J. Appl. Math. & Informatics Vol. **26***(2008), No. 1 - 2, pp. 151 - 160*
Website: http://www.kcam.biz

MULTIOBJECTIVE FRACTIONAL SYMMETRIC DUALITY INVOLVING CONES

I. AHMAD* AND SARITA SHARMA

ABSTRACT. A pair of multiobjective fractional symmetric dual programs is formulated over arbitrary cones. Weak, strong and converse duality theorems are proved under pseudoinvexity assumptions. A self duality theorem is also discussed.

AMS Mathematics Subject Classification : 90C30, 90C32.
Key words and phrases : Multiobjective programming, fractional programming, symmetric duality, weak efficiency, pseudoinvexity.

1. Introduction

Dantzig et al. [5], Dorn [7], and Bazaara and Goode [1] studied symmetric duality for convex/concave functions in nonlinear programming. Subsequently, Mond and Weir [14] presented a distinct pair of symmetric dual programs which admits the relaxation of convexity / concavity assumptions to pseudoconvexity/pseudoconcavity. The duality results discussed in [14] were generalized by Chandra and Kumar [4] in the sense of Bazaraa and Goode [1].

Gulati et al. [8] discussed Wolfe and Mond-Weir type multiobjective symmetric duality results under invexity and pseudoinvexity, respectively, without nonnegative constraints. Kim et al. [11] generalized the results in [8] to arbitrary cones. In [16], Suneja et al. formulated a pair of multiobjective symmetric dual programs over arbitrary cones and proved various duality results by assuming the functions involved to be cone-convex. Recently, Khurana [10] presented Mond-Weir type symmetric duality over arbitrary cones involving cone-pseudoinvex functions.

Chandra et al. [3] formulated a pair of symmetric dual fractional programs under suitable convexity assumptions. Nanda and Das [15] derived symmetric duality results for multiobjective fractional programming problem involving

Received March 8, 2007. *Corresponding author.

cones. Gulati et al. [9] established usual duality results for static and continuous symmetric dual fractional programming problems without nonnegativity constraints using the notion of invexity. In [17], Weir presented a pair of multiobjective fractional symmetric dual programs and derived symmetric duality theorems under convexity assumptions. Lalitha et al. [12] generalized the symmetric duality results in [17] involving invex functions without nonnegativity constraints. Recently, Yang et al. [18] presented the nondifferentiable multiobjective symmetric fractional programs and proved appropriate duality relations under invexity assumptions.

In this paper, we formulate a pair of multiobjective fractional symmetric dual programs over arbitrary cones and establish appropriate duality theorems under pseudoinvexity assumptions. At the end, a self duality theorem is also discussed.

2. Notations and preliminaries

Let R^n denotes the n-dimensional Euclidean space and R^n_+ be its non-negative orthant. The following conventions for vectors x and y in R^n will be followed throughout this paper: $x < y \Leftrightarrow x_i < y_i, i = 1, 2, \ldots, n$, $x \leq y \Leftrightarrow x_i \leq y_i$, $i = 1, 2, \ldots, n$, $x \preceq y \Leftrightarrow x_i \leq y_i$, $i = 1, 2, \ldots, n$, *but* $x \neq y$. The index set $K = \{1, 2, \ldots, k\}$.

Definition 1 [2]. Let $x \in C \subset R^n$. Then C is a *cone* if and only if $\lambda x \in C$, for all $\lambda \geq 0$. Moreover, C is called a *convex cone* if it is convex.

Let $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ be open and let C_1, C_2 be closed convex cones with nonempty interiors in S_1 and S_2, respectively. Let $C \subset R^n$ be a cone. Then C^* is said to be a polar of C, if

$$C^* = \{z \in R^n \mid z^T x \leq 0, \ for\ all\ x \in C\}.$$

Definition 2. A real-valued function $\phi : S_1 \times S_2 \to R$ is said to be *pseudoinvex in the first variable at* $u \in S_1$ *for fixed* $v \in S_2$ if there exists a function $\eta : S_1 \times S_1 \to R^n$ such that

$$\eta(x, u)^T [\nabla_x \phi(u, v)] \geq 0 \Rightarrow \phi(x, v) \geq \phi(u, v),$$

and ϕ is said to be *pseudoinvex in the second variable at* $v \in S_2$ *for fixed* $u \in S_1$ if there exists a function $\xi : S_2 \times S_2 \to R^m$ such that

$$\xi(v, y)^T [\nabla_y \phi(u, v)] \geq 0 \Rightarrow \phi(u, y) \geq \phi(u, v).$$

Definition 3. A real-valued function $\phi : S_1 \times S_2 \to R$ is said to be *strictly pseudoinvex in the first variable at* $u \in S_1$ *for fixed* $v \in S_2$ if there exists a function $\eta : S_1 \times S_1 \to R^n$ such that

$$\eta(x, u)^T [\nabla_x \phi(u, v)] \geq 0 \Rightarrow \phi(x, v) > \phi(u, v),$$

and ϕ is said to be strictly pseudoinvex in the second variable at $v \in S_2$ for fixed $u \in S_1$, if there exists a function $\xi : S_2 \times S_2 \to R^m$ such that

$$\xi(v,y)^T[\nabla_y \phi(u,v)] \geq 0 \Rightarrow \phi(u,y) > \phi(u,v).$$

Throughout the paper, $\nabla_x \phi(x,y)$ and $\nabla_{xx}\phi(x,y)$ denote the first and second order gradient vectors of ϕ with respect to the first variable. $\nabla_y \phi(x,y)$, $\nabla_{yy}\phi(x,y)$ and $\nabla_{xy}\phi(x,y)$ are defined similarly.

Consider the multiobjective programming problem:

(P) Minimize $h(x) = (h_1(x), h_2(x), \ldots, h_p(x))$
subject to $g(x) \leq 0$,
where $h : R^n \to R^p$ and $g : R^n \to R^m$.

Definition 4. A feasible point $\bar{x}$ of (P) is said to be *weakly efficient for (P)* if there exists no other feasible point x such that $h(x) < h(\bar{x})$.

3. Symmetric duality

In this section, we formulate the following pair of multiobjective fractional symmetric dual programs and prove corresponding duality results.

(FP) Minimize $\left(\dfrac{f_1(x,y)}{g_1(x,y)}, \ldots, \dfrac{f_k(x,y)}{g_k(x,y)}\right)$
subject to

$$\sum_{i\in K} \lambda_i \Big[g_i(x,y)\nabla_y f_i(x,y) - f_i(x,y)\nabla_y g_i(x,y)\Big] \in C_2^*,$$

$$y^T \sum_{i\in K} \lambda_i \Big[g_i(x,y)\nabla_y f_i(x,y) - f_i(x,y)\nabla_y g_i(x,y)\Big] \geq 0,$$

$$\lambda > 0, x \in C_1.$$

(FD) Maximize $\left(\dfrac{f_1(u,v)}{g_1(u,v)}, \ldots, \dfrac{f_k(u,v)}{g_k(u,v)}\right)$
subject to

$$-\sum_{i\in K} \lambda_i \Big[g_i(u,v)\nabla_x f_i(u,v) - f_i(u,v)\nabla_x g_i(u,v)\Big] \in C_1^*,$$

$$u^T \sum_{i\in K} \lambda_i \Big[g_i(u,v)\nabla_x f_i(u,v) - f_i(u,v)\nabla_x g_i(u,v)\Big] \leq 0,$$

$$\lambda > 0, v \in C_2,$$

where $f_i : C_1 \times C_2 \to R_+$ and $g_i : C_1 \times C_2 \to R_+ \setminus \{0\}$, $i \in K$ are twice differentiable functions.

Remark 1. If $k = 1$ and $g = 1$, then (FP) and (FD) reduce to the symmetric dual programs of Chandra and Kumar [4].

It is convenient to parameterize the problems (FP) and (FD) in the sense of Dinkelbach [6] for validating duality theorems by defining

$$r_i = \frac{f_i(x,y)}{g_i(x,y)} \;\; and \;\; s_i = \frac{f_i(u,v)}{g_i(u,v)}, \; i \in K,$$

and thus express the programs (FP) and (FD) equivalent as :

(FP)′ Minimize $r = (r_1, r_2, \dots, r_k)$
subject to

$$f_i(x,y) - r_i g_i(x,y) = 0, \;\; i \in K, \tag{1}$$

$$\sum_{i \in K} \lambda_i \Big[\nabla_y f_i(x,y) - r_i \nabla_y g_i(x,y)\Big] \in C_2^*, \tag{2}$$

$$y^T \sum_{i \in K} \lambda_i \Big[\nabla_y f_i(x,y) - r_i \nabla_y g_i(x,y)\Big] \geq 0, \tag{3}$$

$$\lambda > 0, x \in C_1. \tag{4}$$

(FD)′ Maximize $s = (s_1, s_2, \dots, s_k)$
subject to

$$f_i(u,v) - s_i g_i(u,v) = 0, \;\; i \in K, \tag{5}$$

$$-\sum_{i \in K} \lambda_i \Big[\nabla_x f_i(u,v) - s_i \nabla_x g_i(u,v)\Big] \in C_1^*, \tag{6}$$

$$u^T \sum_{i \in K} \lambda_i \Big[\nabla_x f_i(u,v) - s_i \nabla_x g_i(u,v)\Big] \leq 0, \tag{7}$$

$$\lambda > 0, v \in C_2. \tag{8}$$

We now establish symmetric duality theorems for $(FP)'$ and $(FD)'$ by taking the following assumptions similar to those taken by Mond and Hanson [13] and Chandra and Kumar [4]:

$$\eta(x,u) + u \in C_1, \;\; for\ all\ x, u \in C_1, \tag{9}$$

$$\xi(v,y) + y \in C_2, \;\; for\ all\ v, y \in C_2, \tag{10}$$

where $\eta : C_1 \times C_1 \to C_1, \; \xi : C_2 \times C_2 \to C_2$.

Theorem 1 (Weak duality). *Let (x,y,λ,r) be feasible for $(FP)'$ and (u,v,λ,s) be feasible for $(FD)'$. If, either*

(i) $\sum_{i \in K} \lambda_i (f_i - s_i g_i)$ *is pseudoinvex in x with respect to η for fixed y and* $-\sum_{i \in K} \lambda_i (f_i - r_i g_i)$ *is strictly pseudoinvex in y with respect to ξ for fixed x; or*

(ii) $\sum_{i \in K} \lambda_i (f_i - s_i g_i)$ *is strictly pseudoinvex in x with respect to η for fixed y and* $-\sum_{i \in K} \lambda_i (f_i - r_i g_i)$ *is pseudoinvex in y with respect to ξ for fixed x,* *then* $r \nleq s$.

Proof (i). By (6) and (9), we have

$$-(\eta(x,u)+u)^T\left[\sum_{i\in K}\lambda_i(\nabla_x f_i(u,v)-s_i\nabla_x g_i(u,v))\right]\leq 0,$$

or

$$\eta(x,u)^T\sum_{i\in K}\lambda_i(\nabla_x f_i(u,v)-s_i\nabla_x g_i(u,v))$$
$$\geq -u^T\sum_{i\in K}\lambda_i(\nabla_x f_i(u,v)-s_i\nabla_x g_i(u,v))\geq 0,\ (using\ (7)).$$

On using pseudoinvexity of $\sum_{i\in K}\lambda_i(f_i-s_ig_i)$ in x, it follows that

$$\sum_{i\in K}\lambda_i(f_i(x,v)-s_ig_i(x,v))\geq\sum_{i\in K}\lambda_i(f_i(u,v)-s_ig_i(u,v)).$$

The above inequality along with (5) gives

$$\sum_{i\in K}\lambda_i(f_i(x,v)-s_ig_i(x,v))\geq 0. \tag{11}$$

Similarly, by (2), (3) and (10), we get

$$-\xi(v,y)^T\sum_{i\in K}\lambda_i(\nabla_y f_i(x,y)-r_i\nabla_y g_i(x,y))\geq 0,$$

which on using strict pseudoinvexity of $-\sum_{i\in K}\lambda_i(f_i-r_ig_i)$ in y implies

$$-\sum_{i\in K}\lambda_i(f_i(x,v)-r_ig_i(x,v))>-\sum_{i\in K}\lambda_i(f_i(x,y)-r_ig_i(x,y))=0,\ (by\ (1)).$$

That is,

$$-\sum_{i\in K}\lambda_i(f_i(x,v)-r_ig_i(x,v))>0. \tag{12}$$

Combining (11) and (12), we get $\sum_{i\in K}\lambda_i(r_i-s_i)g_i(x,v)>0$. Since $\lambda_i>0$ and $g_i(x,v)>0,\ i\in K$, therefore $r\nleq s$.

(ii). Proof follows on the similar lines of (i). □

Theorem 2 (Strong duality). *Let* $(\bar{x},\bar{y},\bar{\lambda},\bar{r})$ *be weakly efficient for* $(FP)'$ *and* $\lambda=\bar{\lambda}$, *fixed in* $(FD)'$. *Assume that*

(A) *the matrix* $\sum_{i\in K}\bar{\lambda}_i(\nabla_{yy}f_i(\bar{x},\bar{y})-\bar{r}_i\nabla_{yy}g_i(\bar{x},\bar{y}))$ *is positive or negative definite;*

(B) *the set* $[(\nabla_y f_1(\bar{x},\bar{y})-\bar{r}_1\nabla_y g_1(\bar{x},\bar{y})),\ldots,(\nabla_y f_k(\bar{x},\bar{y})-\bar{r}_k\nabla_y g_k(\bar{x},\bar{y}))]$ *is linearly independent.*

Then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r})$ is a feasible solution of $(FD)'$ and the objective values of $(FP)'$ and $(FD)'$ are equal.

If, the hypotheses of weak duality (Theorem 1) are satisfied for all feasible solutions of $(FP)'$ and $(FD)'$, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r})$ is weakly efficient for $(FD)'$.

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r})$ is weakly efficient solution for $(FP)'$, then by Fritz John type necessary conditions [16], there exist $\alpha \in R^k, \beta \in R^k, \gamma \in C_2, \eta \in R$ and $\xi \in R^k$ such that

$$\alpha_i - \beta_i g_i - \bar{\lambda}_i (\nabla_y g_i)^T (\gamma - \eta \bar{y}) = 0, \ i \in K, \tag{13}$$

$$\sum_{i \in K} \beta_i [(\nabla_x f_i - \bar{r}_i \nabla_x g_i) + \sum_{i \in K} \bar{\lambda}_i (\nabla_{yx} f_i - \bar{r}_i \nabla_{yx} g_i)(\gamma - \eta \bar{y})](x - \bar{x}) \geq 0, \ \forall x \in C_1, \tag{14}$$

$$\sum_{i \in K} (\beta_i - \eta \bar{\lambda}_i)(\nabla_y f_i - \bar{r}_i \nabla_y g_i) + \sum_{i \in K} \bar{\lambda}_i (\nabla_{yy} f_i - \bar{r}_i \nabla_{yy} g_i)(\gamma - \eta \bar{y}) = 0, \tag{15}$$

$$(\gamma - \eta \bar{y})^T (\nabla_y f_i - \bar{r}_i \nabla_y g_i) - \xi_i = 0, \ i \in K, \tag{16}$$

$$\xi^T \bar{\lambda} = 0, \tag{17}$$

$$(\alpha, \beta, \gamma, \eta, \xi) \neq 0, \tag{18}$$

$$\alpha \geq 0, \gamma \in C_2, \eta \geq 0, \xi \geq 0. \tag{19}$$

In view of $\bar{\lambda} > 0$ and $\xi \geq 0$, it readily follows from (17) that $\xi = 0$. Therefore from (16), we have

$$(\gamma - \eta \bar{y})^T (\nabla_y f_i - \bar{r}_i \nabla_y g_i) = 0, \ i \in K. \tag{20}$$

On multiplying (15) by $(\gamma - \eta \bar{y})$ and using (20), we get

$$(\gamma - \eta \bar{y})^T \left[\sum_{i \in K} \bar{\lambda}_i (\nabla_{yy} f_i - \bar{r}_i \nabla_{yy} g_i) \right] (\gamma - \eta \bar{y}) = 0,$$

which by hypothesis (A) gives

$$\gamma = \eta \bar{y}. \tag{21}$$

Therefore, from (15), we have

$$\sum_{i \in K} (\beta_i - \eta \bar{\lambda}_i)(\nabla_y f_i - \bar{r}_i \nabla_y g_i) = 0.$$

This, in view of hypothesis (B), yields

$$\beta_i = \eta \bar{\lambda}_i, \ i \in K, \ i.e., \ \beta = \eta \bar{\lambda}. \tag{22}$$

If $\eta = 0$, then from (22), $\beta = 0$. From (13) and (21), we have $\alpha_i = 0$, $i \in K$ i.e., $\alpha = 0$. Also, from (21), $\gamma = 0$. This contradicts (18). Hence $\eta > 0$, therefore $\beta > 0$. Thus from (21), $\bar{y} \in C_2$. Further, by (21) and (14), for all $x \in C_1$,

$$\sum_{i \in K} \beta_i [\nabla_x f_i - \bar{r}_i \nabla_x g_i](x - \bar{x}) \geq 0,$$

which along with (22) and $\eta > 0$, gives

$$\sum_{i \in K} \bar{\lambda}_i [\nabla_x f_i - \bar{r}_i \nabla_x g_i](x - \bar{x}) \geq 0. \tag{23}$$

Let $x \in C_1$. Then $x + \bar{x} \in C_1$, and so inequality (23) implies that $\sum_{i \in K} \bar{\lambda}_i [\nabla_x f_i - \bar{r}_i \nabla_x g_i] x \geq 0$, for every $x \in C_1$, i.e., $-\sum_{i \in K} \bar{\lambda}_i [\nabla_x f_i - \bar{r}_i \nabla_x g_i] \in C_1^*$. Also, by letting $x = 0$ and $x = 2\bar{x}$ in inequality (23) simultaneously, we get $\sum_{i \in K} \bar{\lambda}_i [\nabla_x f_i - \bar{r}_i \nabla_x g_i] \bar{x} = 0$. Thus, $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{r})$ is a feasible solution of $(FD)'$ and the objective values are equal. The weak efficiency for $(FD)'$ thus follows from weak duality (Theorem 1). □

We now merely state the following converse duality theorem as its proof would run analogous to Theorem 2.

Theorem 3 (Converse duality). *Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{s})$ be weakly efficient for $(FD)'$ and $\lambda = \bar{\lambda}$, fixed in $(FP)'$. Assume that*

(I) *the matrix $\sum_{i \in K} \bar{\lambda}_i \Big(\nabla_{xx} f_i(\bar{u}, \bar{v}) - \bar{s}_i \nabla_{xx} g_i(\bar{u}, \bar{v}) \Big)$ is positive or negative definite;*

(II) *the set $\Big[(\nabla_x f_1(\bar{u}, \bar{v}) - \bar{s}_1 \nabla_x g_1(\bar{u}, \bar{v})), \ldots, (\nabla_x f_k(\bar{u}, \bar{v}) - \bar{s}_k \nabla_x g_k(\bar{u}, \bar{v})) \Big]$ is linearly independent.*

Then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{s})$ is a feasible solution of $(FP)'$ and the objective values of $(FP)'$ and $(FD)'$ are equal.

If, the hypotheses of weak duality (Theorem 1) are satisfied for all feasible solutions of $(FP)'$ and $(FD)'$, then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{s})$ is weakly efficient for $(FP)'$.

4. Self duality

A mathematical programming problem is said to be self dual, if it is formally identical with its dual, that is, the dual can be recast in the form of the primal. In order to present a better view of the concept of self duality, we shall consider here the original problems (FP) and (FD) instead of their equivalents $(FP)'$ and $(FD)'$.

Let $x, y, u, v \in C$. The function $f_i(u, v)$, $i \in K$ is said to be *skew symmetric* if

$$f_i(u, v) = -f_i(v, u), \ i \in K,$$

for all u, v in the domain of f_i and the function $g_i(u, v)$, $i \in K$ is said to be *symmetric* if $g_i(u, v) = g_i(v, u)$, $i \in K$, for all u, v in the domain of g_i.

Theorem 4 (Self duality). *If f_i, $i \in K$ is skew symmetric and g_i, $i \in K$ is symmetric, then (FP) is a self dual. Also, if (FP) and (FD) are dual problems and (x^0, y^0, λ^0) is a joint weakly efficient solution, then so is (y^0, x^0, λ^0) and*

$$\frac{f_i(x^0, y^0)}{g_i(x^0, y^0)} = 0, \ i \in K.$$

Proof. The dual (FD) can be written as a minimization problem:

$$\text{(FD) Minimize } -\left(\frac{f_1(u,v)}{g_1(u,v)}, \dots, \frac{f_k(u,v)}{g_k(u,v)}\right)$$

subject to

$$-\sum_{i \in K} \lambda_i (g_i(u,v)\nabla_x f_i(u,v) - f_i(u,v)\nabla_x g_i(u,v)) \in C^*,$$

$$u^T \sum_{i \in K} \lambda_i (g_i(u,v)\nabla_x f_i(u,v) - f_i(u,v)\nabla_x g_i(u,v)) \leq 0,$$

$$\lambda > 0, v \in C,$$

which because of $\nabla_x f(u,v) = -\nabla_y f(v,u)$ and $\nabla_x g(u,v) = \nabla_y g(v,u)$ is transformed to

$$\text{(FD)}^* \text{ Minimize } \left(\frac{f_1(v,u)}{g_1(v,u)}, \dots, \frac{f_k(v,u)}{g_k(v,u)}\right)$$

subject to

$$\sum_{i \in K} \lambda_i (g_i(v,u)\nabla_y f_i(v,u) - f_i(v,u)\nabla_y g_i(v,u)) \in C^*,$$

$$u^T \sum_{i \in K} \lambda_i (g_i(v,u)\nabla_y f_i(v,u) - f_i(v,u)\nabla_y g_i(v,u)) \geq 0,$$

$$\lambda > 0, v \in C.$$

This shows that $(FD)^*$ is formally identical to (FP), that is, the objective and the constraints are identical. Thus (FP) becomes self dual.

It is easily shown that the feasibility of (x, y, λ) for (FP) implies the feasibility of (y, x, λ) for (FD) and vice-versa.

Since (x^0, y^0, λ^0) is a joint weakly efficient solution, the extreme values of (FP) and (FD) are equal to

$$\frac{f_i(x^0, y^0)}{g_i(x^0, y^0)}, \ i \in K.$$

From self duality, (y^0, x^0, λ^0) is feasible for (FP) as well as for (FD). Therefore (x^0, y^0, λ^0) is weakly efficient for (FP) implies weak efficiency of (y^0, x^0, λ^0) for

(FD). By similar argument, (y^0, x^0, λ^0) is weakly efficient for (FP). Also, the two objective values are equal to

$$\frac{f_i(y^0,x^0)}{g_i(y^0,x^0)},\ i \in K.$$

Therefore,

$$\frac{f_i(x^0,y^0)}{g_i(x^0,y^0)} = \frac{f_i(y^0,x^0)}{g_i(y^0,x^0)},\ i \in K.$$

$$= -\frac{f_i(x^0,y^0)}{g_i(x^0,y^0)},\ i \in K,$$

(by the skew symmetry of f_i and symmetry of g_i). Hence

$$\frac{f_i(x^0,y^0)}{g_i(x^0,y^0)} = 0,\ i \in K.$$

This completes the proof. □

5. Conclusion

In this paper, a pair of multiobjective fractional symmetric dual programs over arbitrary cones is formulated and duality results are established under generalized invexity assumptions. Self duality theorem for such problems has also been discussed and not appeared in the literature so far. Our results improve and generalize a number of results existing in the literature. It appears that these results can be further extended for minimax mixed integer programs, wherein some of the primal and dual variables are constrained to belong to some arbitrary sets, e.g., the sets of integers.

References

1. M.S. Bazaraa and J.J. Goode, *On symmetric duality in nonlinear programming*, Oper. Res. **21** (1973), 1-9.
2. M.S. Bazaraa, H.D. Sherali and C.M. Shetty, *Nonlinear Programming: Theory and Algorithms*, John Wiley and Sons, New York, 1993.
3. S. Chandra, B.D. Craven and B. Mond, *Symmetric dual fractional programming*, Z. Oper. Res. **29** (1985), 59-64.
4. S. Chandra and V. Kumar, *A note on pseudoinvexity and symmetric duality*, European J. Oper. Res. **105** (1998), 626-629.
5. G.B. Dantzig, E. Eisenberg and R.W. Cottle, *Symmetric dual nonlinear programs*, Pacific J. Math. **15** (1965), 809-812.
6. W. Dinkelbach, *On nonlinear fractional programming*, Manag. Sci. **13** (1967), 492-498.
7. W.S. Dorn, *A symmetric dual theorem for quadratic programs*, J. Oper. Res. Soc. Japan **2** (1960), 93-97.
8. T.R. Gulati, I. Husain and A. Ahmed, *Multiobjective symmetric duality with invexity*, Bull. Austral. Math. Soc. **56** (1997), 25-36.
9. T.R. Gulati, I. Husain and A. Ahmed, *Symmetric duality with invexity in static and continuous fractional programming*, Optimization **40** (1997), 41-56.

10. S. Khurana, *Symmetric duality in multiobjective programming involving generalized cone-invex functions*, European J. Oper. Res. **165** (2005), 592-597.
11. D.S. Kim, Y.B. Yun and W.J. Lee, *Multiobjective symmetric duality with cone constraints*, European J. Oper. Res. **107** (1998), 681-691.
12. C.S. Lalitha, S.K. Suneja and S. Khurana, *Symmetric duality involving invexity in multiobjective fractional programming*, Asia-Pacific J. Oper. Res. **20** (2003), 57-72.
13. B. Mond and M.A. Hanson, *On duality with generalized convexity*, Math. Operationsforsch: u. Statist., Ser. Optimization **15** (1984), 313-317.
14. B. Mond and T. Weir, *Generalized concavity and duality*, in: Generalized Concavity in Optimization and Economics, S. Schaible and W.T. Ziemba (Eds.), Academic Press, New York (1981), 263-279.
15. S. Nanda and L.N. Das, *Pseudoinvexity and symmetric duality in nonlinear fractional programming*, European J. Oper. Res. **73** (1994), 577-582.
16. S.K. Suneja, S. Aggarwal and S. Davar, *Multiobjective symmetric duality involving cones*, European J. Oper. Res. **141** (2002), 471-479.
17. T. Weir, *Symmetric dual multiobjective fractional programming*, J. Austral. Math. Soc. (Ser. A) **50** (1991), 67-74.
18. X.M. Yang, S.Y. Wang and X.T. Deng, *Symmetric duality for a class of multiobjective fractional programming problems*, J. Math. Anal. Appl. **274** (2002), 279-295.

I. Ahmad is a Reader in the Department of Mathematics, Aligarh Muslim University, Aligarh, India. He obtained his M.Sc. in Applied Mathematics and M.Phil. and Ph.D. in Mathematics from University of Roorkee, India (presently I.I.T. Roorkee, India). He has published a number of papers in the area of mathematical programming. Currently, he is working in multiobjective mathematical programming. His publications appeared in Asia-Pacific Journal of Operational Research, Applied Mathematics Letters, European Journal of Operational Research, Journal of Mathematical Analysis and Applications, Journal of Optimization Theory and Applications, Mathematical Methods of Operations Research, Optimization, Information Sciences, Journal of Global Optimization, and Journal of Computational and Applied Mathematics etc.

Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, Aligarh
e-mail: izharamu@hotmail.com

Sarita Sharma is a Research Scholar in the Department of Mathematics, Aligarh Muslim University, Aligarh, India. She obtained her M.Sc. from C.C.S. University, Meerut, India and M.Phil. from Aligarh Muslim University, Aligarh, India. She has published few papers in the area of mathematical programming.

Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, Aligarh
e-mail: ssharma05@hotmail.com