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# AN IMPROVED COMBINATORIAL OPTIMIZATION ALGORITHM FOR THE THREE-DIMENSIONAL LAYOUT PROBLEM WITH BEHAVIORAL CONSTRAINTS

TIE JUN, JINZHI WANG AND ENMIN FENG

ABSTRACT. This paper is motivated by the problem of fitting a group of cuboids into a simplified rotating vessel of the artificial satellite. Here we introduce a combinatorial optimization model which reduces the three-dimensional layout problem with behavioral constraints to a finite enumeration scheme. Moreover, a global combinatorial optimization algorithm is described in detail, which is an improved graph-theoretic heuristic.

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## 1. Introduction

In recent years, the three-dimensional layout problems evoke much attention and a great of work devoted to which have been done because of their great significance both in theory and economy [1,2,3,4,5,14]. But most of these papers referred to above have developed heuristic solution procedures, and none of them gave an analytical model for the three-dimensional container loading problem.

Against the background of the rotating vessel of the artificial satellite, Enmin Feng etc. [6,7] reported different direct analytical mathematical models for two-and three-dimensional layout optimization problems. They successfully transformed the problems into the convex programming with D.C. constraints (i.e. the remainder of two convex functions) and developed some heuristics that attempt to optimize the space utilization.

In fact, the three-dimensional layout problems have wide-ranging applications in many high technological fields, such as the layout design problems of the vessel of satellite, the vessel of spacecraft, rotating building and high-speed train etc. It is well known that the three-dimensional layout problems are NP-hard, because they include the one-dimensional layout problems as their special cases. This means that there is no effective solution theory and method up to now.

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Moreover, as many authors themselves observed, their solution procedures are not efficient for large scale container loading problems, which require a large amount of computation in determining non-overlapping objects. And it is very difficult to construct descent direction.

This paper studies the three-dimensional layout optimization problem of the simplified rotating vessel of satellite to find an optimal strategy for allocating a finite number of apparatus inside the rotating vessel of satellite. Clearly the industrial layout problems are complex and have many constraints. For this problem one is required to consider the following three constraints: (a) capacity constraints: objects do not exceed the await layout space; (b) non-overlap constraints: objects are not overlapped by each other; (c) static non-equilibrium constraints: the centroid of all await layout objects should be as close to the axis of revolution of the vessel as possible.

Furthermore, we introduce graph theory and group theory to construct a combinatorial global optimization algorithm which can reduce the three-dimensional layout problem with constraints to a finite enumeration scheme and can overcome the difficulty rooted in the on-off nature of three-dimensional layout problems, proposed by Enmin Feng , X.L.Wang and H.F.Teng [8] on two-dimensional layout problem.

## 2. A Layout optimization mathematical model.

Suppose that there are n await layout objects (e.g. apparatus) which are generally simplified as a cuboids group, while the given objects have different shapes, volume, weight and qualities. At the same time, we simplify the rotating vessel of the artificial satellite as follows: it is smoothly connected by the up and down parts, where its upper part is a spherical crowns and its lower part is a cylinder. The await layout space is formed of the up and down parts of the vessel with rigidity housing of even thickness. Now we set up a three-dimensional Descartes coordinate system. Assume that the axis of revolution of the vessel coincides with the centre axis of the cylinder which is in the lower part of the vessel. We take the axis that the vessel rotates around as the upward z-axis . And the circular base board $\mathfrak{B}$ of the cylinder is taken as the xy-coordinate plane, which is used to fix objects (e.g. apparatus). Therefore we set up a three-dimensional Descartes coordinate system and make its origin coincide with the center of the base board of the cylinder. Obviously, the z-axis is vertical to the base board $\mathfrak{B}$ and they satisfy the right-handed law.

Thus the layout space can be described as the set

$$\chi = \left\{(x, y, z)|x^2 + y^2 \le a^2, 0 \le z \le \sqrt{A^2 - x^2 - y^2}\right\}(A > a > 0)$$

and the spherical crowns can be described as the set

$$S = \left\{(x, y, z)|x^2 + y^2 \le a^2, z = \sqrt{A^2 - x^2 - y^2}\right\}(A > a > 0),$$

where the circular base board of the cylinder $\mathfrak{B} = \{(\mathfrak{x}, \mathfrak{y}, \mathfrak{o})|\mathfrak{x}^2 + \mathfrak{y}^2 < \mathfrak{a}^2\}$.

Clearly the layout space $\chi$ is a convex set, its interior int $\chi$ is also a convex set. Let $n$ different cuboids $F_i, i = 1, 2, \cdots, n$ are to be allocate on the base board

$\mathfrak{B}$ and their masses $m_i \in R^3_+$, where $m_i$ is the mass of the cuboid $F_i$. Denote the center of form of the it cuboid $F_i$ by $c_i = (x_i, y_i, z_i) \in R^3$ , and suppose that the centroid of $F_i$ coincides with $c_i$. Let $p_i = (\ell_i, w_i, h_i) \in R^3_+$, $\ell_i \geq w_i > 0$, where $2\ell_i$, $2w_i$, $h_i$ signify the length, the width and the height of $F_i$ , respectively. Moreover, denote $F^i_{prj}$ be the projective rectangle of $F_i$ on xoy-coordinate plane. The unite vectors $t_i = (\cos\theta_i, \sin\theta_i, 0)$ and $t_i^\perp = (-\sin\theta_i, \cos\theta_i, 0)$ parallel respectively the long edge and the short edge of $F^i_{prj}$, where the angle $\theta_i = (t_i \hat{,} x) \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ is included between the long edge of $F^i_{prj}$ and x-axis. Then such a cuboid $F_i$ can be uniquely determined by vectors $c_i, p_i, t_i \in R^3$ as

$$\begin{aligned} F_i &= F(c_i, p_i, t_i) \\ &= \left\{ c_i + u_1 t_i + u_2 t_i^\perp + u_3 e_3 \,\middle|\, \begin{array}{ll} u_1 \in [-\ell_i, \ell_i], & u_2 \in [-w_i, w_i], \\ u_3 \in [0, h_i], & e_3 = (0,0,1) \end{array} \right\}. \end{aligned}$$

To reduce the number of optimization variables, now let $W = (-\frac{\pi}{2}, \frac{\pi}{2})$ and let $T = \{t = (x, y, o) | x = \cos\theta, y = \sin\theta, \theta \in W\}$. Then there exists a bijection

$$g : W \to T$$
$$\theta \mapsto t = (\cos\theta, \sin\theta, 0)$$

For a special three-dimensional layout problem, it is clear that $n$, $a$, $A$, $p_i$, $m_i$ ,$i = 1, 2, \cdots, n$are constants. Hence the cuboids $F_i$ can also be determined by $\theta_i \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, $(x_i, y_i, 0) \in \mathfrak{B}$, then we have

**Definition 1.** Let $y_i = (x_i, y_i, \theta_i) \in R^3$ and $Y = (y_1, y_2, \cdots, y_n) \in R^{3n}$, where$(x_i, y_i, 0) \in \mathfrak{B})$, $\theta_i \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, $i \in I_n$. Then $Y = (y_1, y_2, \cdots, y_n) \in R^{3n}$ is called a *layout scheme.*

**Property 1.** *Let $D$ be the set of all of the layout schemes $Y = (y_1, y_2, \cdots, y_n) \in R^{3n}$. Then $D$ is a convex set in $R^{3n}$ .*

*Proof.* Let $\bar{W} = \{(0, 0, \theta_i) | \theta_i \in W\}$, then following from Definition1 we have that $y_i = (x_i, y_i, \theta_i) \in \mathfrak{B} \oplus \bar{W}$ . It is clear that $\mathfrak{B}$ and $\bar{W}$ are convex sets. By Theorem 3.5 in [9] the direct sum $\mathfrak{B} \oplus \bar{W}$ is a convex set in $R^3$ . Moreover, the set of all of the layout schemes $D$ is a convex set. □

**Definition 2.** For a layout scheme $Y \in D$, if no overlap exists among all the cuboids $F_i$ determined by $Y \in D$ , i.e., $\text{int}F_i \bigcap \text{int}F_j =$, $i \neq j, i, j \in I_n$, where int $F_i$ and $F_j$ are the internals of $F_i$ and int $F_j$, respectively, then the layout scheme $Y$ is called a *non-overlap layout scheme.*

Let $D_n$ be the set of all the non-overlap layout schemes. In this paper, the three-dimensional layout problem consists in finding a placing in $D_n$, denoted by $P$, which have the minimal static non-equilirium quantity of the layout problem $P$ about artificial satellite module so that the centroid of all await layout objects should be as close to the axis of revolution of the vessel as possible, i.e.,

$$P := \min H(Y)$$

such that $Y \in D_n$ and

$$\bigcup_{i=1}^{n} F_i \subset \chi$$

where the static non-equilirium quantity

$$H(Y) = \left\| \sum_{i=1}^{n} m_i c_i \right\| \quad \text{for } Y \in D_n.$$

## 3. The graphs of the non-overlap layout schemes and their combinatorial properties

**Definition 3.** Let $F_i$ and $F_j$ be the cuboids determined by a non-overlap layout scheme $Y \in D_n$ . If cuboids $F_i$ and $F_j$ satisfy one of the following conditions:

($i$) There is at least one congruent point on the lateral faces of $F_i$ and $F_j$;

($ii$) under the condition of non-overlap, translate $F_i$ or $F_j$ along the segment between the projections of $c_i$ and $c_j$ on xoy-coordinate plane, during the translation some rotations are permitted such that ($i$) can be satisfied.

Then $F_i$ and $F_j$ are said to be *adjacent cuboids* in the non-overlap layout scheme $Y \in D_n$, denoted by $(F_i, F_j)$.

Assume that there are $r$ cuboids $F_{i_1}, F_{i_2}, \cdots, F_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. Then we have that the set $S = \{i_1, i_2, \cdots, i_r\} \subset I_n = \{1, 2, \cdots, n\}$. Now we arrange the 2-combinations of $S$ in the order of dictionary. For two 2-combinations of $S$ $a_1, a_2$ and $b_1, b_2$, where $a_1 < a_2$ and $b_1 < b_2$ , if $a_1 < b_1$ or $a_1 = b_1$, $a_2 < b_2$, then we say that the 2-combination $a_1, a_2$ is prior to the 2-combination $b_1, b_2$. From now on, we consider the adjacent relations between cuboids $F_i$ and $F_j$ in the order of dictionary as above.

For any non-overlap layout scheme $Y \in D_n$ , let $V = \{F_1, F_2, \cdots, F_n\}$ be the set of vertexes, which is the set of cuboids determined by $Y \in D_n$ in layout problem $P$. Let $E(Y) = \{(F_i, F_j) : F_i, F_j \in V, F_i \text{ and } F_j \text{ are adjacent cuboids}\}$ be the edge set of $Y$. Then we have

**Definition 4.** For any non-overlap layout scheme $Y \in D_n$, $G(Y) = (V, E(Y))$ is called a *non-overlap layout graph* of $Y$. Denote the set of all the non-overlap layout graphs of problem $P$ by $G_n = \{G(Y) : Y \in D_n\}$.

**Property 2.** *Every non-overlap layout graph $G(Y)$ of $Y \in D_n$ is a connected, simple graph, and it is correspondent with a non-overlap layout scheme.*

*Proof.* Since the non-overlap layout graph $G(Y)$ does not exclude loops (lines joining a point to itself) and multiple lines (in parallel), then $G(Y)$ is a simple graph. The connectivity of $G(Y)$ follows from Definition 3 and Definition 4.

It is clearly that, in any $G(Y) = (V, E(Y)) \in G_n$, we have that $2|E(Y)| = \sum_{F_i \in V} d(F_i)$ , where $|E(Y)|$ is the number of edges of the graph $G(Y)$ and $d(F_i)$ is the number of edges joining with the vertex $F_i$ . □

**Property 3.** *Every permutation $\sigma$ of the set of vertexes $V = \{F_1, F_2, \cdots, F_n\}$ can constitute a directed graph such that there is a directed arc from point $F_i$*

*to $\sigma(F_i)$ . Since $\sigma$ is a bijection, then there is just one incident arc and one out-arc.*

**Theorem 1.** *The number $|G_n|$ of all the non-overlap layout graphs, i.e., the number of n connected simple graphs of problem P satisfies*

$$n2^{\binom{n}{2}} = \sum_{k=1}^{n} k|G_k| \cdot 2^{\binom{n-k}{2}} \cdot \binom{n}{k}.$$

*Proof.* By Property 2, the result follows from Theorem 1 and 2 in [10].

For example, when $n = 1, 2, 3, \cdots, 7$, by Theorem 1 we can calculate the number $G_n$ is 1, 1, 4, 38, 728, 26704, 1866256 respectively.

For layout problem $P$, let $S_n$ be a symmetric group with object set $V = \{F_1, F_2, \cdots, F_n\}$. For any $\sigma \in S_n$ , we define the induced mapping:

$$\bar{\sigma} : G_n \to G_n$$
$$G(Y) \mapsto \bar{\sigma}(G(Y))$$

such that $\bar{\sigma}(G(Y)) = (V, \sigma(E(Y)))$, where

$$\sigma(E(Y)) = \{(\sigma(F_i), \sigma(F_j)) : (F_i, F_j) \in E(Y)\}.$$

In fact, the induced mapping $\bar{\sigma}$ is the action of $\sigma$ on non-overlap layout graph $G(Y)$ which preserves adjacency in $G(Y)$. □

**Theorem 2.** *Denote $\bar{S}_n = \{\bar{\sigma} : \forall\sigma \in S_n\}$. Then the induced a group $\bar{S}_n$ is group isomorphic with $S_n$, i.e. $\bar{S}_n \cong S_n$. Moreover, $\bar{S}_n$ is a permutation group whose objects are all the non-overlap layout graph $G(Y) \in G_n$ and $\bar{S}_n$ is a subgroup of the symmetric group $S_{|G_n|}$.*

*Proof.* Let $f : S_n \to \bar{S}_n$ and $\sigma \mapsto \bar{\sigma}$, then $f$ is a bijection. For $\forall\sigma, \tau \in S_n$, we have that $f(\sigma\tau) = \overline{\sigma\tau} = \bar{\sigma}\bar{\tau} = f(\sigma)f(\tau)$ , so $\bar{S}_n \cong S_n$ . Since $\forall\bar{\sigma} \in \bar{S}_n$ is a permutation of $G_n$ , it is clear that $\bar{S}_n$ is a permutation group whose objects are all the non-overlap layout graph $G(Y) \in G_n$ and $\bar{S}_n$ is a subgroup of the symmetric group $S_{|G_n|}$.

Following [8], let the orbit of the induced group $\bar{S}_n$ acting on $G(Y) \in G_n$ be $O_k$ , i.e., $O_k = \{\bar{\sigma}(G(Y)) : \bar{\sigma} \in \bar{S}_n\} \subset G_n$ . It is evident that the orbit $O_k$ of $G_n$ is a partition of $G_n$ and includes only the isomorphic connected graph in $G_n$. Suppose that the number of orbits in the set of graphs $G_n$ determined by the induced group $\bar{S}_n$ is $c_n$, then$G_n = \bigcup_{k=1}^{c_n} O_k$. □

**Theorem 3.** *Let the orbit of the induced group $\bar{S}_n$ acting on $G(Y) \in G_n$ be $O_k$. Then the number $|O_k|$ of the non-overlap layout graphs in the orbit of $O_k$ the induced group $\bar{S}_n$ acting on $G(Y) \in G_n$ satisfies $|O_k| = \frac{n!}{|H_k|}$, where $H_k$ is the stationary kernel, i.e., $H_k = \{\bar{\sigma} : \bar{\sigma} \in \bar{S}_n, \bar{\sigma}(G(Y)) = G(Y)\}$.*

*Proof.* By Theorem 2, we have that $\bar{S}_n \cong S_n$, and thus $|\bar{S}_n| = |S_n| = n!$ . Then the result follows from Theorem A in §6.6 of [11]. □

**Theorem 4.** *Let $O(x)$ be the generating function for the orbits $O_k$ in the set of graphs $G_n$ determined by the induced group $\bar{S}_n$ so that $O(x) = \sum_{n=1}^{\infty} c_n x^n$. Then the number $c_n$ of the orbits $O_k$ in the non-overlap layout graphs $G(Y) \in G_n$ satisfies $pa_p = \sum_{d/p} nc_d$, where $a_p$ satisfies the formula $\sum_{p=1}^{\infty} a_p x^p = \sum_{n=1}^{\infty} c(x^n)/n$.*

*Proof.* Firstly, we extend the definitions of the non-overlap layout graph and its orbit. For the set $V = \{F_1, F_2, \cdots, F_n\}$, there are $\binom{n}{2}$ distinct unordered pairs of these points. Denote the set of all the labeled graphs with point set $V$ by $X$. It is well known that each pair of points are either adjacent or not adjacent. For layout problem $P$, let $S_n$ be a symmetric group with object set $V = \{F_1, F_2, \cdots, F_n\}$, similarly as above we can define the induced group $\bar{S}_n$ acting on $X$.

Let $g(x)$ be the generating function for the orbits of the induced group $\bar{S}_n$ acting on $X$ so that $g(x) = \sum_{n=1}^{\infty} g_n x^n$ , by Theorem4.2 in [10] we have that

$$1 + g(x) = exp \sum_{n=1}^{\infty} c(x^n)/n.$$

Since $\sum_{p=1}^{\infty} a_p x^p = \sum_{n=1}^{\infty} c(x^n)/n = log(1+g(x))$, then the result follows by equating coefficients. □

For example, when $n = 1, 2, 3, \cdots, 7$, By Theorem 3 we can calculate the number of $c_n$ the orbits $O_k$ in the non-overlap layout graphs $G(Y) \in G_n$ which is $1, 1, 2, 6, 21, 112, 853$ respectively.

## 4. An improved combinatorial optimization algorithm

By virtue of the properties and theorems in section2 and section3, following [8, 12, 13] we get the main steps of the combinatorial optimization algorithm improved as follows:

*Step 1.* Input the parameter of the population of the three-dimensional layout problem $P : A, a, n$ and the physical parameter of the await layout cuboids $: P_i = (\ell_i, w_i, h_i) \in R^3_+, m_i \in R^3_+, i \in I_n$.

*Step 2.* By Theorems 1, 3, and 4, compute the number $|G_n|$ of all the non-overlap layout graphs of problem $P$, the number $|O_k|$ of the non-overlap layout graphs in the orbit $|O_k|$ of the induced group $\bar{S}_n$ acting on $G(Y) \in G_n$ and the number $c_n$ of the orbits $O_k$ in the non-overlap layout graphs $G(Y) \in G_n$, respectively. Set $k = 1$.

*Step3.* Generate a element $G_{ki} \in O_k$ and its corresponding non-overlap layout scheme $Y_{ki} \in D_n$ such that $O_k = \{\bar{\sigma}(G_{ki}) : G_{ki} = G(Y_{ki}), \bar{\sigma} \in \bar{S}_n\} \subset G_n$, where $i = 1, 2, \cdots, |O_k|$, then set $i = 1$.

*Step 4.* If the cuboids $F_i$ determined by $Y_{ki}$ satisfies the capacity constraints, i.e. $\bigcup_{i=1}^{n} F_i \subset \chi$, then go to Step7.

*Step 5.* Set $i = i + 1$. If $i \leq |O_k|$, then go to Step .

*Step 6.* Set $k = k + 1$. If $k \leq c_n$ , then go to Step 3, otherwise go to Step 8.

*Step 7.* Let $I(Y_{ki}) = \{Y : E(Y) = E(Y_{ki}), G(Y_{ki}) \in O_k\}$. Then solve the subproblem $P_{ki}$ of the layout problem $P$ on $I(Y_{ki})$, i.e.,

$$P_{ki}: \quad min \quad H(Y)$$

$$s.t. \quad \bigcup_{i=1}^{n} F_i \subset \chi$$

$$Y \in I(Y_{ki}).$$

*Step 8.* Solve problem $P$ to obtain the global optimal solution $Y^*$ and the optimal value $H(Y^*)$.

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