\frac{1}{b-a}\int_a^b g(t)dt\right]\right.\end{aligned}$$

$$+\frac{h}{1-h}\frac{1}{b-a}\left[\frac{f(a)+f(b)}{2}\int_a^b g(t)dt+\frac{g(a)+g(b)}{2}\int_a^b f(t)dt\right]\Bigg| \\ \leq L_h(x)\left[\|f'\|_p\left(\frac{1}{b-a}\int_a^b |g(t)|dt\right)+\|g'\|_p\left(\frac{1}{b-a}\int_a^b |f(t)|dt\right)\right]. \tag{4}$$

where $h \in [0,1]$, $a+h(b-a)/2 \leq x \leq b-h(b-a)/2$ *and*

$$L_h(x)=\frac{1}{(1-h)(q+1)^{1/q}}\left[2\left(\frac{h}{2}\right)^{q+1} + \left(\frac{x-a}{b-a}-\frac{h}{2}\right)^{q+1}+\left(\frac{b-x}{b-a}-\frac{h}{2}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}. \tag{5}$$

And the equality holds if and only if

(1) $\operatorname{sgn} f'(t)=\operatorname{sgn} r(x,t)=\operatorname{sgn} g'(t)$, $t\in[a,b]$.

(2) $|f'(t)|^p/|r(x,t)|^q=\text{const}$, $|g'(t)|^p/|r(x,t)|^q=\text{const}$ *with* $r(x,t)$ *defined by* (6).

Proof. Let $r:[a,b]^2\to\mathbb{R}$ be given by

$$r(x,t)=\begin{cases} t-[a+h(b-a)/2], & t\in[a,x],\\ t-[b-h(b-a)/2], & t\in[x,b].\end{cases} \tag{6}$$

Integrating by parts, we have

$$f(x)-\frac{1}{1-h}\frac{1}{b-a}\int_a^b f(t)dt+\frac{h}{1-h}\frac{f(a)+f(b)}{2} \\ =\frac{1}{1-h}\frac{1}{b-a}\int_a^b r(x,t)f'(t)dt, \tag{7}$$

$$g(x)-\frac{1}{1-h}\frac{1}{b-a}\int_a^b g(t)dt+\frac{h}{1-h}\frac{g(a)+g(b)}{2} \\ =\frac{1}{1-h}\frac{1}{b-a}\int_a^b r(x,t)g'(t)dt. \tag{8}$$

Multiplying both sides of (7) and (8) by $\frac{1}{b-a}\int_a^b g(t)dt$ and $\frac{1}{b-a}\int_a^b f(t)dt$ respectively and adding we get

$$f(x)\left[\frac{1}{b-a}\int_a^b g(t)dt\right]+g(x)\left[\frac{1}{b-a}\int_a^b f(t)dt\right] \\ -\frac{2}{1-h}\left[\frac{1}{b-a}\int_a^b f(t)dt\right]\left[\frac{1}{b-a}\int_a^b g(t)dt\right]$$

$$
\begin{aligned}
&+\frac{h}{1-h}\frac{1}{b-a}\left[\frac{f(a)+f(b)}{2}\int_a^b g(t)dt+\frac{g(a)+g(b)}{2}\int_a^b f(t)dt\right] \\
&=\frac{1}{1-h}\left\{\left[\frac{1}{b-a}\int_a^b r(x,t)f'(t)dt\right]\left[\frac{1}{b-a}\int_a^b g(t)dt\right]\right. \\
&\quad\left.+\left[\frac{1}{b-a}\int_a^b r(x,t)g'(t)dt\right]\left[\frac{1}{b-a}\int_a^b f(t)dt\right]\right\}.
\end{aligned}
\tag{9}
$$

From (9) and using the properties of modulus and Hölder integral inequality, the absolute value of the right-hand side becomes

$$
\begin{aligned}
|\text{R.H.S.}\,(9)| \le \frac{1}{1-h}\frac{1}{b-a}\|r\|_q &\left[\|f'\|_p\left(\frac{1}{b-a}\int_a^b |g(t)|dt\right)\right. \\
&\left.+\|g'\|_p\left(\frac{1}{b-a}\int_a^b |f(t)|dt\right)\right],
\end{aligned}
\tag{10}
$$

where $\|r\|_q=\left(\int_a^b |r(x,t)|^q dt\right)^{1/q}$, $1/p+1/q=1$, and the equality holds if and only if (1) and (2) in Theorem 4 hold.

Notice that for $c \le d \le e$,

$$
\int_c^e |t-d|^q dt=\int_c^d |t-d|^q dt+\int_d^e |t-d|^q dt=\frac{1}{q+1}\left[(d-c)^{q+1}+(e-d)^{q+1}\right],
\tag{11}
$$

We have

$$
\begin{aligned}
\|r\|_q &= \left[\int_a^x |r(x,t)|^q dt+\int_x^b |r(x,t)|^q dt\right]^{1/q} \\
&=\frac{1}{(q+1)^{1/q}}\left[\left(\frac{h(b-a)}{2}\right)^{q+1}+\left(x-a-\frac{h(b-a)}{2}\right)^{q+1}\right. \\
&\quad\left.+\left(b-x-\frac{h(b-a)}{2}\right)^{q+1}+\left(\frac{h(b-a)}{2}\right)^{q+1}\right]^{1/q} \\
&=\frac{1}{(q+1)^{1/q}}\left[2\left(\frac{h}{2}\right)^{q+1}\right. \\
&\quad\left.+\left(\frac{x-a}{b-a}-\frac{h}{2}\right)^{q+1}+\left(\frac{b-x}{b-a}-\frac{h}{2}\right)^{q+1}\right]^{1/q}(b-a)^{1+1/q}.
\end{aligned}
\tag{12}
$$

From (10) and (12), we obtain

$$|\text{R.H.S. (9)}| \le L_h(x) \left[\|f'\|_p \left(\frac{1}{b-a} \int_a^b |g(t)|dt \right) + \|g'\|_p \left(\frac{1}{b-a} \int_a^b |f(t)|dt \right) \right]. \tag{13}$$

□

Remark 1. We note that in the special cases, if we take $g(x) = 1$ and $h = 0$ in Theorem 4, we get Theorem 2. If we choose $g(x) = 1$ or $f(x) = g(x)$ in Theorem 4, we obtain Theorem 3.1 in [6].

Corollary 5. *Under the assumptions of Theorem 4 and with $h = 0$, we have the inequality*

$$\begin{aligned} &\left| f(x) \left[\frac{1}{b-a} \int_a^b g(t)dt \right] + g(x) \left[\frac{1}{b-a} \int_a^b f(t)dt \right] \right. \\ &\qquad \left. -2 \left[\frac{1}{b-a} \int_a^b f(t)dt \right] \left[\frac{1}{b-a} \int_a^b g(t)dt \right] \right| \\ &\le L_0(x) \left[\|f'\|_p \left(\frac{1}{b-a} \int_a^b |g(t)|dt \right) + \|g'\|_p \left(\frac{1}{b-a} \int_a^b |f(t)|dt \right) \right]. \end{aligned}$$

where $a \le x \le b$ and

$$L_0(x) = \frac{1}{(q+1)^{1/q}} \left[\left(\frac{x-a}{b-a} \right)^{q+1} + \left(\frac{b-x}{b-a} \right)^{q+1} \right]^{1/q} (b-a)^{1/q}.$$

Let $p \to \infty, q \to 1$ in Theorem 4, we have

Corollary 6. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b), and its derivative $f', g' : (a, b) \to \mathbb{R}$ are bounded in (a, b), that is $\|f'\|_\infty < +\infty$ and $\|g'\|_\infty < +\infty$. Then*

$$\begin{aligned} &\left| f(x) \left[\frac{1}{b-a} \int_a^b g(t)dt \right] + g(x) \left[\frac{1}{b-a} \int_a^b f(t)dt \right] \right. \\ &\qquad -\frac{2}{1-h} \left[\frac{1}{b-a} \int_a^b f(t)dt \right] \left[\frac{1}{b-a} \int_a^b g(t)dt \right] \\ &\qquad \left. +\frac{h}{1-h} \frac{1}{b-a} \left[\frac{f(a)+f(b)}{2} \int_a^b g(t)dt + \frac{g(a)+g(b)}{2} \int_a^b f(t)dt \right] \right| \\ &\le L_h^\infty(x) \left[\|f'\|_\infty \left(\frac{1}{b-a} \int_a^b |g(t)|dt \right) + \|g'\|_\infty \left(\frac{1}{b-a} \int_a^b |f(t)|dt \right) \right]. \end{aligned}$$

where $h \in [0, 1]$, $a + h(b-a)/2 \le x \le b - h(b-a)/2$ and

$$\begin{aligned} L_h^\infty(x) &= \frac{1}{1-h}\left[\left(\frac{h}{2}\right)^2+\frac{1}{2}\left(\frac{x-a}{b-a}-\frac{h}{2}\right)^2+\frac{1}{2}\left(\frac{b-x}{b-a}-\frac{h}{2}\right)^2\right](b-a) \\ &= \frac{1}{(1-h)(b-a)}\left[\frac{1}{4}(b-a)^2[h^2+(h-1)^2]+\left(x-\frac{a+b}{2}\right)^2\right]. \end{aligned}$$

Remark 2. In the special cases, if we take $g(x)=1$ and in Corollary 6, we get Theorem 2 in [1].

Take $h=0$ in Corollary 6, we have

Corollary 7. *Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b), and its derivative $f',g':(a,b)\to\mathbb{R}$ are bounded in (a,b), that is $\|f'\|_\infty<+\infty$ and $\|g'\|_\infty<+\infty$. Then*

$$\begin{aligned} &\left|f(x)\left[\frac{1}{b-a}\int_a^b g(t)dt\right]+g(x)\left[\frac{1}{b-a}\int_a^b f(t)dt\right]\right. \\ &\left.\quad -2\left[\frac{1}{b-a}\int_a^b f(t)dt\right]\left[\frac{1}{b-a}\int_a^b g(t)dt\right]\right| \\ &\le L_0^\infty(x)\left[\|f'\|_\infty\left(\frac{1}{b-a}\int_a^b |g(t)|dt\right)+\|g'\|_\infty\left(\frac{1}{b-a}\int_a^b |f(t)|dt\right)\right]. \end{aligned}$$

where $a\le x\le b$ and

$$L_0^\infty(x)=\frac{1}{b-a}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]=\frac{1}{2(b-a)}[(x-a)^2+(b-x)^2].$$

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References

1. S. S. Dragomir, P. Cerone and J. Roumeliotis, *A new generalization of Ostrowski's intergral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means*, Appl. Math. Lett. **13** (2000), 19-25.
2. S. S. Dragomia and S. Wang, *Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules*, Appl. Math. Lett. **11** (1998), 105-109.
3. W. J. Liu, C. C. Li and J. W. Dong, *Note on Qi's Inequality and Bougoffa's Inequality*, J. Inequal. Pure Appl. Math. **7**(4) (2006), Art.129.

4. D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordricht-Boston-London, 1994.
5. B. G. Pachpatte, *A note on Ostrowski like Inequalities*, J. Inequal. Pure and Appl. Math. **6** (4) (2005), Art.114.
6. X. J. Yang, *Refinement of Hölder inequality and application to Ostrowski inequality*, Appl. Math. Lett. **138** (2003), 455-461.

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