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RUIN PROBABILITIES IN THE RISK MODEL WITH TWO COMPOUND BINOMIAL PROCESSES

MAO-JUN ZHANG*, JIANG-XIA NAN AND SEN WANG

ABSTRACT. In this paper, we consider an insurance risk model governed by a compound Binomial arrival claim process and by a compound Binomial arrival premium process. Some formulas for the probabilities of ruin and the distribution of ruin time are given, we also prove the integral equation of the ultimate ruin probability and obtain the Lundberg inequality by the discrete martingale approach.

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1. Introduction

Insurance companies are in the business of risk because they must pool together risks faced by individuals or companies, who in the event of a loss are compensated by the insurer to reduce the financial burden. In its simplest form, when certain events occur, an insurance contract will provide the policyholder the right to claim all or a portion of the loss. In exchange for this entitlement, the policyholder pays a specified amount called the premium and the insurer is obligated to honor its promises when events come out.

In order to ensure that it will be able to pay its promised obligations. The company generally accumulate surplus from possible excesses of premiums collected over claim amounts paid during long periods of time. The surplus process studied in classical risk theory is a very important stochastic framework for understanding the company's capital or surplus over time.The company's surplus at time t is given by

$$U(t) = u_0 + ct - s(t), \tag{1}$$

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where u_0 is an initial surplus, c is the constant premium rate and $s(t)$ is the aggregate claims paid up to the time t, which is a compound poisson process.

A quantity of interest here is usually the so-called probability of ruin which provides a measure of how certain the company will be able to support its book of business. The time to ruin is defined to be the first time surplus in (1) becomes negative and is therefore

$$T = \inf\{t : U(t) < 0\},$$

if $U(t) \geq 0$ for all $t > 0$, that is the surplus never reaches zero, then we define $T = \infty$. This enable us to define ruin probabilities. The finite-horizon probability of ruin is

$$\psi(u_0, t) = P(T < t | U_0 = u_0),$$

and the infinite-horizon probability of ruin is

$$\psi(u_0) = P(T < \infty | U_0 = u_0),$$

this is more often called the ultimate probability of ruin. The surplus process with probabilities of ruin have been extensively studied in the actuarial literature. See Gerber(1979). Dufresne and Gerber(1988) gave rise to the compound binomial risk model, in which the claim process is assumed be a compound Binomial process as follows

$$U_n = u + cn - \sum_{i=1}^{N(n)} X_i,$$

where u is an initial surplus, c is the constant premium rate, and $N(n)$ follows a Binomial process which expresses the number of claims. In the present studies, many people focus exclusively on generalization which the occurrence of the claims may be described by a more general point process than a poisson process, such as a renewal process and a Cox process and a stationary process. See Grandell(1991)and Asmussen(2000). But there are so many kinds of premium policies in the insurance company that the constant premium rate is not satisfied with the actual situation of the company. So we consider the fact that the process of premiums received is also a compound binomial process. As we shall point out below.

The remainder of this paper is organized as follows. Our model is given in section 2, and main results which composed of the probability of eventual ruin, the finite-horizon probability of ruin, the distribution of ruin time, the distribution of surplus immediately before ruin and Lundlerg Inequality are presented in section 3.

2. Model

Let $(\Omega, \Re, P)$ be a complete probability space, we consider a discrete-time risk model which composed of two stochastic processes.

2.1. The premium process

Supposed that the number of insurance policies is governed by a Binomial process $\{M(n), n = 0, 1, 2, \cdots\}$. In any time period, the probability of a insurance policy is p_1 and the probability of no insurance policy is $1 - p_1$. The occurrences of a insurance policy in different periods are independent events. The individual premium amounts $\{Y_n, n = 0, 1, 2, \cdots\}$ are independent, identically distributed and positive value random variables, which are independent of the Binomial process $\{M(n), n = 0, 1, 2, \cdots\}$. Let $Y = Y_1$, $G(y) = P(Y \leq y)$ and $\gamma = E(Y) < \infty$. so that the premium process is expressed as

$$C_n = Y_1 + Y_2 + .. + Y_{M(n)},$$

for $n = 0, 1, 2, \cdots$.

2.2. The claim process

By a similar approach, the aggregate claims process is represented as

$$S_n = X_1 + X_2 + \cdots + X_{N(n)},$$

for $n = 0, 1, 2, \cdots$, where $N(n)$ is the number of insurance claims over the period n. In any time period, the probability of a claim is p_2 and the probability of no claim is $1 - p_2$. And the individual claim amount $\{X_n, n = 0, 1, 2 \cdots\}$ are commonly assumed to be independent and identically distributed and positive value random variables. Let $X = X_1$, $F(x) = P(X \leq x)$ and $\mu = E(X) < \infty$. So the insurance company's surplus at time n is given by

$$U_n = u + C_n - S_n, \tag{2}$$

for $n = 0, 1, 2, \cdots$, where u is an initial surplus in the company.

2.3. Assumptions

Assumptions in the paper is given as follows

(1) $U_n = u - V_n$, where $V_n = S_n - C_n$,

(2) $E[C_n] > E[S_n]$,

(3) S_n is independent of C_n, and $N(n)$ is independent of $M(n)$,

(4) there exists a positive solution of equation $E[r^{V_1}] = 1$, the solution is defined as the adjustment coefficient.

3. Main results

3.1 The probability of eventual ruin

We define the probability of eventual ruin as

$$\psi(u) = P\{T < \infty | U_0 = u\},$$

where u is an initial surplus and T is the time to ruin, which is the first time that the surplus in (2) becomes negative and is therefore

$$T = \inf\{n > 0, U_n < 0\}.$$

Theorem 1. *Let $\psi(u)$ be the probability of eventual ruin. Then for any $u > 0$, it follows the equation*

$$\begin{aligned}\delta_1\psi(u) = \quad & e(u) + p_2q_1\int_0^u \psi(u-x)dF(x) + p_1q_2\int_0^\infty \psi(u+y)dG(y) \\ & +\delta_2\left(\int\int_{x-y<u} \psi(u+y-x)dF(x)G(y) + \int\int_{x-y>u} dF(x)G(y)\right) \end{aligned} \quad (3)$$

where $e(u) = p_2q_1(1-F(u))$, $\delta_1 = 1 - q_1q_2$, $\delta_2 = p_1p_2$.

Proof. There exists the four possible cases for U_n in a small time interval $[0, \Delta]$ as follows

A_1. neither claim or premium occurs in $[0, \Delta]$,
A_2. one premium occurs, but no claim occurs in $[0, \Delta]$,
A_3. no premium occurs, but one claim occurs in $[0, \Delta]$,
A_4. both premium and claim occur in $[0, \Delta]$.

Since $N(n)$ is independent of $M(n)$, we have

$$P(A_1) = q_1q_2\ ,\ P(A_2) = p_1q_2\ ,\ P(A_3) = q_1p_2\ ,\ P(A_4) = p_1p_2.$$

By the law of total probability, we get

$$\psi(u) = P(T < \infty) = \sum_{i=1}^{4} P(T < \infty | A_i)P(A_i). \quad (4)$$

Moreover, one has

$$P(T < \infty|A_1)P(A_1) = P(T < \infty)|U_0 = u)P(A_1) = \psi(u)q_1q_2, \quad (5)$$

$$\begin{aligned} P(T < \infty|A_2)P(A_2) &= E\{P(T < \infty)|(U_1 = u + Y)I_{(Y\geq 0)}\}P(A_2) \\ &= E\{\psi(u+Y)I_{(Y\geq 0)}\}P(A_2) \\ &= p_1q_2\int_0^\infty \psi(u+y)dG(y), \end{aligned} \quad (6)$$

Let $B = \{0 \leq X \leq u\}$ and $B^- = \{X > u\}$ be random events. Then $P(T < \infty|A_3B^-) = 1$, A_3 is independent of B and B^-. So we have

$$\begin{aligned} P(T < \infty|A_3)P(A_3) &= P(T < \infty|A_3B)P(AB) + P(T < \infty|A_3B^-)P(A_3B^-) \\ &= E[I_{(T<\infty)}I_{(A_3B)}] + P(A_3B^-) \\ &= E\{E[I_{(T<\infty)}|X]I_{(A_3B)}\} + P(A_3B^-) \\ &= E\{P(T < \infty|U_2 = u - X)I_{(A_3B)}\} + P(A_3B^-) \\ &= E\{\psi(u-X)I_{(A_3B)}\} + P(A_3B^-) \\ &= P(A_3)E\{\psi(u-X)I_{(B)}\} + P(A_3)P(B^-) \\ &= q_1p_2\left\{\int_0^u \psi(u-x)dF(x) + (1-F(u))\right\}. \end{aligned} \quad (7)$$

Let $Q = \{X - Y \le u\}$ and $Q^- = \{X - Y > u\}$ be random events. Then $P(T < \infty | A_4 Q^-) = 1$, A_4 is independent of Q and Q^-. So we have

$$\begin{aligned} P(T<\infty|A_4)P(A_4) &= P(T<\infty|A_4Q)P(A_4Q) + P(T<\infty|A_4Q^-)P(A_4Q^-) \\ &= E\{P(T<\infty|U_3 = u+Y-X)I_{(A_4Q)}\} + P(A_4Q^-) \\ &= P(A_4)E\{\psi(u+Y-X)I_{(Q)}\} + P(A_4)P(Q^-) \\ &= p_2p_1\Big\{\int\int_{x-y<u}\psi(u+y-x)dF(x)dG(y) \\ &\quad + \int\int_{x-y>u} dF(x)dG(y)\Big\}. \end{aligned} \tag{8}$$

From (4) to (8), we have

$$\begin{aligned} \psi(u) = \;& \psi(u)q_1q_2 + p_1q_2\textstyle\int_0^\infty \psi(u+y)dG(y) \\ &+q_1p_2\left\{\int_0^u \psi(u-x)dF(x) + (1-F(u))\right\} \\ &+p_1p_2\Big\{\int\int_{x-y<u}\psi(u+y-x)dF(x)dG(y) \\ &+\int\int_{x-y>u} dF(x)dG(y)\Big\}, \end{aligned} \tag{9}$$

the above equation can be written as

$$\begin{aligned} (1-q_1q_2)\psi(u) &= p_2q_1(1-F(u)) + q_1p_2\int_0^u \psi(u-x)dF(x) \\ &+p_1q_2\textstyle\int_0^\infty \psi(u+y)dG(y) \\ &+p_1p_2\Big\{\int\int_{x-y<u}\psi(u+y-x)dF(x)dG(y) \\ &+\int\int_{x-y>u} dF(x)dG(y)\Big\}. \end{aligned} \tag{10}$$

□

Example: Both exponential premium-size distribution and exponential claim-size distribution

This section presents a differential equation for $\psi(u)$ under the assumptions on both premium-size distributions and claim-size distributions.

Let $F(x)$ be the exponential distribution function of premium-sizes, $G(y)$ be the exponential distribution of claim-sizes. Then we have the following theorem.

Theorem 2. *Assume $F^{'}(x) = \alpha e^{\alpha x}$ $(x \ge 0, \alpha > 0)$, $G^{'}(y) = \beta e^{\beta y}$ $(y \ge 0, \beta > 0)$, and let $\psi(u)$ be differential, and p_1, p_2 and $\psi(0)$ be given. Then $\psi(u)$ is the only solution of the following differential equation*

$$a_1\psi^{'''}(u) + a_2\psi^{''}(u) + a_3\psi^{'}(u) + a_4\psi(u) + a_5 = 0, \tag{11}$$

where $a_1 = (1 - q_1q_2)$, $a_2 = (\alpha p_1 - \beta p_2)$, $a_3 = -\alpha^2 p_2 q_1$, $a_4 = [\alpha^2\beta p_2 q_1 - p_1 q_2 \beta^2(\alpha + \beta)]$, $a_5 = -\beta p_2 q_1 \alpha^2$.

Proof. Substituting $F^{'}(x) = \alpha e^{\alpha x}$ and $G^{'}(y) = \beta e^{\beta y}$ into (10), and we have

$$\begin{aligned} a_1\psi(u) &= p_2q_1e^{-\alpha u}(1 + \alpha \textstyle\int_0^u \psi(t)e^{\alpha t}dt) + p_1q_2\beta e^{\beta u} \int_u^\infty \psi(m)e^{-\beta m}dm \\ &+p_1p_2\beta e^{-\alpha u}(\alpha \textstyle\int_0^\infty \int_0^{u+y} \psi(h)e^{\alpha h}e^{-(\alpha+\beta)y}dhdy + \dfrac{1}{\alpha+\beta}), \end{aligned} \tag{12}$$

where $a_1 = (1 - q_1q_2)$, differentiate (12) with respect to u, and we get

$$\begin{aligned} a_1\psi^{'}(u) &= -\alpha p_2q_1e^{-\alpha u} - q_1p_2\alpha^2e^{-\alpha u} \textstyle\int_0^u \psi(t)e^{\alpha t}dt \\ &+\alpha p_2q_1\psi(u) + p_1q_2\beta^2e^{\beta u} \textstyle\int_u^\infty \psi(m)e^{-\beta m}dm - \beta p_1q_2\psi(u) \\ &-p_1p_2\alpha^2\beta e^{-\alpha u} \textstyle\int_0^\infty \int_0^{u+y} \psi(h)e^{\alpha h}e^{-(\alpha+\beta)y}dhdy \\ &+p_1p_2\alpha\beta \textstyle\int_0^\infty \psi(u+y)e^{-\beta y}dy - \dfrac{p_1\alpha\beta p_2}{\alpha+\beta}e^{-\alpha u}, \end{aligned} \tag{13}$$

differentiate (13) with respect to u, and we therefore get

$$\begin{aligned} a_1\psi^{''}(u) &= \alpha^2p_2q_1e^{-\alpha u} - q_1p_2\alpha^2 + (\alpha p_2q_1 - p_1q_2\beta)\psi^{'}(u) \\ &-(p_1q_2\beta^2 + p_1q_2\beta\alpha)\psi(u) + p_1q_2\beta^3e^{\beta u} \textstyle\int_u^\infty \psi(m)e^{-\beta m}dm \\ &+\alpha\beta p_1p_2\nu \textstyle\int_0^\infty \psi(u+y)e^{-\beta y}dy + q_1p_2\alpha^3 \int_0^u \psi(t)e^{\alpha(t-u)}dt \\ &+p_1p_2\beta e^{-\alpha u}\left(\alpha^3 \textstyle\int_0^\infty \int_0^{u+y} \psi(h)e^{\alpha h}e^{-(\alpha+\beta)y}dhdy + \dfrac{\alpha^2}{\alpha+\beta}\right) \end{aligned} \tag{14}$$

where $\nu_1 = \beta - \alpha$.

By (13)$\times\alpha$+ (12), we have

$$\begin{aligned} a_1\psi^{''}(u) + (\alpha p_1 + \beta q_2p_1)\psi^{'}(u) &= -q_1p_2\alpha^2 + (q_1p_2\alpha^2 - \beta^2q_2p_1 - \alpha\beta p_1)\psi(u) \\ &+\beta^2q_2p_1(\alpha+\beta)e^{\beta u} \textstyle\int_u^\infty \psi(x)e^{-\beta x}dx \\ &+\beta^2\alpha p_2p_1 \textstyle\int_0^\infty \psi(u+y)e^{-\beta y}dy, \end{aligned} \tag{15}$$

differentiate (15) with respect to u, we get

$$\begin{aligned} a_1\psi^{'''}(u) + \nu_2\psi^{''}(u) = &\ (q_1p_2\alpha^2 - \beta^2q_2p_1 - \alpha\beta p_1)\psi^{'}(u) \\ &+\beta^3q_2p_1(\alpha+\beta)e^{\beta u} \textstyle\int_u^\infty \psi(x)e^{-\beta x}dx \\ &-p_1p_2\alpha\beta^2(\psi(u) - \beta \textstyle\int_0^\infty \psi(u+y)e^{-\beta y}dy), \end{aligned} \tag{16}$$

where $\nu_2 = \alpha p_1 + \beta q_2p_1$. By (16)-(15)$\times\beta$, we obtain (11). □

3.2 The finite-horizon probability of ruin

The probability of ruin within infinite time is considered above, however, Some insurance companies may be bankrupt in the finite time, we consider the ruin probability of the finite time.

Let a time n be given and T denote the time of ruin. Then the finite-time ruin probability $\psi_n(u)$ is defined by

$$\psi_n(u) = P(T \leq n).$$

It is somewhat easier to work with the non-ruin distribution until the time n

$$\phi_n = 1 - \psi_n(u) = P(T > n),$$

for $u < 0$, $\phi_n = 0$.

We prove that $\{V_n, n = 0, 1, \cdots\}$ has stationary and independent increment, in which V_n is a continuous random variable when n is given. Let $W_{n+1} = V_{n+1} - V_n$. Then $W_0, W_1, W_2, \cdots, W_n$ are independent and identically distributed random variables, which have the same distribution as V_1. Let $H(x)$ be the distribution of W_n, i.e.,

$$H(x) = P(W_n < x),$$

for $n = 0, 1, 2 \cdots$

On the basis of the above discuss, we have

$$\phi_n = P(T > n) = P(U_1 \geq 0, U_2 \geq 0, ..., U_n \geq 0) = P(V_1 \leq u, V_2 \leq u, ..., V_n \leq u),$$

and the following recursive formulas are given by

$$\phi_1(u) = P(T > 1) = P(V_1 \leq u) = H(u),$$

$$\begin{aligned} \phi_2(u) = P(T > 2) &= P(V_1 \leq u, V_2 \leq u) = P(V_1 \leq u, V_1 + W_2 \leq u) \\ &= P(V_1 \leq u, W_2 \leq u - V_1) = \textstyle\int_{-\infty}^{u} H(u-y)dH(y) \\ &= \textstyle\int_{-\infty}^{u} \phi_1(u-y)dH(y), \end{aligned} \tag{17}$$

$$\begin{aligned} \phi_3(u) = P(T > 3) &= P(V_1 \leq u, V_2 \leq u, V_3 \leq u) \\ &= P(V_1 \leq u, V_1 + W_2 \leq u, V_1 + W_2 + W_3 \leq u) \\ &= \textstyle\int_{-\infty}^{u} P(W_2 \leq u - y, W_2 + W_3 \leq u - y)dH(y) \\ &= \textstyle\int_{-\infty}^{u} \phi_2(u-y)dH(y), \end{aligned} \tag{18}$$

by this way to recurse, and we have

$$P(T > n) = \int_{-\infty}^{u} \phi_{n-1}(u-y)dF(y). \tag{19}$$

So the recursive formula of the probability of the finite-horizon ruin is given as follows

$$\psi_1(u) = 1 - \phi_1,$$

$$\begin{aligned} \psi_2(u) &= 1 - \phi_2 = 1 - \textstyle\int_{-\infty}^{u} \phi_1(u-y)dH(y) \\ &= 1 - \textstyle\int_{-\infty}^{u} 1 - \psi_1(u-y)dH(y) \\ &= 1 - H(u) + \textstyle\int_{-\infty}^{u} \psi_1(u-y)dH(y), \end{aligned} \tag{20}$$

for the time n, we have

$$\psi_n(u) = 1 - H(u) + \int_{-\infty}^{u} \psi_{n-1}(u-y)dH(y). \tag{21}$$

From the recursive formula and an initial condition, if we know the distribution of premium and claim, then the recursive formula of the probability of the finite-horizon ruin can be computed.

3.3. The distribution of ruin time

This section presents the recursive formula for the ruin time by making the best use of the finite-time probability of ruin.

Let T be ruin time.Then its distribution is expressed as

$$\varphi_n(u) = P(T = n).$$

By using above the probability of finite-time ruin, we can obtain the recursive formula of the ruin time distribution as follows:

$$\varphi_1(u) = P(T = 1) = P(u - V_1 < 0),$$

$$\begin{aligned} \varphi_2(u) &= P(T = 2) = P(V_1 \leq u, V_1 + W_2 > u) \\ &= \int_{-\infty}^{u} P(W_2 > u - y) dH(y) \\ &= \int_{-\infty}^{u} \varphi_1(u - y) dH(y). \end{aligned} \tag{22}$$

For $n = 3$ we have

$$\begin{aligned} \varphi_3(u) &= P(T = 3) \\ &= P(V_1 \leq u, V_1 + W_2 \leq u, V_1 + W_2 + W_3 > u) \\ &= \int_{-\infty}^{u} P(W_2 \leq u - y, ..., W_2 + W_3 > u - y) dH(y) \\ &= \int_{-\infty}^{u} \varphi_2(u - y) dH(y), \end{aligned} \tag{23}$$

and when $T = n$, the distribution of ruin time is

$$\varphi_n(u) = P(T = n) = \int_{-\infty}^{u} \varphi_{n-1}(u - y) dH(y).$$

3.4. The distribution of surplus immediately before ruin

The distribution of surplus immediately before ruin in the insurance company is very important for the insurer and the policyholder. It has become a interesting research object in the risk theory since Dufresne and Gerber (1988) firstly gave rise to the distribution. we will consider the case in our model

Let U_{T_-} be the surplus immediately before ruin and T_- be the time immediately before ruin. Then distribution of U_{T-} is given by

$$P(U_{T_-} \leq x, T < \infty | U_0 = u) = \psi(u) - P(U_{T_-} > x, T < \infty | U_0 = u),$$

where x is a positive real value and $\psi(u)$ is the distribution of eventual ruin. Let $P(U_{T_-} > x, T < \infty | U_0 = u) = F(u, x)$.

Since T is a discrete integral random variable, we have

$$\begin{aligned} F(u,x) &= P(U_{T_-} > x, T < \infty) = \sum_{n=1}^{\infty} P(U_{T_-} > x, T = n) \\ &= \sum_{n=1}^{\infty} P(V_n > u, V_{n-1} < u - x, V_{n-2} \le u, ..., V_1 \le u) \\ &= \sum_{n=1}^{\infty} f_n(u,x), \end{aligned} \tag{24}$$

where$f_1(u,x) = P(V_1 > u, V_0 < u - x) = 1 - H(u) = \begin{cases} \bar{H}(u), & x \le u \\ 0, & x > u, \end{cases}$

$$\begin{aligned} f_2(u,x) &= P(V_2 > u, V_1 < u - x) = P(V_1 + W_2 > u, V_1 < u - x) \\ &= \int_{-\infty}^{u-x} P(W_2 > u - s) dH(s) = \int_{-\infty}^{u-x} \overline{H}(u-s) dH(s) \\ &= \int_{-\infty}^{u} \overline{H}(u-s) dH(s) - \int_{u-x}^{u} \overline{H}(u-s) dH(s), \end{aligned} \tag{25}$$

$$\begin{aligned} f_3(u,x) &= P(V_3 > u, V_2 < u - x, V_1 \le u) \\ &= P(V_1 + W_2 + W_3 > u, V_1 + W_2 < u - x, V_1 \le u) \\ &= \int_{-\infty}^{u} P(W_2 + W_3 > u - s, W_2 < u - x - s, V_1 \le u) dH(s) \\ &= \int_{-\infty}^{u} f_2(u-s, x) dH(s), \end{aligned} \tag{26}$$

for $n \ge 4$, we have

$$\begin{aligned} f_n(u,x) &= P(V_n > u, V_{n-1} < u - x, V_{n-2} \le u, ..., V_1 \le u) \\ &= \int_{-\infty}^{u} f_{n-1}(u-s, x) dF(s). \end{aligned} \tag{27}$$

Since

$$\sum_{n=1}^{\infty} f_n(u,x) = P(U_{T-1} > x, T < \infty | U_0 = u) \le 1, \tag{28}$$

we know that $\sum_{n=1}^{\infty} f_n(u,x)$ is convergent, and

$$\begin{aligned} F(u,x) &= \sum_{n=1}^{\infty} f_n(u,x) = f_1(u,x) + f_2(u,x) + \sum_{n=3}^{\infty} f_n(u,x) \\ &= f_1(u,x) + f_2(u,x) + \sum_{n=3}^{\infty} \int_{-\infty}^{u} f_{n-1}(u-s,x) dH(s). \end{aligned} \tag{29}$$

We consider the case that $x > u$ and $x \le u$.

For $x > u$,

$$\begin{aligned} F(u,x) &= \sum_{n=1}^{\infty} f_n(u,x) = f_2(u,x) + \sum_{n=3}^{\infty} \int_{-\infty}^{u} f_{n-1}(u-s,x) dH(s) \\ &= \int_{-\infty}^{u} \overline{H}(u-s) dH(s) + \int_{-\infty}^{u} \sum_{n=2}^{\infty} f_n(u-s,x) dH(s) \\ &= \int_{-\infty}^{u} \overline{H}(u-s) dH(s) + \int_{-\infty}^{u} F(u-s,x) dH(s), \end{aligned} \tag{30}$$

and for $x \leq u$,

$$\begin{aligned} F(u,x) &= \sum_{n=1}^{\infty} f_n(u,x) \\ &= f_1(u,x) + \int_{-\infty}^{u-x} f_1(u-s)dH(s) + \int_{-\infty}^{u} \sum_{n=2}^{\infty} f_n(u-s,x)dH(s) \\ &= \overline{H}(u,x) - \int_{u-x}^{u} \overline{H}(u-s)dH(s) + \int_{-\infty}^{u} \sum_{n=2}^{\infty} f_n(u-s,x)dH(s) \\ &= \overline{H}(u,x) - \int_{u-x}^{u} \overline{H}(u-s)dF(s) + \int_{-\infty}^{u} F(u-s,x)dH(s), \end{aligned} \tag{31}$$

So we get (30) and (31) which are the formulas for computing the distribution of surplus immediately before ruin.

3.5. Lundlerg inequality

Theorem 3. *Let $R > 1$ be a positive solution to $E[r^{V_1}] = 1$. Then $\{R^{-U_n}, n \geq 0\}$ is a positive martingale.*

Proof. Let $A_n = \{U_k, k \leq n\}$ be a σ-algebra and $U_{n+1} = U_n - W_{n+1}$, where $W_{n+1} = -V_{n+1} + V_n$ is independent of $\{U_0, U_1, ...U_n\}$ and identically distributed with V_1.

According to the assumption of the model, it is easy to know that $\{V_n, n \geq 0\}$ has stationary and independent increments. By its condition, we have

$$E[R^{-U_n}] = E[R^{-u+V_n}] = R^{-u}E[R^{V_n}] = R^{-u}\{E[R^{V_1}]\}^n = R^{-u} < \infty,$$

and

$$E\{R^{-U_{n+1}}|F_n\} = E\{R^{-U_n-W_{n+1}}|F_n\} = R^{-U_n}ER^{V_1} = R^{-U_n},$$

such that $\{R^{-U_n}; n \geq 0\}$ is a positive martingale.

□

Theorem 4. *If R is a adjustment coefficient, then*

$$\psi(u) = \frac{R^{-u}}{E\{R^{-U_T}|U_0 = u, T < \infty\}}.$$

Proof. Since $\{V_n, n \geq 0\}$ has stationary and independent increments,

$$E[V_1] = p_1\gamma - p_2\mu < \infty,$$

according to the strong law of large numbers, we have $U_\infty = \lim_{n\to\infty} U_n = \infty$, a.s, and thus $\lim_{n\to\infty} R^{-U_n} = 0$, a.s

On the other hand, $T \wedge n$ is a stop-time because T is a stop-time; by stop-time theorem, we have

$$\begin{aligned} R^{-u} &= E[R^{-U_{T\wedge n}}|U_0 = u] \\ &= E\{R^{-U_{T\wedge n}}|U_0 = u, T \leq n\}P\{T \leq n|U_0 = u\} \\ &\quad + E\{R^{-U_{T\wedge n}}|U_0 = u, T > n\}P\{T > n|U_0 = u\} \\ &= E\{R^{-U_T}|U_0 = u, T \leq n\}P\{T \leq n|U_0 = u\} \\ &\quad + E\{R^{-U_n}|U_0 = u, T > n\}P\{T > n|U_0 = u\}. \end{aligned} \tag{32}$$

By the monotone convergence theorem and the control convergence theorem, we have

$$E\{R^{-U_\infty}|U_0, T=\infty\}P(T=\infty|U_0)=0.$$

So (32) satisfies

$$R^{-u}=E\{e^{-RU_T}|T<\infty\}P(T<\infty|U_0=u).$$

i.e.,

$$\psi(u)=\frac{R^{-u}}{E\{R^{-U_T}|T<\infty\}}$$

□

Corollary: Lundberg inequality. $\psi(u)\le R^{-u}$, $(R>1)$.

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Mao-jun Zhang received Ph.D from Dalian University of technology, China in 2007 under the direction of Prof. Zun-quan Xia. He is now with department of economics, Dalian University of technology, P.R. China. His research interests focus on financial optimization, risk theory, and stochastic programming and its applications.

Department of economics, Dalian University of technology, Dalian 116024, P. R. China
e-mail: zmj@dlut.edu.cn

Jiang-xia Nan is now a doctor student under the direction of Prof. Deng-feng Li. She is now with Information and engineering college, Dalian University, P.R. China. Her research interests focus on game theory and its applications.

Department of mathematics, Information and engineering college, Dalian University, Dalian 116022, P. R. China
e-mail: jiangxia1107@163.com

Sen Wang is now a master student under the direction of Prof. Li-Wei Zhang. He is now with Information and engineering college, Dalian University, P.R. China. His research interests focus on optimization theory and its applications.

Department of mathematics, Information and engineering college, Dalian University, Dalian 116022, P. R. China