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CONVERGENCE AND PERIODICITY IN A DISCRETE-TIME NEURAL NETWORK MODEL OF TWO NEURONS

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ABSTRACT. For the discrete version of an artificial neural network of two neurons with piecewise constant argument, we obtain some sufficient conditions that every solution is either periodic or convergent.

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1. Introduction

Consider the following difference system

$$\begin{cases} x_{n+1} = \lambda x_n + f(y_n), \\ y_{n+1} = \lambda y_n - f(x_n), \end{cases} \qquad n = 0, 1, 2, \ldots, \tag{1}$$

where $\lambda \in (0,1)$ is a constant, $f : \mathbf{R} \to \mathbf{R}$ is given by

$$f(u) = \begin{cases} 1, & u \in [a,b], \\ 0, & u \notin [a,b]. \end{cases} \tag{2}$$

Here $a, b \in \mathbf{R}$ are constants.

The system (1) includes the discrete version of the following two-neuron network model with piecewise constant argument

$$\begin{cases} \dfrac{\mathrm{dx}}{\mathrm{dt}} = -\alpha x + \beta f(y([t])), \\ \dfrac{\mathrm{dy}}{\mathrm{dt}} = -\alpha y - \beta f(x([t])), \end{cases} \tag{3}$$

where $[\cdot]$ denotes the greatest integer function, the constant $\alpha > 0$ represents the internal decay rate, the constant $\beta > 0$ measures the synaptic strength, $x(t)$ and $y(t)$ denote the activations of the corresponding neurons, respectively, f is the activation function.

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System (3) describes the evolution of two neurons network with excitatory interactions, which has some interesting applications in, for example, image processing of moving objects and has been studied in [7]. Moreover, there are some interesting results for system (3) in [9] and references therein.

In fact, we can rewrite (3) in the following form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(x(t)e^{\alpha t}) &= \beta e^{\alpha t} f(y([t])), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(y(t)e^{\alpha t}) &= -\beta e^{\alpha t} f(x([t])). \end{cases} \tag{4}$$

Let n be a positive integer. Integrate (4) from n to $t \in [n, n+1)$ and obtain

$$\begin{cases} x(t)e^{\alpha t} - x(n)e^{\alpha n} &= \dfrac{\beta}{\alpha}(e^{\alpha t} - e^{\alpha n}) f(y(n)), \\ y(t)e^{\alpha t} - y(n)e^{\alpha n} &= -\dfrac{\beta}{\alpha}(e^{\alpha t} - e^{\alpha n}) f(x(n)). \end{cases} \tag{5}$$

For any nonnegative integer k, we denote $x(k)$ and $y(k)$ by x_k and y_k, respectively. Let $t \to n+1$ in (5), then

$$\begin{cases} x_{n+1} = \dfrac{1}{e^{\alpha}} x_n + \dfrac{\beta}{\alpha}\Big(1 - \dfrac{1}{e^{\alpha}}\Big) f(y_n), \\ y_{n+1} = \dfrac{1}{e^{\alpha}} y_n - \dfrac{\beta}{\alpha}\Big(1 - \dfrac{1}{e^{\alpha}}\Big) f(x_n), \end{cases} \qquad n = 0, 1, 2, \ldots. \tag{6}$$

Let

$$f^*(u) = f\Big(\frac{\beta(e^{\alpha} - 1)}{\alpha e^{\alpha}} u\Big), \qquad a^* = \frac{\alpha e^{\alpha}}{\beta(e^{\alpha} - 1)} a, \qquad b^* = \frac{\alpha e^{\alpha}}{\beta(e^{\alpha} - 1)} b,$$
$$x_n^* = \frac{\alpha e^{\alpha}}{\beta(e^{\alpha} - 1)} x_n, \qquad y_n^* = \frac{\alpha e^{\alpha}}{\beta(e^{\alpha} - 1)} y_n, \qquad n = 0, 1, 2, \cdots.$$

Then it follows from (6) with dropping the $*$ that

$$\begin{cases} x_{n+1} = \dfrac{1}{e^{\alpha}} x_n + f(y_n), \\ y_{n+1} = \dfrac{1}{e^{\alpha}} y_n - f(x_n), \end{cases} \qquad n = 0, 1, 2, \ldots. \tag{7}$$

Obviously, the system (7) is a special form of the system (1) with $\lambda = \frac{1}{e^{\alpha}}$.

The dynamics of the systems (1) and (3) have been extensively studied in the literature. However, most of the existing work is concentrated on the case where the function f is piecewise linear or a smooth sigmoid (see [2-5], [9]). To the best of our knowledge, few results are given for the dynamics of (1) and (3) when f is not continuous. As the signal function f is of the following piecewise constant McCulloch-Pitts nonlinearity,

$$f(u) = \begin{cases} 1, & u \le \sigma, \\ -1, & u > \sigma, \end{cases} \qquad \sigma \in \mathbf{R},$$

systems (1) and (3) have been studied by some authors in [6-8], [10-12] and references therein. The aim of this paper is to study the convergence and periodicity of solutions of (1) when f is of the form (2), which describes the input-output relation of a neuron.

We denote by $\mathbf{N}$ the set of all nonnegative integers. For any $m, n \in \mathbf{N}$ and $m \leq n$, define $N(m) = \{m, m+1, m+2, \dots\}, N(m,n) = \{m, m+1, \dots n\}$. Let $\gamma_k = \frac{b}{\lambda^k}, \nu_k = \frac{a}{\lambda^k}, (b > 0, a < 0, k \in N(0))$. Then $\lim_{k\to+\infty} \gamma_k = +\infty, \lim_{k\to+\infty} \nu_k = -\infty$. Let

$$I_{11} = \{(x,y); x < a, y < a\}, \qquad I_{12} = \{(x,y); x < a, y \in [a,b]\},$$
$$I_{13} = \{(x,y); x < a, y > b\}, \qquad I_{21} = \{(x,y); x \in [a,b], y < a\},$$
$$I_{22} = \{(x,y); x \in [a,b], y \in [a,b]\}, \qquad I_{23} = \{(x,y); x \in [a,b], y > b\},$$
$$I_{31} = \{(x,y); x > b, y < a\}, \qquad I_{32} = \{(x,y); x > b, y \in [a,b]\},$$
$$I_{33} = \{(x,y); x > b, y > b\}, \qquad \Upsilon = \bigcup_{k\in N} ((\gamma_{k+1}, +\infty) \times (\gamma_k, \gamma_{k+1}]),$$
$$\Xi = \bigcup_{k\in N} ([\nu_{k+1}, \nu_k) \times (-\infty, \nu_{k+1})), \qquad \Theta = \bigcup_{k\in N} ((\gamma_k, \gamma_{k+1}] \times [\nu_{k+1}, \nu_k)),$$
$$\Lambda = \bigcup_{k\in N} ((\gamma_{k+1}, +\infty) \times [\nu_{k+1}, \nu_k)), \qquad \Omega = \bigcup_{k\in N} ((\gamma_k, \gamma_{k+1}] \times (-\infty, \nu_{k+1})).$$

Obviously, $\bigcup_{i,j=1}^{3} I_{i,j} = \mathbf{R}^2, \quad \Upsilon \subset I_{33}, \quad \Xi \subset I_{11}, \quad \Theta \cup \Lambda \cup \Omega = I_{31}.$

By a solution of the system (1), we mean a sequence $\{(x_n, y_n)\}$ of points in $\mathbf{R}^2$ that is defined for all $n \in N(0)$ and satisfies (1) for $n \in N(1)$. Clearly, for any $(x_0, y_0) \in \mathbf{R}^2$, the system (1) has an unique solution $\{(x_n, y_n)\}$ satisfying the initial condition $(x_n, y_n)|_{n=0} = (x_0, y_0)$.

For the general background of difference equations, we refer to [1].

2. Main results

Throughout this paper, $\{(x_n, y_n)\}$ denotes the unique solution of the system (1) with initial value $(x_0, y_0) \in \mathbf{R}^2$. Our main results state as the following.

Theorem 1. (1) *If* $b < 0$, *then* $(x_n, y_n) \to (0,0)$ *as* $n \to \infty$.

(2) *If* $a > 0$, *then* $(x_n, y_n) \to (0,0)$ *as* $n \to \infty$.

(3) *If* $a < -\frac{1}{1-\lambda} < 0 < b < \frac{1}{1-\lambda}$, *then* $(x_n, y_n) \to (\frac{1}{1-\lambda}, 0)$ *as* $n \to \infty$.

(4) *If* $-\frac{1}{1-\lambda} < a < 0 < \frac{1}{1-\lambda} < b$, *then* $(x_n, y_n) \to (0, -\frac{1}{1-\lambda})$ *as* $n \to \infty$.

(5) *If* $a < -\frac{1}{1-\lambda} < \frac{1}{1-\lambda} < b$, *then* $(x_n, y_n) \to (\frac{1}{1-\lambda}, -\frac{1}{1-\lambda})$ *as* $n \to \infty$.

(6) *If* $-\frac{1}{1-\lambda} < a < 0 < b < \frac{1}{1-\lambda}$ *and* $(x_0, y_0) \in I_{32} \cup \Upsilon \cup \Lambda$, *then* $(x_n, y_n) \to (\frac{1}{1-\lambda}, 0)$ *as* $n \to \infty$.

(7) *If* $-\frac{1}{1-\lambda} < a < 0 < b < \frac{1}{1-\lambda}$ *and* $(x_0, y_0) \in I_{21} \cup \Xi \cup \Omega$, *then* $(x_n, y_n) \to (0, -\frac{1}{1-\lambda})$ *as* $n \to \infty$.

Theorem 2. *(1) $(x_n, y_n) \to (\frac{1}{1-\lambda}, 0)$ as $n \to \infty$, if $(x_0, y_0) \in (b, \lambda b + 1] \times [\lambda a - 1, a) \subset I_{31}$ and one of the following conditions holds:*

(i) $-\frac{1}{1-\lambda} < a < -\frac{1}{1-\lambda^2} < 0 < b < \frac{\lambda}{1-\lambda^2}$;

(ii) $-\frac{1}{1-\lambda^2} < a < -\frac{\lambda}{1-\lambda^2} < 0 < b < \frac{\lambda}{1-\lambda^2}$;

(iii) $-\frac{1}{1-\lambda} < a < -\frac{1}{1-\lambda^2} < \frac{\lambda}{1-\lambda^2} < b < \frac{1}{1-\lambda^2}$.

(2) $(x_n, y_n) \to (0, -\frac{1}{1-\lambda})$ as $n \to \infty$, if $(x_0, y_0) \in (b, \lambda b+1] \times [\lambda a - 1, a) \subset I_{31}$ and one of the following conditions holds:

(i) $-\frac{\lambda}{1-\lambda^2} < a < 0 < \frac{1}{1-\lambda^2} < b < \frac{1}{1-\lambda}$;

(ii) $-\frac{\lambda}{1-\lambda^2} < a < 0 < \frac{\lambda}{1-\lambda^2} < b < \frac{1}{1-\lambda^2}$;

(iii) $-\frac{1}{1-\lambda^2} < a < -\frac{\lambda}{1-\lambda^2} < \frac{1}{1-\lambda^2} < b < \frac{1}{1-\lambda}$.

Remark 1. By analysis, we can find that the solution $\{(x_n, y_n)\}$ of (1) with the initial value $(x_0, y_0) \in \mathbf{R}^2$ will be in the region $I_{21} \cup I_{32} \cup I_{31}$ eventually when $-\frac{1}{1-\lambda} < a < 0 < b < \frac{1}{1-\lambda}$. Note that $\Theta \cup \Lambda \cup \Omega = I_{31}$, by Theorems 1-2, it remains to consider the case where $(x_0, y_0) \in \Theta$ for $-\frac{1}{1-\lambda} < a < 0 < b < \frac{1}{1-\lambda}$.

For $m \in N(1)$ and $l \in N(0)$, let

$$\begin{aligned}
\delta_m &= \frac{1}{1-\lambda} - \frac{\lambda^{m-1}}{1-\lambda^{m+1}}, \qquad \epsilon_m = \frac{1}{1-\lambda} - \frac{\lambda^m}{1-\lambda^{m+1}}, \\
\theta_{m,l} &= \frac{\lambda^{(m+2)(l+2)-2}(1-\lambda) + (1-\lambda^m)(1+\lambda^{(m+2)(l+1)-1})}{(1-\lambda)(1-\lambda^{(m+2)(l+2)-1})} \\
&\quad + \frac{\lambda^{m+1}(1-\lambda^{m+1})(1-\lambda^{(m+2)l})/(1-\lambda^{m+2})}{(1-\lambda)(1-\lambda^{(m+2)(l+2)-1})}, \\
\mu_{m,l} &= \frac{1-\lambda^m + \lambda^{m+1}(1-\lambda^{m+1})(1-\lambda^{(m+2)(l+1)})/(1-\lambda^{m+2})}{(1-\lambda)(1-\lambda^{(m+2)(l+2)-1})}
\end{aligned}$$

and

$$\xi_{m,l} = \frac{(1-\lambda^m)(1+\lambda^{m+1} - \lambda^{(m+2)(l+2)-1}) + \lambda^{2m+2}(1-\lambda^{(m+2)(l+1)})\frac{1-\lambda^{m+1}}{1-\lambda^{m+2}}}{(1-\lambda^{(m+2)(l+2)-1})(1-\lambda)}.$$

Theorem 3. *Let $-\frac{1}{1-\lambda} < a \le -\frac{\lambda}{1-\lambda^2} < \frac{\lambda}{1-\lambda^2} \le b < \frac{1}{1-\lambda}$, $x^* = \frac{b-(1-\lambda^m)/(1-\lambda)}{\lambda^m}$, $y^* = \frac{a+(1-\lambda^m)/(1-\lambda)}{\lambda^m}$.*

(1) If $a \in (-\epsilon_m, -\delta_m]$, $b \in [\delta_m, \epsilon_m)$, then the solution $\{(\bar{x}_n, \bar{y}_n)\}$ of the system (1) with the initial value $(\epsilon_m, -\epsilon_m)$ is periodic with minimal period $m+1$.

Moreover, for any solution $\{(x_n, y_n)\}$ *of (1) with initial value* $(x_0, y_0) \in (b, \lambda b + 1] \times [\lambda a - 1, a)$, $\lim_{n\to\infty}(x_n - \bar{x}_n) = \lim_{n\to\infty}(y_n - \bar{y}_n) = 0$.

(2) *If* $a \in (-\mu_{m,l}, -\theta_{m,l}]$, $b \in [\theta_{m,l}, \mu_{m,l})$, *then there exists a* $(\bar{x}_0, \bar{y}_0) \in (x^*, \lambda b + 1] \times [\lambda a - 1, y^*)$ *such that the solution* $\{(\bar{x}_n, \bar{y}_n)\}$ *of (1) with the initial value* $(\bar{x}_0, \bar{y}_0)$ *is periodic with minimal period* $(m+2)(l+2)-1$. *Moreover, for any solution* $\{(x_n, y_n)\}$ *of (1) with the initial value* $(x_0, y_0) \in (x^*, \lambda b+1] \times [\lambda a - 1, y^*)$, $\lim_{n\to\infty}(x_n - \bar{x}_n) = \lim_{n\to\infty}(y_n - \bar{y}_n) = 0$.

(3) *If* $a \in (-\mu_{m,l}, \xi_{m,l}]$, $b \in [\xi_{m,l}, \mu_{m,l})$, *then there exists a* $(\bar{x}_0, \bar{y}_0) \in (b, x^*] \times [y^*, a)$ *such that solution* $\{(\bar{x}_n, \bar{y}_n)\}$ *of (1) with* $(\bar{x}_0, \bar{y}_0)$ *is periodic with minimal period* $(m+2)(l+2)-1$. *Moreover, for any solution* $\{(x_n, y_n)\}$ *of (1) with initial value* $(x_0, y_0) \in (b, x^*] \times [y^*, a)$, $\lim_{n\to\infty}(x_n - \bar{x}_n) = \lim_{n\to\infty}(y_n - \bar{y}_n) = 0$.

Remark 2. For $m \in N(1), l \in N(0)$, it is easy to see that $[\theta_{m,l}, \mu_{m,l}) \subseteq (\epsilon_m, \delta_{m+1})$, $[\xi_{m,l}, \mu_{m,l}) \subseteq (\epsilon_m, \delta_{m+1})$,

$$\frac{\lambda}{1-\lambda^2} = \delta_1 < \epsilon_1 < \delta_2 < \epsilon_2 < \delta_3 < \cdots < \delta_m < \epsilon_m < \cdots,$$

$$\epsilon_m < \theta_{m,0} < \mu_{m,0} < \theta_{m,1} < \cdots < \mu_{m,l} < \theta_{m,l+1} < \mu_{m,l+1} < \cdots < \delta_{m+1},$$

$$\epsilon_m < \xi_{m,0} < \mu_{m,0} < \xi_{m,1} < \cdots < \xi_{m,l} < \mu_{m,l} < \cdots < \delta_{m+1},$$

and $\lim_{m\to\infty} \epsilon_m = \frac{1}{1-\lambda}$, $\lim_{l\to\infty} \mu_{m,l} = \delta_{m+1}$.

For $m \in N(2)$ and $l \in N(0)$, let

$$\begin{aligned}
\zeta_m &= \frac{\lambda^m}{1-\lambda^{m+1}}, \qquad \eta_m = \frac{\lambda^{m-1}}{1-\lambda^{m+1}}, \\
\rho_{m,l} &= \frac{\lambda^m[1+\lambda^{(m+1)(l+1)+1} + \lambda^{m+1}(1-\lambda^{(m+1)l})/(1-\lambda^{m+1})]}{1-\lambda^{(m+1)(l+2)+1}}, \\
\tau_{m,l} &= \frac{\lambda^m[1+\lambda^{m+1}(1-\lambda^{(m+1)(l+1)})/(1-\lambda^{m+1})]}{1-\lambda^{(m+1)(l+2)+1}}, \\
\omega_{m,l} &= \lambda^m + \frac{\lambda^{2m+2}[1+\lambda^{m+1}(1-\lambda^{(m+1)(l+1)})/(1-\lambda^{m+1})]}{1-\lambda^{(m+1)(l+2)+1}}.
\end{aligned}$$

Theorem 4. *Let* $-\frac{\lambda}{1-\lambda^2} < a < 0 < b < \frac{\lambda}{1-\lambda^2}$, $\bar{x}^* = (b-\lambda^m)/(\lambda^{m+2})$, $\bar{y}^* = (a+\lambda^m)/(\lambda^{m+2})$.

(1) *If* $a \in (-\eta_m, -\zeta_m]$, $b \in [\zeta_m, \eta_m)$, *then the solution* $\{(\bar{x}_n, \bar{y}_n)\}$ *of the system (1) with the initial value* $(\eta_m, -\eta_m)$ *is periodic with minimal period* $m+1$. *Moreover, for any solution* $\{(x_n, y_n)\}$ *of (1) with initial value* $(x_0, y_0) \in (b, \frac{b}{\lambda}] \times [\frac{a}{\lambda}, a)$, $\lim_{n\to\infty}(x_n - \bar{x}_n) = \lim_{n\to\infty}(y_n - \bar{y}_n) = 0$.

(2) *If* $a \in (-\tau_{m,l}, -\rho_{m,l}]$, $b \in [\rho_{m,l}, \tau_{m,l})$, *then there exists a* $(\bar{x}_0, \bar{y}_0) \in (\bar{x}^*, \frac{b}{\lambda}] \times [\frac{a}{\lambda}, \bar{y}^*)$ *such that solution* $\{(\bar{x}_n, \bar{y}_n)\}$ *of (1) with* $(\bar{x}_0, \bar{y}_0)$ *is periodic with minimal period* $(m+1)(l+2)+1$. *Moreover, for any solution* $\{(x_n, y_n)\}$ *of (1) with the initial value* $(x_0, y_0) \in (\bar{x}^*, \frac{b}{\lambda}] \times [\frac{a}{\lambda}, \bar{y}^*)$, $\lim_{n\to\infty}(x_n - \bar{x}_n) = \lim_{n\to\infty}(y_n - \bar{y}_n) = 0$.

(3) *If* $a \in (-\tau_{m,l}, -\omega_{m,l}]$, $b \in [\omega_{m,l}, \tau_{m,l})$, *then there exists* $(\bar{x}_0, \bar{y}_0) \in (b, \bar{x}^*] \times [\bar{y}^*, a)$ *such that the solution* $\{(\bar{x}_n, \bar{y}_n)\}$ *of (1) with the initial value* $(\bar{x}_0, \bar{y}_0)$ *is periodic with minimal period* $(m+1)(l+2)+1$. *Moreover, for any solution* $\{(x_n, y_n)\}$ *of (1) with the initial value* $(x_0, y_0) \in (b, \bar{x}^*] \times [\bar{y}^*, a)$, $\lim_{n\to\infty}(x_n - \bar{x}_n) = \lim_{n\to\infty}(y_n - \bar{y}_n) = 0$.

Remark 3. Obviously, for $m \in N(2)$ and $l \in N$, we have $[\rho_{m,l}, \tau_{m,l}) \subseteq (\eta_{m+1}, \zeta_m)$, $[\omega_{m,l}, \tau_{m,l}) \subseteq (\eta_{m+1}, \zeta_m)$,

$$\frac{\lambda}{1-\lambda^2} = \zeta_1 > \eta_2 > \zeta_2 > \cdots > \eta_{m-1} > \zeta_{m-1} > \eta_m > \cdots,$$

$$\eta_{m+1} < \rho_{m,0} < \tau_{m,0} < \rho_{m,1} < \tau_{m,1} < \cdots < \rho_{m,l} < \tau_{m,l} < \cdots < \zeta_m,$$

$$\eta_{m+1} < \omega_{m,0} < \tau_{m,0} < \omega_{m,1} < \cdots \omega_{m,l} < \tau_{m,l} < \cdots < \zeta_m,$$

and $\lim_{m\to\infty} \eta_m = 0$, $\lim_{l\to\infty} \tau_{m,l} = \zeta_m$.

3. Proofs of main results

From (1) and (2), it is easy to see that the system (1) has an obvious connection with the following linear difference systems

$$\begin{cases} x_{n+1} = \lambda x_n + 1, \\ y_{n+1} = \lambda y_n - 1; \end{cases} \begin{cases} x_{n+1} = \lambda x_n, \\ y_{n+1} = \lambda y_n; \end{cases} \begin{cases} x_{n+1} = \lambda x_n, \\ y_{n+1} = \lambda y_n - 1; \end{cases} \begin{cases} x_{n+1} = \lambda x_n + 1, \\ y_{n+1} = \lambda y_n. \end{cases} \tag{8}$$

Therefore, we first consider the following relating equations

$$u_{n+1} = \lambda u_n + 1,\ v_{n+1} = \lambda v_n,\ w_{n+1} = \lambda w_n - 1. \tag{9}$$

By induction, for $n \in N(n_0)$, the solutions of equations in (9) with the initial value c are given by

$$u_n = \lambda^{n-n_0}c + \frac{1-\lambda^{n-n_0}}{1-\lambda},\ v_n = \lambda^{n-n_0}c,\ w_n = \lambda^{n-n_0}c - \frac{1-\lambda^{n-n_0}}{1-\lambda}. \tag{10}$$

Note that $\lambda \in (0,1)$, it follows from (10) that

$$\lim_{n\to\infty} u_n = \frac{1}{1-\lambda},\ \lim_{n\to\infty} v_n = 0,\ \lim_{n\to\infty} w_n = -\frac{1}{1-\lambda}. \tag{11}$$

We first give the proof of Theorem 1.

Proof. We only show the conclusion (1). The others are similar and omitted.

By (8), it follows that system (1) has four limited points $A(0,0)$, $B(\frac{1}{1-\lambda}, 0)$, $C(0, -\frac{1}{1-\lambda})$ and $D(\frac{1}{1-\lambda}, -\frac{1}{1-\lambda})$. Note that $b < 0$, there are $A, B \in I_{33}$, $C, D \in I_{33}$ if $b < -\frac{1}{1-\lambda}$, or $C, D \in I_{32}$ if $a \le -\frac{1}{1-\lambda} \le b < 0$, or $C, D \in I_{31}$ if $-\frac{1}{1-\lambda} < a < b < 0$. Therefore, in view of the distribution of the four limited points, for any initial value $(x_0, y_0) \in \mathbf{R}^2$, there exists a $m_0 \in N(1)$ such that $(x_n, y_n) \in I_{31} \cup I_{32} \cup I_{33}$ for $n \in N(m_0)$. For initial value $(x_0, y_0) \in I_{32}$ and $(x_0, y_0) \in I_{31}$, the limited points of system (1) are B and A, respectively. Therefore, there exists a $m \in N(1)$ such that $(x_n, y_n) \in I_{33}$ for $n \in N(m)$, and by (8)-(11), $(x_n, y_n) \to (0,0)$ as $n \to \infty$. □

We next give the proof of Theorem 2 and only show the conclusion (1) as assumption (i) holds. The others are similar and omitted.

Proof. Note $-\frac{1}{1-\lambda} < a < -\frac{1}{1-\lambda^2} < 0 < b < \frac{\lambda}{1-\lambda^2}$. It follows that

$$\frac{a}{\lambda} < \lambda a - 1 < a < \frac{a+1}{\lambda} < \lambda a, \qquad \frac{b-1}{\lambda} < \lambda b < b < \frac{b}{\lambda} < \lambda b + 1.$$

Since $0 < b < \frac{\lambda}{1-\lambda^2}$, there exists a $m \in N(1)$ such that

$$b \leq \frac{1-\lambda^{2m}}{1-\lambda^{2m+1}} \cdot \frac{\lambda}{1-\lambda^2} < \frac{\lambda}{1-\lambda^2}. \tag{12}$$

If we take the initial value $(x_0, y_0) \in (\frac{b}{\lambda}, \lambda b + 1] \times [\lambda a - 1, a)$, then by the second in (8), we have

$$x_1 = \lambda x_0 > b, \quad y_1 = \lambda y_0 > a, \quad (x_1, y_1) \in I_{32}. \tag{13}$$

If we restrict $(x_0, y_0) \in (b, \frac{b}{\lambda}] \times [\lambda a - 1, a)$, for $n \in N(1, m-1)$, by (12), we have

$$x_{2n} = \lambda^{2n} x_0 + \frac{1-\lambda^{2n}}{1-\lambda^2} \in (b, \lambda b + 1], \ x_{2n+1} = \lambda^{2n+1} x_0 + \frac{\lambda(1-\lambda^{2n})}{1-\lambda^2} \in (\lambda b, b],$$

and $x_{2n+2} > x_{2n}, x_{2n+1} > x_{2n-1}$, $y_{2n} \in [\lambda a - 1, a), y_{2n+1} \in [a, \lambda a)$. Thus, for $n \in N(1, m+1)$,

$$(x_{2n}, y_{2n}) \in I_{31}, \ (x_{2n+1}, y_{2n+1}) \in I_{22}. \tag{14}$$

Case (1). Assume that $x_{2n} \in (b, \frac{b}{\lambda}]$ for some $n \in N(1, m-1)$. If there is a $l \leq m$ such that, for $n \in N(1, l-1)$,

$$y_{2l-1} = \lambda^{2l-1} y_0 - \frac{\lambda(1-\lambda^{2l-2})}{1-\lambda^2} \geq \frac{a+1}{\lambda}, \ y_{2n-1} \in [a, \frac{a+1}{\lambda}),$$

then $(x_{2l-1}, y_{2l-1}) \in (\lambda b, b] \times [\frac{a+1}{\lambda}, \lambda a) \subseteq I_{22}$. By the first in (8), we have

$$(x_{2l}, y_{2l}) = (\lambda x_{2l-1} + 1, \lambda y_{2l-1} - 1) \in (b, \lambda b + 1] \times [a, b] \subseteq I_{32}. \tag{15}$$

If $y_{2n-1} \in [a, \frac{a+1}{\lambda})$ for all $n \in N(1, m)$, by (14) we have $(x_{2m}, y_{2m}) \in (\frac{b}{\lambda}, \lambda b + 1] \times [\lambda a - 1, a)$ and

$$x_{2m+1} > b, \quad y_{2m+1} > a, \quad (x_{2m+1}, y_{2m+1}) \in I_{32}. \tag{16}$$

Case (2). Assume $x_{2n} > \frac{b}{\lambda}$ foe some $n \in N(1, m)$, by the second in (8), we have

$$(x_{2n+1}, y_{2n+1}) \in I_{32}. \tag{17}$$

Therefore, in terms of (13,(15)-(17), for any $(x_0, y_0) \in (b, \lambda b + 1] \times [\lambda a - 1, a)$, there exists a $k \in N(1)$ such that $(x_k, y_k) \in I_{32}$. By the conclusion (6) in Theorem 1, we have $(x_n, y_n) \to (\frac{1}{1-\lambda}, 0)$ as $n \to \infty$. □

We sketch the proof of Theorem 3 as following.

Proof. By $-\frac{1}{1-\lambda} < a \leq -\frac{\lambda}{1-\lambda^2} < \frac{\lambda}{1-\lambda^2} \leq b < \frac{1}{1-\lambda}$, it follows that

$$\frac{a}{\lambda} \leq \lambda a - 1 < a < \lambda a, \quad \lambda b < b < \lambda b + 1 \leq \frac{b}{\lambda}.$$

We first prove the conclusion (1). Let $(x_0, y_0) \in (b, \lambda b + 1] \times [\lambda a - 1, a) \subseteq I_{31}$. Then

$$x_1 = \lambda x_0 < b, \quad y_1 = \lambda y_0 > a, \quad (x_1, y_1) \in (\lambda b, b] \times [a, \lambda a) \subseteq I_{22}.$$

In view of the first in (8), there exists a $n_1 \in N(1)$ such that

$$(x_n, y_n) \in I_{22} \text{ for } n \in N(1, n_1), \quad (x_{n_1+1}, y_{n_1+1}) \notin I_{22},$$

where

$$x_{n_1} = \lambda^{n_1} x_0 + \frac{1-\lambda^{n_1-1}}{1-\lambda} \leq b, \quad y_{n_1} = \lambda^{n_1} y_0 - \frac{1-\lambda^{n_1-1}}{1-\lambda} \geq a.$$

Since $a \in (-\epsilon_m, -\delta_m]$, $b \in [\delta_m, \epsilon_m)$, we have

$$(x_m, y_m) \in I_{22}, \quad (x_{m+1}, y_{m+1}) \in (b, \lambda b + 1] \times [\lambda a - 1, a) \subseteq I_{31},$$

that is, $n_1 = m$. Repeating the above proceeding, by induction, for $l \in N(0)$ and $k \in N(1, m)$, we have

$$(x_{(m+1)l}, y_{(m+1)l}) \in (b, \lambda b + 1] \times [\lambda a - 1, a), (x_{(m+1)l+k}, y_{(m+1)l+k}) \in I_{22}.$$

By (9), we define

$$f_1(x) = \lambda x + 1, \quad f_2(x) = \lambda x, \quad f_3(x) = \lambda x - 1 \tag{18}$$

and denote

$$F_{m+1}(x, y) = ((f_1^m \circ f_2)(x), (f_3^m \circ f_2)(y)), \ (x, y) \in (b, \lambda b + 1] \times [\lambda a - 1, a)$$

where $f^{n+1} = f \circ f^n$. It yields that

$$F_{m+1}(x, y) = \left(\lambda^{m+1} x + \frac{1-\lambda^m}{1-\lambda}, \lambda^{m+1} y - \frac{1-\lambda^m}{1-\lambda}\right),$$

$$F_{m+1}^n(x, y)$$
$$= \left(\lambda^{n(m+1)} x + \frac{1-\lambda^m}{1-\lambda} \cdot \frac{1-\lambda^{n(m+1)}}{1-\lambda^{m+1}}, \lambda^{n(m+1)} y - \frac{1-\lambda^m}{1-\lambda} \cdot \frac{1-\lambda^{n(m+1)}}{1-\lambda^{m+1}}\right)$$

and $\lim_{n\to\infty} F_{m+1}^n(x, y) = (\epsilon_m, -\epsilon_m)$. Moreover,

$$(x_{(m+1)n}, y_{(m+1)n}) = F_{m+1}^n(x_0, y_0), \quad (x_0, y_0) \in (b, \lambda b + 1] \times [\lambda a - 1, a).$$

In fact, $(\epsilon_m, -\epsilon_m)$ is the unique fixed point of $F_{m+1}(x, y)$, and the solution $\{(\bar{x}_n, \bar{y}_n)\}$ of (1.1) with the initial value $(\epsilon_m, -\epsilon_m)$ is periodic with minimal period $m + 1$. Thus, for any solution $\{(x_n, y_n)\}$ of (1) with the initial value $(x_0, y_0) \in (b, \lambda b + 1] \times [\lambda a - 1, a)$, we have $\lim_{n\to\infty}(x_n - \bar{x}_n) = \lim_{n\to\infty}(y_n - \bar{y}_n) = 0$.

Next we prove the conclusion (2). By (18), for $(x,y) \in (b, \lambda b+1] \times [\lambda a-1, a)$, we define

$$P_{m+1}(x) = (f_1^m \circ f_2)(x),\ Q_{m+1}(y) = (f_3^m \circ f_2)(y).$$

It follows from (18) that, for $m \in N(1)$,

$$P_{m+1}(x) = \lambda^{m+1}x + \frac{1-\lambda^m}{1-\lambda},\ Q_{m+1}(y) = \lambda^{m+1}y - \frac{1-\lambda^m}{1-\lambda}.$$

Note $a \in (-\delta_{m+1}, -\epsilon_m)$, $b \in (\epsilon_m, \delta_{m+1})$, we have $-1/(1-\lambda) < y < 0 < x < 1/(1-\lambda)$ and $P_m(x) < P_{m+1}(x)$, $Q_{m+1}(y) < Q_m(y)$. Moreover,

$$P_{m+1}(x^*) = b, P_{m+2}(x^*) = \lambda b+1, \quad Q_{m+1}(y^*) = a, Q_{m+2}(y^*) = \lambda a - 1.$$

Therefore, for $(x,y) \in (x^*, \lambda b+1] \times [\lambda a-1, y^*)$,

$$P_{m+1}(x) \in (b, \lambda b+1], \quad Q_{m+1}(y) \in [\lambda a-1, a)$$

and for $(x,y) \in (b, x^*] \times [y^*, a)$,

$$P_{m+2}(x) \in (b, \lambda b+1], \quad Q_{m+2}(y) \in [\lambda a-1, a).$$

In view of $a \le -\theta_{m,0}$, $b \ge \theta_{m,0}$, we have $P_{m+1}(\lambda b+1) \le x^*$, and $P_{m+1}((x^*, \lambda b+1]) \subseteq (b, x^*]$, $Q_{m+1}(\lambda a-1) \ge y^*$ and $Q_{m+1}([\lambda a-1, y^*)) \subseteq [y^*, a)$.

By $a \in (-\mu_{m,l}, -\theta_{m,l}]$, $b \in [\theta_{m,l}, \mu_{m,l})$, it follows that

$$P_{m+2}^l \circ P_{m+1}(\lambda b+1) \le x^* \quad \text{and} \quad P_{m+2}^{l+1} \circ P_{m+1}(x^*) > x^*,$$

$$Q_{m+2}^l \circ P_{m+1}(\lambda a-1) \ge y^* \quad \text{and} \quad Q_{m+2}^{l+1} \circ P_{m+1}(y^*) < y^*.$$

Let $(x_0, y_0) \in (x^*, \lambda b+1] \times [\lambda a-1, y^*)$, and denote $\{(x_n, y_n)\}$ by the solution of (1) with the initial value (x_0, y_0). For $n \in N(1)$ and $k \in N(0,l)$, we have

$$(x_{m+1+(m+2)n}, y_{m+1+(m+2)n}) = (P_{m+2}^n \circ P_{m+1}(x_0), Q_{m+2}^n \circ Q_{m+1}(y_0)),$$

$$(x_{m+1+(m+2)k}, y_{m+1+(m+2)k}) \in (b, x^*] \times [y^*, a)$$

and

$$(x_{m+1+(m+2)(l+1)}, y_{m+1+(m+2)(l+1)}) \in (x^*, \lambda b+1] \times (x^*, \lambda b+1].$$

On the other hand, for $(x,y) \in (x^*, \lambda b+1] \times [\lambda a-1, y^*)$, we define

$$H(x,y) = \left(P_{m+2}^{l+1} \circ P_{m+1}(x), Q_{m+2}^{l+1} \circ Q_{m+1}(y)\right).$$

Then, for $(x_0, y_0) \in (x^*, \lambda b+1] \times [\lambda a-1, y^*)$, it follows that

$$(x_{(m+2)(l+2)-1}, y_{(m+2)(l+2)-1}) = H(x_0, y_0).$$

Obviously, there exists a $(\bar{x}_0, \bar{y}_0) \in (x^*, \lambda b+1] \times [\lambda a-1, y^*)$ such that, for $(x,y) \in (x^*, \lambda b+1] \times [\lambda a - 1y^*)$,

$$\lim_{n\to\infty} H^{(n)}(x,y) = (\bar{x}_0, \bar{y}_0)$$

where $(\bar{x}_0, \bar{y}_0)$ is the unique fixed point of H. Therefore, the solution $\{(\bar{x}_n, \bar{y}_n)\}$ of (1) with the initial value $(\bar{x}_0, \bar{y}_0) \in (x^*, \lambda b+1] \times [\lambda a-1, y^*)$ is periodic with minimal period $(m+2)(l+2)-1$. Moreover, for any solution $\{(x_n, y_n)\}$ of (1) with the initial value $(x_0, y_0) \in (x^*, \lambda b + 1] \times [\lambda a - 1, y^*)$, $\lim_{n\to\infty}(x_n - \bar{x}_n) = \lim_{n\to\infty}(y_n - \bar{y}_n) = 0$.

The proof of conclusion (3) is similar to the above, and is omitted. □

Finally, *we sketch the proof of Theorem 4* and only prove the conclusion (1). The others are similar to that of Theorem 3 and omitted.

Proof. By $-\frac{\lambda}{1-\lambda^2} < a < 0 < b < \frac{\lambda}{1-\lambda^2}$, we have

$$\lambda a - 1 < \frac{a}{\lambda} < a < \lambda a < \frac{a+1}{\lambda}, \quad \frac{b-1}{\lambda} < \lambda b < b < \frac{b}{\lambda} < \lambda b + 1.$$

Let $(x_0, y_0) \in (b, b/\lambda] \times [a/\lambda, a) \subseteq I_{31}$. Then $x_1 = \lambda x_0, y_1 = \lambda y_0, x_2 = \lambda^2 x_0 + 1, y_2 = \lambda^2 y_0 - 1$, where $(x_1, y_1) \in I_{22}, (x_2, y_2) \in (b, \lambda b + 1] \times [\lambda a - 1, a) \subseteq I_{31}$. Since $a \in (-\eta_m, -\zeta_m]$, $b \in [\zeta_m, \eta_m)$, we have

$$(x_n, y_n) \in \left(\frac{b}{\lambda}, \lambda b + 1\right] \times \left[\lambda a - 1, \frac{a}{\lambda}\right), \ n \in N(2, m),$$

$$(x_{m+1}, y_{m+1}) \in \left(b, \frac{b}{\lambda}\right] \times \left[\frac{a}{\lambda}, a\right), \ (x_{m+2}, y_{m+2}) \in I_{22}, \quad m \in N(2),$$

where $x_n = \lambda^{n-2}x_2 = \lambda^n x_0 + \lambda^{n-2}$, $y_n = \lambda^{n-2}y_2 = \lambda^n y_0 - \lambda^{n-2}$, $n \in N(2, m+1)$. For $l \in N(0)$, and $k \in N(2, m)$, repeating the above proceeding, we have

$$(x_{(m+1)l+k}, y_{(m+1)l+k}) \in \left(\frac{b}{\lambda}, \lambda b + 1\right] \times \left[\lambda a - 1, \frac{a}{\lambda}\right),$$

$$(x_{(m+1)l+1}, y_{(m+1)l+1}) \in I_{22}, \quad (x_{(m+1)l}, y_{(m+1)l}) \in \left(b, \frac{b}{\lambda}\right] \times \left[\frac{a}{\lambda}, a\right).$$

By means of (18), for $(x, y) \in (b, b/\lambda] \times [a/\lambda, a)$, we define

$$G_{m+1}(x, y) = ((f_2^{m-1} \circ f_1 \circ f_2)(x), (f_2^{m-1} \circ f_3 \circ f_2)(y)).$$

It yields that

$$G_{m+1}(x, y) = (\lambda^{m+1}x + \lambda^{m-1}, \lambda^{m+1}y - \lambda^{m-1}),$$

$$G^n_{m+1}(x, y) = \left(\lambda^{n(m+1)}x + \frac{\lambda^{m-1}(1 - \lambda^{n(m+1)})}{1 - \lambda^{m+1}}, \lambda^{n(m+1)}y - \frac{\lambda^{m-1}(1 - \lambda^{n(m+1)})}{1 - \lambda^{m+1}}\right)$$

and $\lim_{n\to\infty} G^n_{m+1}(x, y) = (\eta_m, -\eta_m)$. Moreover, for $(x_0, y_0) \in (b, b/\lambda] \times [a/\lambda, a)$,

$$(x_{(m+1)n}, y_{(m+1)n}) = G^n_{m+1}(x_0, y_0).$$

Obviously, $(\eta_m, -\eta_m)$ is the unique fixed point of G_{m+1} and the solution $\{(\bar{x}_n, \bar{y}_n)\}$ of (1.1) with the initial value $(\eta_m, -\eta_m)$ is periodic with minimal period $m + 1$. Moreover, for any solution $\{(x_n, y_n)\}$ of (1) with the initial value $(x_0, y_0) \in (b, b/\lambda] \times [a/\lambda, a)$, we have $\lim_{n\to\infty}(x_n - \bar{x}_n) = \lim_{n\to\infty}(y_n - \bar{y}_n) = 0$. □

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