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SECOND ORDER DUALITY IN VECTOR OPTIMIZATION OVER CONES

S.K. SUNEJA, SUNILA SHARMA* AND VANI

ABSTRACT. In this paper second order cone convex, second order cone pseudoconvex, second order strongly cone pseudoconvex and second order cone quasiconvex functions are introduced and their interrelations are discussed. Further a Mond Weir Type second order dual is associated with the Vector Minimization Problem and the weak and strong duality theorems are established under these new generalized convexity assumptions.

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1. Introduction

Convex functions are extremely important in Mathematical Programming because they are among the few functions for which optimality conditions and duality can be given. Various generalizations of convexity have been considered in literature. In 1987, Weir, Mond and Craven [12] introduced the class of cone convex functions. Cambini [2] introduced several classes of vector valued functions which are possible extensions of scalar generalized concavity. These classes are defined by using three order relations generated by a cone or the interior of a cone or the cone without the origin. The aim of this paper is to define second order cone convexity and its generalizations, where all the functions involved are twice differentiable.

Second order dual of the primal nonlinear programming problem was first formulated by Mangasarian [6] that involves second derivatives of the functions constituting the primal problem. He also derived the duality results for this pair of problems by using the inclusion condition. One advantage of second

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order duality is that if a feasible point in the primal is given and first order duality conditions do not apply, then we can use second order duality to provide a lower bound of the value of the primal programming problem. Mond [8] proved the second order duality theorems by making use of second order type convexity. Mahajan and Varatak [5] used the second order convexity of the Lagrangian to establish the second order duality results. Till now many authors have given second order duality results. Hanson [4] introduced second order type-1 functions and applied to second order duality theorems in Mathematical Programming. Bector, Chandra and Husain [1] extended the concept of an invex function to a second order invex function called binvex function. They introduced four models of second order duality for the minimax problem and established duality theorem for each of them under generalized binvexity assumption on the functions involved. Egudo and Hanson [3] gave second order duality results for multiobjective programs using proper efficiency and assuming that the functions satisfy some type of second order invexity. Mishra [7] defined second order type 1 function, second order quasi type 1 function and their generalizations and applied to second order duality results for several mathematical programs. Srivastava and Govil [10] formulated second order Mond-Weir type dual for a multiobjective nonlinear programming problem by defining second order (F, ρ, σ)-type 1 functions and their generalizations and established various duality results.

In this paper, we introduce second order cone convex, second order cone pseudoconvex, second order strongly cone pseudoconvex and second order cone quasiconvex functions and discuss the relations among these functions. Further we associate a Mond-Weir type second order dual to a Vector Minimization Problem over cones and obtain weak and strong duality theorems under these new concepts of second order cone convexity and its generalizations.

2. Notations and definitions

Let $K \subseteq \mathbb{R}^m$ be a closed convex pointed cone with nonempty interior and let $\operatorname{int} K$ denote the interior of K. The positive dual cone K^+ of K is defined as

$$K^+ = \{y^* \in \mathbb{R}^m \mid \langle y, y^* \rangle \geq 0, \text{for all } y \in K\}.$$

We now introduce the definitions of second order cone convex functions and their generalizations. Let f_i, $i = 1, 2, \ldots, m$ be twice continuously differentiable real valued functions defined on a nonempty open subset X of $\mathbb{R}^n$ and $f = (f_1, f_2, \ldots, f_m)$.

Definition 1. f is said to be *second order K-convex at $u \in X$ with respect to $p \in \mathbb{R}^n$* if for every $x \in X$,

$$\Big[f_1(x) - f_1(u) - (x-u)^T \nabla f_1(u) - (x-u)^T \nabla^2 f_1(u) p + \frac{1}{2} p^T \nabla^2 f_1(u) p, \ldots,$$

$$f_m(x) - f_m(u) - (x-u)^T \nabla f_m(u) - (x-u)^T \nabla^2 f_m(u) p + \frac{1}{2} p^T \nabla^2 f_m(u) p\Big] \in K.$$

Now we give an example of a second order K-convex function.

Example 1. Let $K = \{x = (x_1, x_2) \mid x_1 \le 0, x_2 \ge x_1\}$ be a cone in $\mathbb{R}^2$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ as $f = (f_1, f_2)$, where

$$f_1(x_1, x_2) = x_1 - x_2^2, \quad f_2(x_1, x_2) = -x_2^2$$

$$\nabla f_1 = \begin{pmatrix} 1 \\ -2x_2 \end{pmatrix}, \qquad \nabla f_2 = \begin{pmatrix} 0 \\ -2x_2 \end{pmatrix},$$

$$\nabla^2 f_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \qquad \nabla^2 f_2 = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then f is second order K-convex at $u = (0,0)$ for any $p = (p_1, p_2) \in \mathbb{R}^2$ because

$$\Big[f_1(x) - f_1(u) - (x-u)^T \nabla f_1(u) - (x-u)^T \nabla^2 f_1(u) p + \frac{1}{2} p^T \nabla^2 f_1(u) p,$$
$$f_2(x) - f_2(u) - (x-u)^T \nabla f_2(u) - (x-u)^T \nabla^2 f_2(u) p + \frac{1}{2} p^T \nabla^2 f_2(u) p\Big]$$
$$= \quad (-(x_2 - p_2)^2, -(x_2 - p_2)^2) \in K.$$

Definition 2. f is said to be *second order K-pseudoconvex at $u \in X$ with respect to $p \in \mathbb{R}^n$* if for every $x \in X$

$$[-(x-u)^T (\nabla f_1(u) + \nabla^2 f_1(u) p), \ldots, -(x-u)^T (\nabla f_m(u) + \nabla^2 f_m(u) p)] \notin \operatorname{int} K$$
$$\Longrightarrow \Big[-\big(f_1(x) - f_1(u) + \frac{1}{2} p^T \nabla^2 f_1(u) p\big), \ldots, -\big(f_m(x) - f_m(u) + \frac{1}{2} p^T \nabla^2 f_m(u) p\big)\Big] \notin \operatorname{int} K.$$

Remark 1. Every second order K-convex function at a point is second order K-pseudoconvex at the same point. But the converse is not true as can be seen from the following example.

Example 2. Let $K = \{x = (x_1, x_2) \mid x_2 \le 0, x_1 \ge x_2\}$ be a cone in $\mathbb{R}^2$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ as $f = (f_1, f_2)$, where

$$f_1(x_1, x_2) = -x_1^3, \qquad f_2(x_1, x_2) = -x_1^3 - x_2$$

$$\nabla f_1 = \begin{pmatrix} -3x_1^2 \\ 0 \end{pmatrix}, \qquad \nabla f_2 = \begin{pmatrix} -3x_1^2 \\ -1 \end{pmatrix},$$

$$\nabla^2 f_1 = \begin{pmatrix} -6x_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \nabla^2 f_2 = \begin{pmatrix} -6x_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then f is second order K-pseudoconvex at $u = (0,0)$ for any $p = (p_1, p_2) \in \mathbb{R}^2$. But f is not second order K-convex at $u = (0,0)$, because for $x = (-1, 1)$

$$\Big[f_1(x) - f_1(u) - (x-u)^T \nabla f_1(u) - (x-u)^T \nabla^2 f_1(u) p + \frac{1}{2} p^T \nabla^2 f_1(u) p,$$
$$f_2(x) - f_2(u) - (x-u)^T \nabla f_2(u) - (x-u)^T \nabla^2 f_2(u) p + \frac{1}{2} p^T \nabla^2 f_2(u) p\Big]$$
$$= \quad (-x_1^3, -x_1^3) = (1,1) \notin K.$$

Definition 3. f is said to be *second order strongly K-pseudoconvex at $u \in X$ with respect to $p \in \mathbb{R}^n$* if for every $x \in X$

$$[-(x-u)^T(\nabla f_1(u) + \nabla^2 f_1(u)p), \ldots, -(x-u)^T(\nabla f_m(u) + \nabla^2 f_m(u)p)] \notin \operatorname{int} K$$

$$\Longrightarrow \left[f_1(x) - f_1(u) + \frac{1}{2}p^T\nabla^2 f_1(u)p, \ldots, f_m(x) - f_m(u) + \frac{1}{2}p^T\nabla^2 f_m(u)p\right] \in K\,.$$

Remark 2. Every second order strongly K-pseudoconvex function at a point is second order K-pseudoconvex at the same point. But the converse is not true as can be seen from Example 2, where f is second order K-pseudoconvex at $u = (0,0)$, but not second order strongly K-pseudoconvex at $u = (0,0)$, because for $x = (-2,1)$

$$\begin{aligned}&[-(x-u)^T(\nabla f_1(u) + \nabla^2 f_1(u)p), -(x-u)^T(\nabla f_2(u) + \nabla^2 f_2(u)p)]\\ &\quad = (0, x_2) = (0,1) \notin \operatorname{int} K\,,\end{aligned}$$

and

$$\begin{aligned}&\left[f_1(x) - f_1(u) + \frac{1}{2}p^T\nabla^2 f_1(u)p, f_2(x) - f_2(u) + \frac{1}{2}p^T\nabla^2 f_2(u)p\right]\\ &\quad = (-x_1^3, -x_1^3 - x_2) = (8,7) \notin K\,.\end{aligned}$$

Definition 4. f is said to be *second order K-quasiconvex at $u \in X$ with respect to $p \in \mathbb{R}^n$* if for every $x \in X$

$$\left[f_1(x) - f_1(u) + \frac{1}{2}p^T\nabla^2 f_1(u)p, \ldots, f_m(x) - f_m(u) + \frac{1}{2}p^T\nabla^2 f_m(u)p\right] \notin \operatorname{int} K$$

$$\Longrightarrow [-(x-u)^T(\nabla f_1(u) + \nabla^2 f_1(u)p), \ldots, -(x-u)^T(\nabla f_m(u) + \nabla^2 f_m(u)p)] \in K\,.$$

Remark 3. If $K = R_+$ or R_-, then every second order K-convex function at a point is second order K-quasiconvex at the same point. Otherwise every second order K-convex function at a point may not be second order K-quasiconvex at the same point, as can be seen from the following example.

Example 3. Let $K = \{x = (x_1, x_2) \,|\, x_1 \le 0, x_2 \ge x_1\}$ be a cone in $\mathbb{R}^2$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ as $f = (f_1, f_2)$, where

$$f_1(x_1,x_2) = -x_1^2, \quad f_2(x_1,x_2) = x_2, \quad \nabla f_1 = \begin{pmatrix}-2x_1\\ 0\end{pmatrix}, \quad \nabla f_2 = \begin{pmatrix}0\\ 1\end{pmatrix},$$

$$\nabla^2 f_1 = \begin{pmatrix}-2 & 0\\ 0 & 0\end{pmatrix}, \quad \nabla^2 f_2 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Then

$$\begin{aligned}&\Big[f_1(x) - f_1(u) - (x-u)^T\nabla f_1(u) - (x-u)^T\nabla^2 f_1(u)p + \frac{1}{2}p^T\nabla^2 f_1(u)p,\\ &\quad f_2(x) - f_2(u) - (x-u)^T\nabla f_2(u) - (x-u)^T\nabla^2 f_2(u)p + \frac{1}{2}p^T\nabla^2 f_2(u)p\Big]\\ &\quad = (-(x_1-p_1)^2, 0) \in K \quad \text{for any } p = (p_1,p_2) \in \mathbb{R}^2\,.\end{aligned}$$

Therefore f is second order K-convex at $u = (0,0)$ but f is not second order K-quasiconvex at $u = (0,0)$, because for $x = (1,-3)$ and $p = (1,2)$

$$\begin{aligned}&\left[f_1(x) - f_1(u) + \frac{1}{2}p^T\nabla^2 f_1(u)p, f_2(x) - f_2(u) + \frac{1}{2}p^T\nabla^2 f_2(u)p\right]\\ &= \quad (-(x_1^2+1), x_2) = (-2,-3) \notin \operatorname{int} K\,,\end{aligned}$$

and

$$\begin{aligned}&[-(x-u)^T(\nabla f_1(u) + \nabla^2 f_1(u)p), -(x-u)^T(\nabla f_2(u) + \nabla^2 f_2(u)p)]\\ &= \quad (2x_1, -x_2) = (2,3) \notin K\,.\end{aligned}$$

Next example shows that there exist functions which are second order K-quasiconvex at a point but not second order K-convex at the same point.

Example 4. Let $K = \{x = (x_1,x_2)\,|\,x_1 \le 0, x_1 \ge x_2\}$ be a cone in $\mathbb{R}^2$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ as $f = (f_1, f_2)$, where

$$f_1(x_1,x_2) = x_1^4(x_2^2+1)\,, \qquad f_2(x_1,x_2) = x_1 + x_1^3$$

$$\nabla f_1 = \begin{pmatrix} 4x_1^3(x_2^2+1) \\ 2x_1^4 x_2 \end{pmatrix}, \qquad \nabla f_2 = \begin{pmatrix} 1+3x_1^2 \\ 0 \end{pmatrix},$$

$$\nabla^2 f_1 = \begin{pmatrix} 12x_1^2(x_2^2+1) & 8x_1^3 x_2 \\ 8x_1^3 x_2 & 2x_1^4 \end{pmatrix}, \quad \nabla^2 f_2 = \begin{pmatrix} 6x_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then f is second order K-quasiconvex at $u = (0,0)$ for any $p = (p_1,p_2) \in \mathbb{R}^2$. But f is not second order K-convex at $u = (0,0)$, because for $x = (1/2, 0)$

$$\begin{aligned}&\Big[f_1(x) - f_1(u) - (x-u)^T\nabla f_1(u) - (x-u)^T\nabla^2 f_1(u)p + \frac{1}{2}p^T\nabla^2 f_1(u)p,\\ &f_2(x) - f_2(u) - (x-u)^T\nabla f_2(u) - (x-u)^T\nabla^2 f_2(u)p + \frac{1}{2}p^T\nabla^2 f_2(u)p\Big]\\ &= \quad (x_1^4(x_2^2+1), x_1^3) = \left(\frac{1}{16}, \frac{1}{8}\right) \notin K\,.\end{aligned}$$

Remark 4. Note that there exist functions which are second order K-pseudoconvex at a point but not second order K-quasiconvex at the same point. For example, the function f considered in Example 2, is second order K-pseudoconvex at $u = (0,0)$ but f is not second order K-quasiconvex at $u = (0,0)$, because for $x = (-3,1)$

$$\begin{aligned}&\left[f_1(x) - f_1(u) + \frac{1}{2}p^T\nabla^2 f_1(u)p, f_2(x) - f_2(u) + \frac{1}{2}p^T\nabla^2 f_2(u)p\right]\\ &= \quad (-x_1^3, -x_1^3 - x_2) = (27,26) \notin \operatorname{int} K\,,\end{aligned}$$

and

$$\begin{aligned}&\Big[-(x-u)^T(\nabla f_1(u) + \nabla^2 f_1(u)p), -(x-u)^T(\nabla f_2(u) + \nabla^2 f_2(u)p)]\\ &= \quad (0, x_2) = (0,1) \notin K\,.\end{aligned}$$

Next example shows that there exist functions which are second order K-quasiconvex at a point but not second order K-pseudoconvex at the same point.

Example 5. Let $K = R_+ \times R_+$. Define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ as $f = (f_1, f_2)$, where

$$f_1(x_1, x_2) = (x_1 + x_2)^3, \qquad f_2(x_1, x_2) = x_1 + x_2 + 1$$

$$\nabla f_1 = \begin{pmatrix} 3(x_1 + x_2)^2 \\ 3(x_1 + x_2)^2 \end{pmatrix}, \qquad \nabla f_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\nabla^2 f_1 = \begin{pmatrix} 6(x_1 + x_2) & 6(x_1 + x_2) \\ 6(x_1 + x_2) & 6(x_1 + x_2) \end{pmatrix}, \quad \nabla^2 f_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

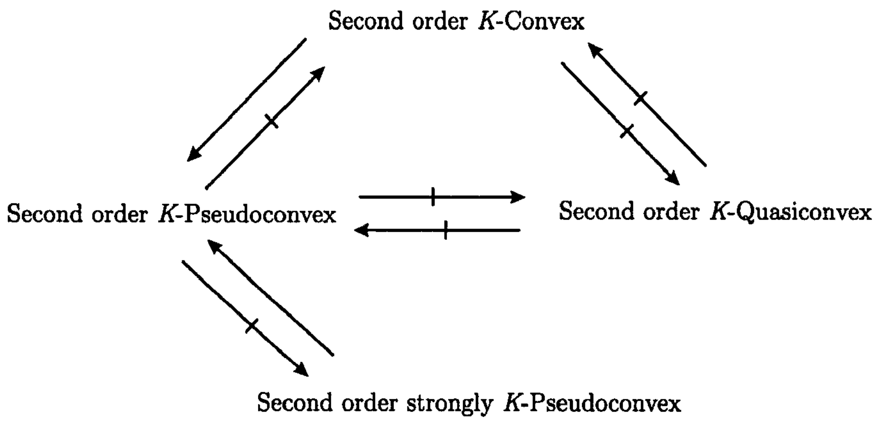
Then f is second order K-quasiconvex at $u = (0,0)$ for any $p = (p_1, p_2) \in \mathbb{R}^2$. But f is not second order K-pseudoconvex at $u = (0,0)$, because for $x = (1,-2)$

$$\begin{aligned} &[-(x-u)^T(\nabla f_1(u) + \nabla^2 f_1(u)p), -(x-u)^T(\nabla f_2(u) + \nabla^2 f_2(u)p)] \\ &= \quad (0, -(x_1 + x_2)) = (0,1) \notin \operatorname{int} K \end{aligned}$$

and

$$\begin{aligned} &\left[-\left(f_1(x) - f_1(u) + \frac{1}{2} p^T \nabla^2 f_1(u) p \right), -\left(f_2(x) - f_2(u) + \frac{1}{2} p^T \nabla^2 f_2(u) p \right) \right] \\ &= \quad -((x_1 + x_2)^3, x_1 + x_2) = (1,1) \in \operatorname{int} K . \end{aligned}$$

Interrelations between second order K-convex functions and its generalizations:



Remark 5. If $p = 0$, then a second order K-convex function at u is said to be K-convex at u. Similarly we have the notions of K-pseudoconvex, strongly K-pseudoconvex and K-quasiconvex functions at u if $p = 0$, in the definitions of second order K-pseudoconvex, second order strongly K-pseudoconvex and second order K-quasiconvex respectively.

Consider the following vector minimization problem

$$\text{(VP)} \qquad K\text{-Minimize } f(x)$$
$$\text{subject to } -g(x) \in Q$$

where f_i, g_j, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, \ell$ are real valued twice differentiable functions defined on a nonempty open subset X of $\mathbb{R}^n$ and $f = (f_1, f_2, \ldots, f_m)$ and $g = (g_1, g_2, \ldots, g_\ell)$. Let $X_0 = \{x \in X \mid -g(x) \in Q\}$ denote the feasible set of (VP).

Definition 5. A point $\bar{x} \in X_0$ is called a *weak minimum of(VP)* if for all $x \in X_0$

$$f(\bar{x}) - f(x) \notin \operatorname{int} K .$$

3. Duality

Mond and Weir [9] associated the following first order dual with problem (VP)

(D1) K-Maximize $(f_1(u), \ldots, f_m(u))$

$$\text{subject to } (x-u)^T \nabla(\tau^T f(u) + \lambda^T g(u)) \geq 0 \quad \text{for all } x \in X_0$$

$$\lambda^T g(u) \geq 0,$$

where $\tau \in K^+$, $\lambda \in Q^+$, $u \in X$.

We associate the following second order dual problem with (VP).

(D2) K-Maximize $\left(f_1(u) - \frac{1}{2}p^T \nabla^2 f_1(u)p, \ldots, f_m(u) - \frac{1}{2}p^T \nabla^2 f_m(u)p\right)$

$$\text{subject to } (x-u)^T \nabla(\tau^T f(u) + \lambda^T g(u)) + (x-u)^T \nabla^2(\tau^T f(u) + \lambda^T g(u))p \geq 0, \text{ for all } x \in X_0 . \quad (3.1)$$

$$\lambda^T g(u) - \frac{1}{2}p^T \nabla^2(\lambda^T g)(u)p \geq 0, \quad (3.2)$$

where $0 \neq \tau \in K^+$, $\lambda \in Q^+$, $p \in \mathbb{R}^n$, $u \in X$.

Now we will establish the weak duality relation between feasible points of the primal (VP) and the second order dual (D2).

Theorem 1 (Weak Duality). *If x is feasible for (VP) and (u, τ, λ, p) is feasible for (D2), f is second order K-convex at $u \in X$ and g is second order Q-convex at $u \in X$, then*

$$\left[f_1(u) - \frac{1}{2}p^T \nabla^2 f_1(u)p - f_1(x), \ldots, f_m(u) - \frac{1}{2}p^T \nabla^2 f_m(u)p - f_m(x)\right] \notin \operatorname{int} K.$$

Proof. Suppose that

$$\left[f_1(u) - \frac{1}{2}p^T \nabla^2 f_1(u)p - f_1(x), \ldots, f_m(u) - \frac{1}{2}p^T \nabla^2 f_m(u)p - f_m(x)\right] \in \operatorname{int} K. \quad (3.3)$$

Since f is second order K-convex and g is second order Q-convex at $u \in X$, we get

$$\Big[f_1(x) - f_1(u) - (x-u)^T \nabla f_1(u) - (x-u)^T \nabla^2 f_1(u)p + \frac{1}{2}p^T \nabla^2 f_1(u)p, \ldots,$$

$$f_m(x) - f_m(u) - (x-u)^T \nabla f_m(u) - (x-u)^T \nabla^2 f_m(u)p + \frac{1}{2}p^T \nabla^2 f_m(u)p\Big] \in K, \quad (3.4)$$

and

$$\Big[g_1(x) - g_1(u) - (x-u)^T\nabla g_1(u) - (x-u)^T\nabla^2 g_1(u)p + \frac{1}{2}p^T\nabla^2 g_1(u)p, \ldots,$$

$$g_\ell(x) - g_\ell(u) - (x-u)^T\nabla g_\ell(u) - (x-u)^T\nabla^2 g_\ell(u)p + \frac{1}{2}p^T\nabla^2 g_\ell(u)p\Big] \in Q, \quad (3.5)$$

Adding (3) and (4), we obtain

$$[-(x-u)^T\nabla f_1(u) - (x-u)^T\nabla^2 f_1(u)p, \ldots,$$
$$-(x-u)^T\nabla f_m(u) - (x-u)^T\nabla^2 f_m(u)p] \in \operatorname{int} K$$

Since $0 \neq \tau \in K^+$, we get

$$(x-u)^T\nabla(\tau^T f)(u) + (x-u)^T\nabla^2(\tau^T f)(u)p < 0.$$

Now feasibility of (u, τ, λ, p) for (D2) gives

$$(x-u)^T\nabla(\lambda^T g)(u) + (x-u)^T\nabla^2(\lambda^T g)(u)p > 0. \quad (3.6)$$

From (5), since $\lambda \in Q^+$, we obtain

$$\lambda^T g(x) - \lambda^T g(u) - (x-u)^T\nabla(\lambda^T g)(u) - (x-u)^T\nabla^2(\lambda^T g)(u)p$$
$$+\frac{1}{2}p^T\nabla^2(\lambda^T g)(u)p \geq 0. \quad (3.7)$$

Adding (6) and (7), we get

$$\lambda^T g(x) - \lambda^T g(u) + \frac{1}{2}p^T\nabla^2(\lambda^T g)(u)p > 0,$$

which is equivalent to

$$\lambda^T g(u) - \frac{1}{2}p^T\nabla^2(\lambda^T g)(u)p < \lambda^T g(x) \leq 0.$$

This contradicts (2). Hence

$$\left[f_1(u) - \frac{1}{2}p^T\nabla^2 f_1(u)p - f_1(x), \ldots, f_m(u) - \frac{1}{2}p^T\nabla^2 f_m(u)p - f_m(x)\right] \notin \operatorname{int} K.$$

□

Theorem 2 (Weak Duality). *If x is feasible for* (VP) *and* (u, τ, λ, p) *is feasible for* (D2) *and f is second order K-pseudoconvex and g is second order Q-quasiconvex at $u \in X$, then*

$$\left[f_1(u) - \frac{1}{2}p^T\nabla^2 f_1(u)p - f_1(x), \ldots, f_m(u) - \frac{1}{2}p^T\nabla^2 f_m(u)p - f_m(x)\right] \notin \operatorname{int} K.$$

Proof. Since x is feasible for (VP) and (u, τ, λ, p) is feasible for (D2), we get

$$\lambda^T g(x) - \lambda^T g(u) + \frac{1}{2}p^T\nabla^2(\lambda^T g)(u)p \leq 0. \quad (3.8)$$

Now we claim that

$$(x-u)^T(\nabla(\lambda^T g)(u) + \nabla^2(\lambda^T g)(u)p) \leq 0. \quad (3.9)$$

If $\lambda = 0$, then (9) trivially holds. If $\lambda \neq 0$, then from (8), we get

$$\left[g_1(x) - g_1(u) + \frac{1}{2}p^T\nabla^2 g_1(u)p, \ldots, g_\ell(x) - g_\ell(u) + \frac{1}{2}p^T\nabla^2 g_\ell(u)p\right] \notin \operatorname{int} Q.$$

Now g is second order Q-quasiconvex at $u \in X$, therefore we get

$[-(x-u)^T(\nabla g_1(u) + \nabla^2 g_1(u)p), \ldots, -(x-u)^T(\nabla g_\ell(u) + \nabla^2 g_\ell(u)p)] \in Q,$

which implies that (9) holds. On using (1) in (9), we get

$(x-u)^T(\nabla(\tau^T f)(u) + \nabla^2(\tau^T f)(u)p) \geq 0\,.$

Now $0 \neq \tau \in K^+$ gives that

$[-(x-u)^T(\nabla f_1(u) + \nabla^2 f_1(u)p), \ldots, -(x-u)^T(\nabla f_m(u) + \nabla^2 f_m(u)p)] \notin \operatorname{int} K.$

Since f is second order K-pseudoconvex at $u \in X$, therefore it follows that

$$\left[f_1(u) - f_1(x) - \frac{1}{2}p^T\nabla^2 f_1(u)p, \ldots, f_m(u) - f_m(x) - \frac{1}{2}p^T\nabla^2 f_m(u)p\right] \notin \operatorname{int} K. \ \square$$

We shall be using the following constraint qualifications for proving the Strong Duality Theorems for (D2).

Definition 6. The function g is said to satisfy *Slater's type constraint qualification at $\bar{x}$*

(CQ1) if g is Q-convex at $\bar{x}$ and there exists $x^* \in X$ such that $-g(x^*) \in \operatorname{int} Q$,

(CQ2) if g is strongly Q-pseudoconvex at $\bar{x}$ and there exists $x^* \in X$ such that $-g(x^*) \in \operatorname{int} Q$.

In order to prove the strong duality theorem, we will make use of the following lemma which gives generalized form of Fritz John optimality conditions for a point to be a weak minimum of (VP), established by Suneja, Aggarwal and Davar [11].

Lemma 1. *If $\bar{x}$ is a weak minimum of (VP), then there exist $\bar{\tau} \in K^+$, $\bar{\lambda} \in Q^+$ not both zero such that $(x-\bar{x})^T(\bar{\tau}^T\nabla f(\bar{x}) + \bar{\lambda}^T\nabla g(\bar{x})) \geq 0$, for all $x \in X$ and $\bar{\lambda}^T g(\bar{x}) = 0$.*

Theorem 3 (Strong Duality). *Let $\bar{x}$ be a weak minimum for (VP) at which the Slater's type constraint qualification (CQ1) is satisfied. Then there exist $0 \neq \bar{\tau} \in K^+$ and $\bar{\lambda} \in Q^+$ such that $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is feasible for the second order dual problem (D2) and both the objective functions are equal. Moreover if f is second order K-convex and g is second order Q-convex at $\bar{x} \in X$, then $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is weak maximum for (D2).*

Proof. Since $\bar{x}$ is a weak minimum of (VP), by Lemma 1, there exist $\bar{\tau} \in K^+$, $\bar{\lambda} \in Q^+$ not both zero such that

$$(x-\bar{x})^T(\bar{\tau}^T\nabla f(\bar{x}) + \bar{\lambda}^T\nabla g(\bar{x})) \geq 0, \quad \text{for all } x \in X, \tag{3.10}$$

and

$$\bar{\lambda}^T g(\bar{x}) = 0\,. \tag{3.11}$$

We assert that $\bar{\tau} \neq 0$. On the contrary suppose that $\bar{\tau} = 0$, then $\bar{\lambda} \neq 0$ and from (10), we get

$$(x-\bar{x})^T(\bar{\lambda}^T\nabla g(\bar{x})) \geq 0, \qquad \text{for all } x \in X\,. \tag{3.12}$$

Since the Slater's type constraint qualification (CQ1) is satisfied, it follows that there exists $x^* \in X$ such that $-g(x^*) \in \operatorname{int} Q$.

Now $0 \neq \bar{\lambda} \in Q^+$ gives that

$$\bar{\lambda}^T g(x^*) < 0. \tag{3.13}$$

Also since g is Q-convex at $\bar{x}$, we get

$$[g_1(x) - g_1(\bar{x}) - (x-\bar{x})^T \nabla g_1(\bar{x}), \ldots, g_\ell(x) - g_\ell(\bar{x}) - (x-\bar{x})^T \nabla g_\ell(\bar{x})] \in Q,$$

for all $x \in X$.

Since $\bar{\lambda} \in Q^+$, we get $\bar{\lambda}^T g(x) - \bar{\lambda}^T g(\bar{x}) - (x-\bar{x})^T \bar{\lambda}^T \nabla g(\bar{x}) \geq 0$ for all $x \in X$. Using (11) and (12), we obtain $\bar{\lambda}^T g(x) \geq 0$ for all $x \in X$. In particular for $x = x^*$, $\bar{\lambda}^T g(x^*) \geq 0$, which contradicts (13). Hence $\bar{\tau} \neq 0$.

Thus (10) and (11) give that $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is a feasible solution for (D2). Both the objective functions coincide as $\bar{p} = 0$. Suppose that $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is not a weak maximum for (D2), then there exists a feasible solution (u, p, τ, λ) of (D2) such that

$$\Big[f_1(u) - \frac{1}{2}p^T \nabla^2 f_1(u)p - f_1(\bar{x}) + \frac{1}{2}\bar{p}^T \nabla^2 f_1(\bar{x})\bar{p}, \ldots, f_m(u) - \frac{1}{2}p^T \nabla^2 f_m(u)p$$
$$-f_m(\bar{x}) + \frac{1}{2}\bar{p}^T \nabla^2 f_m(\bar{x})\bar{p}\Big] \in \operatorname{int} K.$$

Since $\bar{p} = 0$, we get

$$\Big[f_1(u) - \frac{1}{2}p^T \nabla^2 f_1(u)p - f_1(\bar{x}), \ldots, f_m(u) - \frac{1}{2}p^T \nabla^2 f_m(u)p - f_m(\bar{x})\Big] \in \operatorname{int} K$$

which contradicts Weak Duality Theorem 1. Hence $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is a weak maximum for (D2). □

Theorem 4 (Strong Duality). *Let $\bar{x}$ be a weak minimum for* (VP) *at which the Slater's type constraint qualification* (CQ2) *is satisfied. Then there exist $0 \neq \bar{\tau} \in K^+$ and $\bar{\lambda} \in Q^+$ such that $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is feasible for the second order dual problem* (D2) *and both the objective functions are equal. Moreover if f is second order K-pseudoconvex and g is second order Q-quasiconvex at $\bar{x} \in X$, then $(\bar{x}, \bar{p} = 0, \bar{\tau}, \bar{\lambda})$ is weak maximum for* (D2).

Proof. Proceeding on the same lines as in the proof of Theorem 3, we get that (10) and (11) hold. We assert that $\bar{\tau} \neq 0$. On the contrary suppose that $\bar{\tau} = 0$, then $\bar{\lambda} \neq 0$ and from (10), we get $(x-\bar{x})^T(\bar{\lambda}^T \nabla g(\bar{x})) \geq 0$ for all $x \in X$.

Since $0 \neq \bar{\lambda} \in Q^+$, we get $[-(x-\bar{x})^T \nabla g_1(\bar{x}), \ldots, -(x-\bar{x})^T \nabla g_\ell(\bar{x})] \notin \operatorname{int} Q$ for all $x \in X$. Now since Slater's type constraint qualification (CQ2) holds, therefore g is strongly Q-pseudoconvex at $\bar{x}$, so we get $g(x) - g(\bar{x}) \in Q$ for all $x \in X$, which gives that

$$\bar{\lambda}^T (g(x) - g(\bar{x})) \geq 0, \qquad \text{for all } x \in X. \tag{3.14}$$

Since the Slater's type constraint qualification (CQ2) is satisfied at $\bar{x}$, it follows that there exists $x^* \in X$ such that $-g(x^*) \in \operatorname{int} Q$, which gives that $\bar{\lambda}^T g(x^*) < 0$. Using (11), we get $\bar{\lambda}^T (g(x^*) - g(\bar{x})) < 0$, which contradicts (14). Hence $\bar{\tau} \neq 0$. Rest of the proof follows on the lines of Theorem 3. □

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S.K. Suneja is a Reader in the Department of Mathematics, Miranda House, University of Delhi, India. She completed her M.Sc. in 1969 and Ph.D. in 1984 from the University of Delhi. She has a total of fifty publications in various journals of repute. Her areas of interest include generalized convexity and optimization.

Department of Mathematics, Miranda House, University of Delhi, Delhi-110007, India
e-mail: `surjeetsuneja@gmail.com`

Sunila Sharma is a Reader in the Department of Mathematics, Miranda House, University of Delhi, India She completed her M.A. in 1987 and Ph.D. in 1999 from the University of Delhi. Her areas of interest includes generalized convexity and optimization.

Department of Mathematics, Miranda House, University of Delhi, Delhi-110007, India

Vani is Senior Lecturer in the Department of Mathematics, Satyawati College, University of Delhi, India. She completed her M.Sc. in 1994 and M.Phil. in 1996 from the University of Delhi. Recently, she has submitted her Ph.D. thesis in Mathematical Programming at the University of Delhi. Her areas of interest includes generalized convexity and optimization.

Department of Mathematics, Satyawati College, University of Delhi, Ashok Vihar, Delhi-110052, India
e-mail: `vani5@rediffmail.com`