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CONTINUOUS PROGRAMMING CONTAINING SUPPORT FUNCTIONS

I. HUSAIN AND Z. JABEEN

ABSTRACT. In this paper, we derive necessary optimality conditions for a continuous programming problem in which both objective and constraint functions contain support functions and is, therefore, nondifferentiable. It is shown that under generalized invexity of functionals, Karush-Kuhn-Tucker type optimality conditions for the continuous programming problem are also sufficient. Using these optimality conditions, we construct dual problems of both Wolfe and Mond-Weir types and validate appropriate duality theorems under invexity and generalized invexity. A mixed type dual is also proposed and duality results are validated under generalized invexity. A special case which often occurs in mathematical programming is that in which the support function is the square root of a positive semidefinite quadratic form. Further, it is also pointed out that our results can be considered as dynamic generalizations of those of (static) nonlinear programming with support functions recently incorporated in the literature.

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1. Introduction

Chandra, Craven and Husain [4] obtained necessary optimality conditions for a constrained continuous programming problem having term with a square root of a quadratic form in the objective function, and using these optimality conditions formulated Wolfe type dual and established weak, strong and Huard [13] type converse duality theorems under convexity of functions. Subsequently,

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for the problem of [4], Bector, Chandra and Husain [1] constructed a Mond-Weir type dual which allows weakening of convexity hypotheses of [4] and derived various duality results under generalized convexity of functionals. Mond, Chandra and Husain [15], and Mond and Smart [16] examined invexity in continuous programming. As the concept of invexity has allowed the convexity requirements in a variety of mathematical programming problems to be weakened, Mond and Smart [16] derived the duality results for a class of nondifferentiable continuous programming problems considered in [4] with Wolfe and Mond-Weir type duals. The popularity of this type of problems seems to originate from the fact that, even though the objective function and or / constraint functions are non-smooth, a simple representation of the dual problem may be found. The theory of nonsmooth mathematical programming deals with much more general types of functions by means of generalized sub-differentials [5] and quasi-differentials [10]. However, the square root of a positive semidefinite quadratic form is one of the few cases of a nondifferentiable function for which one can write down the sub or quasi-differentials explicitly. In this research, we replace the square root of quadratic form by the support function of a compact convex set that is somewhat more general and for which the subdifferential may be simply expressed.

In this exposition, we derive Fritz John and Karush-Kuhn-Tucker type necessary optimality conditions for a nondifferentiable continuous programming problem in which nondifferentiability enters due to appearance of support functions in the integrand of the objective functional as well as in each constraint function. The Karush-Kuhn-Tucker optimality conditions are also shown to be sufficient under suitable generalized invexity hypotheses. As an application of these necessary optimality conditions, Wolfe and Mond-Weir type duals are formulated, and weak, strong and Mangasarian [13] type strict converse duality theorems are proved. Further, in the spirit of Bector et al. [2] and Xu [18], a mixed type dual is proposed and suitable duality theorems are established. It is also indicated that our results can be regarded as dynamic generalizations of those of (static) non-linear programming recently studied.

2. Prerequisites

Let $I = [a, b]$ be a real interval; $f : I \times R^n \times R^n \to R$ and $g : I \times R^n \times R^n \to R^m$ be continuously differentiable functions. In order to consider $f(t, x(t), \dot{x}(t))$, where $x : I \to R^n$ is differentiable with derivative $\dot{x}$, denote the partial derivatives of f with respect to x and $\dot{x}$ respectively by

$$f_x = \left[\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right]^T \quad \text{and} \quad f_{\dot{x}} = \left[\frac{\partial f}{\partial \dot{x}^1}, \ldots, \frac{\partial f}{\partial \dot{x}^n}\right]^T .$$

The partial derivatives of g are similarly defined using matrix with m rows instead of one. Let $C(I, R^m)$ denote the space of continuous functions $\phi : I \to R^m$, with uniform norm; let $C_+(I, R^m)$ denote the cone of non-negative functions in $C(I, R^m)$. Let X denote the space of piecewise smooth functions

$x : I \to R^n$ with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$y = Dx \Longleftrightarrow x(t) = \alpha + \int_a^t y(s)ds$$

and $x(a) = \alpha$, $x(b) = \beta$ are given boundary values. The boundary conditions $x(a) = \alpha$, $x(b) = \beta$ on X may be replaced by $x(a) = 0 = x(b)$, by a shift of origin. It is convenient to so define X in the proof of Theorem 1 of the forthcoming section; the original problem is recovered by a converse shift of origin.

We review some well known facts about a support function for easy reference. Let K be a compact convex set in R^n then the support function of K is defined by

$$s(x(t)|K) = \max\{x(t)^T v(t) : v(t) \in K, t \in I\}.$$

A support function, being convex and everywhere finite, has a subdifferential in the sense of convex analysis. From [17] subdifferential of $s(x(t)|K)$ is given by

$$\partial s(x(t)|K) = \{z(t) \in K, t \in I \text{ such that } x(t)^T z(t) = s(x(t)|K)\}.$$

In the subsequent analysis of this research, we shall need the following definitions of invexity and generalized invexity.

Definition 1. If there exist a vector function $\eta \equiv \eta(t, x(t), u(t)) \in R^n$ with $\eta = 0$ at t if $x(t) = u(t)$ for $t \in I$ such that for a scalar function $h(t, x(t), \dot{x}(t))$, the functional $H(x) = \int_I h(t, x(t), \dot{x}(t))dt$ satisfies.

$$H(x) - H(u) \geq \int_I \left[\eta^T h_x(t, u(t), \dot{u}(t)) + (D\eta)^T h_{\dot{x}}(t, u(t), \dot{u}(t))\right] dt,$$

then H is said to be *invex at u with respect to η*.

H is said to be *strictly invex with respect to η at u* if for all $x \in X$, $x \neq u$,

$$H(x) - H(u) > \int_I \left[\eta^T h_x(t, u(t), \dot{u}(t)) + (D\eta)^T h_{\dot{x}}(t, u(t), \dot{u}(t))\right] dt.$$

The functional H is said to be *pseudoinvex at u with respect to η* if

$$\int_I \left[\eta^T h_x(t, u(t), \dot{u}(t)) + (D\eta)^T h_{\dot{x}}(t, u(t), \dot{u}(t))\right] dt \geq 0$$
$$\Longrightarrow \quad H(x) > H(u)$$

and H is *strictly pseudoinvex with to η at u* if, for all $x \in X$ and $x \neq u$,

$$\int_I \left[\eta^T h_x(t, u(t), \dot{u}(t) + (D\eta)^T h_{\dot{x}}(t, u(t), \dot{u}(t))\right] dt \geq 0$$
$$\Longrightarrow \quad H(x) \geq H(u).$$

The functional H is *quasi-invex with respect to η* if

$$H(x) \leq H(u)$$

$$\Longrightarrow \qquad \int_I \left[\eta^T h_x(t,u(t),\dot{u}(t) + (D\eta)^T h_{\dot{x}}(t,u(t),\dot{u}(t))\right] dt \leq 0.$$

As in Ben-Israel and Mond [3], all pseudoinvex fucntionals are also invex.

3. Continuous programming problem and optimality

Consider the following nondifferentiable continuous programming problem whose necessary optimality conditions of both Fritz Hohn and Karush-Kuhn-Tucker types will be derived in this section.

Problem (CP) : Minimize $\int_I [f(t,x(t),\dot{x}(t)) + s(x(t)|K)]dt$,

$x \in X$

subject to

$$x(a) = \alpha, \ x(b) = \beta, \tag{3.1}$$

$$g^j(t,x(t),\dot{x}(t)) + s(x(t)|C^j)) \leq 0, \ j = 1,2,\dots,m \in I, \tag{3.2}$$

where f and g are continuously differentiable and each C^j, $(j = 1,2,\dots,m)$ is a compact convex set in R^n.

Theorem 1 (Fritz John Necessary Optimality Conditions). *If the problem* (CP) *attains a minimum at $x = \bar{x} \in X$, then there exist $\gamma \in R$ and piecewise smooth vector functions $\bar{\lambda} : I \to R^m$ with $\bar{\lambda}(t) =: (\bar{\lambda}'(t), \dots, \bar{\lambda}^m(t))^T$, $\bar{z} : I \to R^n$ and $\bar{w}^j : I \to R^n$, $j = 1,\dots,m$ such that*

$$\bar{\gamma}[f_x(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{z}(t)] + \sum_{j=1}^m \bar{\lambda}^j(t)[g_x^j(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{w}^j(t)]$$

$$= D[f_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))], \quad t \in I, \tag{3.3}$$

$$\sum_{j=1}^m \bar{\lambda}^j(t)[g^j(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{x}(t)^T \bar{w}^j(t)] = 0, \quad t \in I, \tag{3.4}$$

$$\bar{x}(t)^T \bar{z}(t) = s(\bar{x}(t)IK), \quad t \in I, \tag{3.5}$$

$$\bar{x}(t)^T \bar{w}^j(t) = s(\bar{x}(t)|C^j), \quad j = 1,2,\dots m, \quad t \in I, \tag{3.6}$$

$$\bar{z}(t) \in K, \bar{w}^j(t) \in C^j, \quad t \in I, \quad j = 1,2,\dots,m, \tag{3.7}$$

$$(\bar{\gamma}, \lambda(t)) \geq 0, \quad t \in I, \tag{3.8}$$

$$(\bar{\gamma}, \lambda(t)) \neq 0, \quad t \in I. \tag{3.9}$$

Proof. The problem (CP) may be expressed as (PE).

Problem (PE) : Minimize $\phi(x) = F(x) + Q(x)$,

subject to

$G(x) \in \Gamma$

$x \in X$

in which

$$F(x) = \int_I f(t, x(t), \dot{x}(t))dt;$$
$$Q(x) = \int_I s(x(t)|K)dt; \quad G : X \to C(I, R^m)$$

is given by (for all $t \in I$, $x \in X$), $G^j(x)(t) = g^j(t, x(t), \dot{x}(t)) + s(x(t)|C^j)$, $j = 1, 2, \ldots, m$ and $\Gamma = C_+(I, R^m)$.

From [9, Theorem 3], the Fritz John necessary optimality conditions for (PE) to attain a minimum at $x = \bar{x}$ are the existence of Lagrange multipliers $\gamma \in R$, $\rho = (\rho', \ldots, \rho^m)^T \in \Gamma^*$ (where Γ^* is the dual cone of convex cone Γ) satisfying

$$0 \in \gamma \partial\phi(\bar{x}) + \sum_{j=1}^{m} \partial(\rho^j G^j)(\bar{x}), \; 0 = \sum_{j=1}^{m} \rho^j G^j(\bar{x}), \; (\gamma, \rho) \geq 0, \; (\gamma, \rho) \neq 0. \tag{3.10}$$

The cited theorem requires certain convex sets to be weak * compact; this is automatic for (CP). Here $\partial(\rho^T G)(\bar{x})$ denotes subgradient for nearly convex function (see [9]).

Since $f(t, \ldots)$ is continuously differentiable, $F'(\bar{x})$ is given ([8], p. 16) by

$$(\forall\, v \in X), F'(\bar{x})\bar{v} = \int_I [f_x(t, \bar{x}(t), \dot{\bar{x}}(t))\bar{v}(t) + f_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t))\dot{\bar{v}}(t)]dt \tag{3.11}$$

Assume now, subject to later validation, that $\rho \in \Gamma^*$can be represented by a measurable function $\bar{\lambda} : I \to R^m$ with $\bar{\lambda}(t) = (\bar{\lambda}'(t), \ldots, \bar{\lambda}^m(t))^T$ satisfying,

$$(\forall\, \zeta \in C(I, R^m)), \langle \rho, \zeta \rangle = \int_I \lambda(t)^T \zeta(t)dt. \tag{3.12}$$

Define the convex function $\eta_t : R^n \to R$ by $\eta_t(\nu) = s(\nu|\, K)$. From [17], its subdifferential,

$$\partial\eta_t(\nu) = \left\{ zIz \in K, \; \eta_t(\nu) = \nu^T z \right\}. \tag{3.13}$$

Now $Q(x) = \int_I \eta_t(x)dt$. From [6, Theorem 3], we have,

$$y \in \partial Q(\bar{x}) \Leftrightarrow \left\{ (\forall t \in I), \sigma(t) \in \partial\eta_t(\bar{x}), \langle y, v \rangle = \int_I \sigma(t)^T v(t)dt \right\} \tag{3.14}$$

with $\sigma : I \to R^n$ measurable, namely $\sigma(t)^T = z(t)$ $t \in I$, from (13).

Let $\xi(t, x(t)) = s(x(t)|C^{(\cdot)})$, for $x \in X$, $t \in I$, where $s(x(t)|C^{(\cdot)})$ denotes the vector support function whose jth component is $s(x(t)|C^j)$. Then

$$\rho^T \xi(\cdot, x) = \int_I \lambda(t)^T \xi(t, x(t))dt. \tag{3.15}$$

Denote by ∂_c the Clarke generalized gradient [5] with respect to x. Then

$$\begin{aligned}\partial_c(\lambda(t)^T\xi(t,x(t))) &\subset \sum_{j=1}^{m}\partial_c(\lambda^j(t)(\xi^j(t,x(t))),\\ &= \sum_{j=1}^{m}|\lambda^j(t)|\partial_c(\mathrm{sgn}(\lambda^j(t))\xi^j(t,x(t))\\ &= \sum_{j=1}^{m}|\lambda^j(t)|\,\mathrm{sgn}(\lambda^j(t))\partial_c(\xi^j(t,x(t))). \qquad (3.16)\end{aligned}$$

The above is possible by using the representation of $\partial_c(\cdot)$ as the convex hull of limit of points of gradients at smooth points near x. Here $\sum$ denotes the algebraic sum of sets. Since $\xi^j(t,.) = s((\cdot)|C^j)$ is convex, we have for each $j \in \{1,2,\ldots,m\}$,

$$\begin{aligned}\partial_c\xi^j(t,x(t)) &= \partial\xi^j(t,x(t))\\ &= \{w^j(t)|w^j(t)\in C^j, \xi^j(t,x(t)) = \bar{x}(t)^Tw^j(t), t\in I\}. \qquad (3.17)\end{aligned}$$

From [6], it implies that $q \in \partial(\rho^T\xi)(\cdot,x)$ if and only there exist a measurable function $\mu : I \to R^m$ such that

$$(\forall\ t\in I), \mu(t)\in\partial_c\xi(t,x(t)); (\forall\ v\in X), \langle q,v\rangle = \int_I \lambda(t)^T\mu(t)v(t)dt.$$

Here from (16) and (17), $\mu^j(t) = w^j(t)$, $t\in I$, $j = 1,2,\ldots,m$. Therefore

$$\begin{aligned}\partial(\rho^TG)(\bar{x})v &= \{\theta^Tv|\theta\in\partial(\rho^TG)(\bar{x})\}\\ &\subset \sum_{j=1}^{m}\int_I\lambda(t)^T\Big[g_x(t,\bar{x}(t),\dot{\bar{x}}(t))v(t) + g_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))\dot{v}(t)\\ &\qquad + w^j(t)^Tv(t)\Big]dt. \qquad (3.18)\end{aligned}$$

Using (11), (13), (14) and (18), the relation

$0 \in \bar{\gamma}\partial\phi(\bar{x}) + \sum_{j=1}^{m}\partial(\bar{\rho}^jG^j)(\bar{x})$ of (10) yields that for each $v\in X$,

$$\int_I\Bigg[\Bigg\{\bar{\gamma}(f_x(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{z}(t)) + \sum_{j=1}^{m}\bar{\lambda}^j(t)(g^j_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{w}^I(t)))\Bigg\}v(t)$$
$$+(\bar{\gamma}f_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{\lambda}(t)^Tg_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))\dot{v}(t)\Bigg]dt = 0.$$

This, on integration by parts gives,

$$\int_I\Bigg[\bar{\gamma}(f_x(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{z}(t)) + \sum_{j=1}^{m}\bar{\lambda}^j(t)(g^I_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{w}^I(t))$$

$$-D(\bar{\gamma} f_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))\Big] v(t)dt$$

$$+(\bar{\gamma} f_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)))v(t)\Big|_{t=a}^{t=b} = 0$$

which by using $v(a) = 0 = v(b)$ implies

$$\int_I \Big[\bar{\gamma}(f_x(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{z}(t)) + \sum_{j=1}^{m} \bar{\lambda}^j(t)(g_{\dot{x}}^I(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{w}^I(t))$$
$$- D(\bar{\gamma} f_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)) + +\bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))\Big] v(t)dt = 0 \qquad (3.19)$$

Since this integral vanishes for any $v \in X$ by Lemma 2 ([7] p. 500), it follows that

$$\bar{\gamma}\{f_x(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{z}(t)\} + \sum_{j=1}^{m} \bar{\lambda}^I(t)\{g_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)\dot{v}(t)) + \bar{w}^j(t)\}$$
$$= \quad D\{\bar{\gamma} f_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x}(t),\dot{\bar{x}}(t))\}, \quad t \in I.$$

The cited Lemma 2 [7] assumes that the expression in the square bracket of (19) is piecewise continuous, *but this readily extends to measurable.* This validates (3).

Also $\sum_{j=1}^{m} \bar{\rho}^j G^j(\bar{x}) = 0$ alongwith $\bar{x}(t)^T \bar{w}^j(t) = (\bar{x}(t)|C^j)$ of (17) yields

$$\sum_{j=1}^{m} \int_I \bar{\lambda}^j(t)[g^i(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{x}(t)^T w^j(t)]v(t)dt = 0.$$

By the application of the above-cited lemma, this gives

$$\sum_{j=1}^{m} \lambda^I(t)\Big[g^i(t,\bar{x}(t),\dot{\bar{x}}(t)) + \bar{x}(t)^T w^j(t)\Big] = 0, \quad t \in I.$$

This proves (4).

In order to validate the representation of ρ by a function $\lambda(\cdot)$, it is to be noted here that the proof leading to (3) and (4) remains valid, without this assumption, if $\bar{\lambda}(\cdot)$ is regarded as a Schwarz distribution. However, (3) and (4) constitute a system of first order linear differential equations for $\bar{\lambda}(\cdot)$, given $\bar{x}$, $\bar{z}$ and $\bar{w}^j$, $(j = 1, 2, , \ldots, m)$, and therefore, possesses a piecewise smooth solution $\bar{\lambda}(\cdot)$. Then from (3) and (4) $\bar{z}$ and $\bar{w}^j$, $j = 1, 2, \ldots, m$ are also piecewise smooth. The conditions (5), (6) and (7) are obvious from (13) and (17). The conditions (8) and (9) follow as in [8, p. 59].

Hence the above analysis establishes the theorem fully.

The minimum $\bar{x}$ of (P) may be described as normal if $\bar{\gamma} = 1$ so that the Fritz John conditions (3)–(9) reduce to the Karush-Kuhn-Tucker optimality conditions. It suffices for $\bar{\gamma} = 1$ that Slater condition [13] holds at $\bar{x}$. □

Theorem 2 (Sufficient Karush-Kuhn-Tucker optimality conditions). *If there exist $\bar{x} \in X$, feasible for* (CP) *and piecewise smooth vector functions $\bar{z} : I \to R^n$, $\bar{\lambda} : I \to R^m$ with $\bar{\lambda}(t) = (\bar{\lambda}^1(t), \ldots, \bar{\lambda}^m(t))^T$ and $\bar{w}^j : I \to R^n$, $j = 1, 2, \ldots, m$, such that*

$$f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{z}(t) + \sum_{j=1}^{m} \bar{\lambda}^j(t)\left[g_x^j(t, \bar{x}, \dot{\bar{x}}) + \bar{w}^j(t)\right]$$
$$= D\left[f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{\lambda}^T(t) g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})\right], t \in I, \tag{3.20}$$

$$\sum_{j=1}^{m} \bar{\lambda}^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)) = 0,\ t \in I, \tag{3.21}$$

$$\bar{x}(t)^T \bar{z}(t) = s(\bar{x}(t)|K), t \in I \tag{3.22}$$

$$\bar{x}(t)^T \bar{w}^j(t) = s(\bar{x}(t)|C^j), t \in I, j = 1, 2, \ldots, m \tag{3.23}$$

$$\bar{z}(t) \in K, t \in I \tag{3.24}$$

$$\bar{w}^j(t), \in C^j, t \in I\ j = 1, 2, \ldots, m,\ t \in I \tag{3.25}$$

$$\bar{\lambda}(t) \geq 0,\ t \in I, \tag{3.26}$$

and if

(i) $\int_I \left\{ f(t, \cdot, \cdot) + (\cdot)^T \bar{z}(t) + \sum_{j=1}^{m} \bar{\lambda}^j(t)(g^j(t, x, \dot{x} + (\cdot)^T \bar{w}^j(t)) \right\} dt$
is pseudoinvex with respect to $\eta \equiv \eta(t, x, \bar{x})$, for all $\bar{\lambda}^j(t) \in R$ and $\bar{w}^j(t) \in R^n$, $j = 1, 2, \ldots, m$, or

(ii) $\int_I \left[f(t, \cdot, \cdot) + (\cdot)^T \bar{z}(t) \right] dt$ *pseudoinvex and*
$\sum_{j=1}^{m} \int_I \bar{\lambda}^j(t)(g^i(t, \cdot, \cdot) + (\cdot)^T \bar{w}^j(t)) dt$ *is quasi-invex for all $\bar{z}(t) \in R^n$, $\bar{\lambda}^j(t) \in R$ and $\bar{w}^j(t) \in R^n$, $(j = 1, 2, \ldots, m)$ with respect to the same $\eta \equiv \eta(t, x, \bar{x})$, then $\bar{x}$ is optimal for* (CP).

Proof. (i) From (20), we have

$$\int_I \eta^T \left\{ f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{z}(t) + \sum_{j=1}^{m} \bar{\lambda}^j(t)(g_x^j(t, \bar{x}, \dot{\bar{x}}) + \bar{w}^j(t)) \right.$$
$$\left. - D(f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{\lambda}^j(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})) \right\} dt = 0$$

On integrating by parts, this yields,

$$\int_I \Big[\eta^T\Big\{f_x(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t) + \sum_{j=1}^m \bar{\lambda}^j(t)(g_x^j(t,\bar{x},\dot{\bar{x}}) + \bar{w}^j(t))\Big\}$$
$$+(D\eta)^T\{f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{\lambda}^j(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\}\Big]dt$$
$$-\eta^T\{f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\}\Big|_{t=a}^{t=b} = 0.$$

Using the fact that $\eta(t,x,\bar{x}) = 0$ for $x(t) = \bar{x}(t)$ at $t = a$ and $t = b$ in view of $x(a) = \alpha$, $x(b) = \beta$, and $\bar{x}(a) = \alpha$, $\bar{x}(b) = \beta$, the above equation becomes

$$\int_I \Big[\eta^T\Big\{f_x(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t) + \sum_{j=1}^m \bar{\lambda}^j(t)(g_x^j(t,\bar{x},\dot{\bar{x}}) + \bar{w}^j(t))\Big\}$$
$$+(D\eta)^T\{f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})\}\Big]dt = 0.$$

By pseudoinvexity of

$$\int_I \Big\{f(t,\cdot,\cdot) + (\cdot)^T\bar{z}(t) + \sum_{j=1}^m \bar{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}} + (\cdot)^T\bar{w}^j(t))\Big\}dt$$

for $\bar{z}(t) \in R^n, t \in I\bar{\lambda}^j(t) \in R$ and $\bar{w}^j(t) \in R^n, j = 1,2,\dots,m, t \in I$ this implies

$$\int_I \Big\{f(t,x,\dot{x}) + x(t)^T\bar{z}(t) + \sum_{j=1}^m \bar{\lambda}^j(t)(g^j(t,x,\dot{x}) + x(t)^T\bar{w}^j(t))\Big\}dt$$
$$\geq \int_I \Big\{f(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T\bar{z}(t) + \sum_{j=1}^m \bar{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T\bar{w}^j(t))\Big\}dt$$

which, because of $x(t)^T\bar{z}(t) \leq s(x(t)|K)$ and $x(t)^T\bar{w}^j(t) \leq s(x(t)|C^j)$ with $\bar{z}(t) \in K$ and $\bar{w}^j(t) \in C^j$, $(j = 1,2,\dots,m)$ together with (20) and (21) gives

$$\int_I \Big\{f(t,x,\dot{x}) + s(x(t)|K) + \sum_{j=1}^m \bar{\lambda}^j(t)(g^j(t,x,\dot{x}) + s(x(t)|C^j))\Big\}dt$$
$$\geq \int_I \{f(t,\bar{x},\dot{\bar{x}}) + s(\bar{x}(t)|K)\}dt.$$

This, because of (24) and (2) implies that

$$\int_I \{f(t,x,\dot{x}) + s(x(t)|K)\}dt \geq \int_I \{f(t,\bar{x},\dot{\bar{x}}) + s(\bar{x}(t)|K)\}dt.$$

This implies that under the stated pseudinvexity condition, $\bar{x}$ is indeed an optimal solution of (CP).

(ii) Assume that $\bar{x}$ is not optimal for (CP). Then there exists x feasible for (CP), with $x \neq \bar{x}$ such that

$$\int_I [f(t,x,\dot{x}) + s(x(t)|K)]dt < \int_I [f(t,\bar{x},\dot{\bar{x}}) + s(\bar{x}(t)|K)]dt.$$

Since $x(t)^T\bar{z}(t) \leq s(x(t)|K)$ and $\bar{x}(t)^T\bar{z}(t) = s(\bar{x}(t)|K)$, for $\bar{z}(t) \in K,\ \ t \in I$, so this becomes $\int_I [f(t,x,\dot{x}) + x(t)^T\bar{z}(t)]dt < \int_I [f(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T\bar{z}(t)]dt$. As $\int_I [f(t,\cdot,\cdot) + (\cdot)^T\bar{z}(t)]dt$ is pseudoinvex with respect to η, from the above inequality, it follows that that

$$\int_I [\eta^T(f_x(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t)) + (D\eta)^T f_{\dot{x}}(t,\bar{x},\dot{\bar{x}})]dt < 0.$$

This, on integrating by parts, gives

$$\int_I \eta^T[(f_x(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t)) - Df_{\dot{x}}(t,\bar{x},\dot{\bar{x}})]dt + \eta^T f_{\dot{x}}(t,\bar{x},\dot{\bar{x}})|_{t=a}^{t=b} < 0.$$

Using $\eta = 0$ at $t = a$ and $t = b$, this reduces to

$$\int_I \eta^T[(f_x(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t)) + Df_{\dot{x}}(t,\bar{x},\dot{\bar{x}})]dt < 0. \tag{3.27}$$

Now from (2) and (26), we have

$$\sum_{j=1}^{m}\int_I \bar{\lambda}^j(t)(g^j(t,x,\dot{x}) + s(x(t)|C^j)) \leq \sum_{j=1}^{m}\int_I \bar{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T\bar{w}^j(t))dt$$

This, in view of $x(t)^T\bar{w}^j(t) \leq s(x(t)|C^j), w^j(t) \in C^j, t \in I,\ j = 1,2,\ldots,m$ yields,

$$\sum_{j=1}^{m}\int_I \bar{\lambda}^j(t)(g^j(t,x,\dot{x}) + x(t)^T w^j(t))dt$$
$$= \sum_{j=1}^{m}\int_I \bar{\lambda}^j(t)(g^j_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T\bar{w}^j(t))dt \leq 0.$$

This, because of quasi-invexity of $\sum_{j=1}^{m}\int_I \bar{\lambda}^j(t)(g^j(t,\cdot,\cdot) + (\cdot)^T\bar{w}^j(t))dt$ with respect to η implies

$$\int_I \left[\eta^T\left\{\sum_{j=1}^{m}\bar{\lambda}^I(t)(g^j_x(t,\bar{x},\dot{\bar{x}}) + \bar{w}^j(t))\right\} + (D\eta)^T(\bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}}))\right]dt \leq 0.$$

This, as earlier, on integrating by parts and using $\eta = 0$, at $t = a$ and $t = b$, gives

$$\int_I \eta^T \left[\sum_{i=1}^{m} \bar{\lambda}^j(t)(g_x^j(t,\bar{x},\dot{\bar{x}}) + w^j(t)) - D(\bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})) \right] dt \leq 0. \quad (3.28)$$

Combining (27) and (28), we have,

$$\int_I \eta^T \Bigg[f_x(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t) + \sum_{j=1}^{m} \bar{\lambda}^j(t)(g_x^j(t,\bar{x},\dot{\bar{x}}) + \bar{w}^j(t))$$
$$- D(f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})) \Bigg] < 0. \quad (3.29)$$

Now pre-multiplying (18) by η^T and then integrating we have

$$\int_I \eta^T \Bigg[f_x(t,\bar{x},\dot{\bar{x}}) + \bar{z}(t) + \sum_{j=1}^{m} \bar{\lambda}^I(t)(g_x^j(t,\bar{x},\dot{\bar{x}}) + \bar{w}^j(t))$$
$$-D(f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{\lambda}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})) \Bigg] dt = 0$$

This contradicts (29). Hence $\bar{x}$ must be optimal for (CP). □

4. Duality

The following problem is formulated as Wolfe type dual for the problem (CP).

Dual(WCD) : $\underset{u\in X,\lambda,z,w}{\text{Maximize}}\, \psi(u,\lambda,z,w^1,\ldots,w)$

$$= \int_I \Bigg[f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{j=1}^{m} \lambda^j(t)(g^j(t,u,\dot{u}) + u(t)^T w^j(t)) \Bigg] dt$$

subject to

$$u(a) = \alpha, u(b) = \beta \quad (4.30)$$

$$f_{\dot{x}}(t,u,\dot{u}) - z(t) + \sum_{j=1}^{m} \lambda^j(t)(g_x^j(t,u,\dot{u}) + w^j(t))$$
$$= D(f_{\dot{x}}(t,u,\dot{u}) + \lambda(t)^T g_{\dot{x}}(t,u,\dot{u})), t \in I \quad (4.31)$$

$$z(t) \in K_i, w^j(t) \in C^j, j = 1,2,\ldots,m, \quad (4.32)$$

$$\lambda(t) \geq 0, \;\; t \in I \quad (4.33)$$

Theorem 3 (Weak duality). *Let x be feasible for* (CP) *and* $(u,\lambda,z,w^1,\ldots,w^m)$ *feasible for* (WCD). *If for all feasible* $(x,u,\lambda,z,w^1,\ldots,w^m)$ *and with respect to* $\eta \equiv \eta(t,x,u)$

(i) $\int_I \Big[f(t,\cdot,\cdot)+(\cdot)^T z(t)\Big]dt$ *and* $\sum_{j=1}^{m}\int_I \lambda^j(t)\Big(g^j(t,\cdot,\cdot)+(\cdot)^T w^j(t)\Big)dt$ *are invex,*

or

(ii) $\int_I \Big[f(t,\cdot,\cdot)+(\cdot)^T z(t)+\sum_{j=1}^{m}\lambda^j(t)(g^j(t,\cdot,\cdot)+(\cdot)^T w^j(t))\Big]dt$ *is pseudoinvex,*

then $\inf(CP)\geq \sup(WCD)$.

Proof. (i)

$$
\begin{aligned}
&\phi(x)-\psi(u,\lambda,z,w^1,\dots,w^m)\\
&=\int_I[[f(t,x,\dot{x})+s(x(t)|K)]dt\\
&\quad-\int_I\Big[f(t,u,\dot{u})+u(t)^Tz(t)+\sum_{j=1}^{m}\lambda^I(t)(g^j(t,u,\dot{u})+u(t)^Tw^j(t))\Big]dt\\
&\quad-\sum_{j=1}^{m}\int_I\lambda^j(t)(g^j(t,u,\dot{u})+u(t)^Tw^j(t))dt\\
&\geq\int_I\Big[\eta^T(f_{\dot{x}}(t,u,\dot{u})+z(t))+(D\eta)^Tf_{\dot{x}}(t,u,\dot{u})]dt\\
&\quad-\sum_{j=1}^{m}\int_I\lambda^I(t)[g^j(f_{\dot{x}}(t,u,\dot{u})+u(t)^Tw^j(t))]dt\\
&\qquad\qquad\text{(Using invexity of }\int_I[f(t,\cdot,\cdot)+(\cdot)^Tz(t)]dt\\
&\qquad\qquad\quad\text{and }x(t)^Tz(t)\leq s(x(t)|K)\text{ for }z(t)\in K,\ t\in I)\\
&=\int_I\eta^T\{f_{\dot{x}}(t,x,\dot{x})+z(t)-Df_{\dot{x}}(t,u,\dot{u})\}dt+\eta^Tf_{\dot{x}}(t,u,\dot{u})|_{t=a}^{t=b}\\
&\quad-\sum_{j=1}^{m}\int_I\lambda^j(t)(g^j(t,u,\dot{u})+u(t)^Tw^j(t))dt,\\
&\qquad\qquad\text{(using integration by parts).}\\
&=\int_I\eta^T\{f_x(t,u,\dot{u})+z(t)-Df_{\dot{x}}(t,u,\dot{u})\}dt\\
&\quad-\sum_{j=1}^{m}\int_I\lambda^j(t)(g^j(t,u,\dot{u})+u(t)^Tw^j(t))dt,\\
&\qquad\qquad\text{(using }\eta=0\text{ at }t=a,t=b)\\
&=-\int_I\eta^T\Big\{\sum_{j=1}^{m}\lambda^j(t)(g_x^j(t,u,\dot{u})+w^j(t))-D(\lambda^T(t)g_{\dot{x}}(t,u,\dot{u}))\Big\}dt\\
&\quad-\sum_{j=1}^{m}\int_I\lambda^j(t)(g^j(t,u,\dot{u})+u(t)^Tw^j(t))dt,\quad\text{(using (29))}
\end{aligned}
$$

$$= -\int_I \eta^T \left[\sum_{j=1}^m \lambda^j(t)(g_x^j(t,u,\dot{u}) + w^j(t)) + (D\eta)^T(\lambda^T(t) g_{\dot{x}}(t,u,\dot{u}))\right] dt$$

$$-\eta^T(\lambda(t)g_{\dot{x}}(t,u,\dot{u}))|_{t=a}^{t=b} - \sum_{j=1}^m \int_I \lambda^j(t)(g^j(t,u,\dot{u}) + u(t)^T w^j(t))dt,$$

(using integration by parts)

$$= -\int_I \left[\eta^T \sum_{j=1}^m \lambda^j(t)(g_x^j(t,u,\dot{u}) + w^j(t)) + (D\eta)^T(\lambda^T(t) g_{\dot{x}}(t,u,\dot{u}))\right] dt$$

$$-\sum_{j=1}^m \int_I \lambda^j(t)(g_x^j(t,u,\dot{u}) + u(t)^T w^j(t))dt$$

(using $\eta = 0$ at $t = a, t = b$)

$$\geq -\sum_{j=1}^m \int_I \lambda^j(t)(g^j(t,x,\dot{x}) + x(t)^T w^j(t))dt.$$

(using invexity of $\sum_{j=1}^m \int_I \lambda^j(t)(g^j(t,\cdot,\cdot) + (\cdot)^T w^j(t))dt$)

$$\geq -\sum_{j=1}^m \int_I \lambda^j(t)(g^j(t,x,\dot{x}) + s(x(t)|C^j))dt$$

(using the fact that $x(t)^T w^j(t) \leq s(x(t)|C^j)$,
for $w^j(t) \in C^j, j = 1,2,\ldots,m$)

Since $\lambda^j \geq 0$, $g^j(t,x,\dot{x}) + s(x(t|C^j) \leq 0$, $t \in I$, $j = 1,2,\ldots,m$, the above inequality implies $\phi(x) - \psi(u,\lambda,z,w',\ldots,w^m) \geq 0$. That is, $\inf(CP) \geq \sup(WCD)$.

(ii) From (31), we have

$$0 = \int_I \eta^T \left[f_x(t,u,\dot{u}) + z(t) + \sum_{j=1}^m \lambda^j(t)(g_x^j(t,u,\dot{u}) + w^j(t)) \right.$$
$$\left. -D\{f_{\dot{x}}(t,u,\dot{u}) + \lambda(t)^T g_{\dot{x}}(t,u,\dot{u})\}\right] dt$$

$$= \int_I \left[\eta^T \left\{ f_x(t,u,\dot{u}) + z(t) + \sum_{j=1}^m \lambda^j(t)(g_x^j(t,u,\dot{u}) + w^j(t))\right\}\right.$$
$$\left. +(D\eta)^T\{f_x(t,u,\dot{u}) + \lambda(t)^T g_x(t,u,\dot{u})\}\right] dt$$
$$-\eta^T\{f_{\dot{x}}(t,u,\dot{u}) + \lambda(t)^T g_{\dot{x}}(t,u,\dot{u})\}|_{t=a}^{t=b}$$

(by integrating by parts)

$$= \int_I \left[\eta^T \left\{ f_x(t,u,\dot{u}) + z(t) + \sum_{j=1}^m \bar{\lambda}^j(t)(g_x^j(t,u,\dot{u}) + w^j(t))\right\}\right.$$

$$+(D\eta)^T\{f_{\dot{x}}(t,u,\dot{u})+\lambda(t)^T g_{\dot{x}}(t,u,\dot{u})\}\Big]dt.$$
(using $\eta=0$ at $t=a$ and $t=b$).

By pseudoinvexity of

$$\int_I\left\{f(t,\cdot,\cdot)+(\cdot)^T z(t)+\sum_{j=1}^m \lambda^j(t)(g^j(t,x,\dot{x})+(\cdot)^T w^j(t))\right\}dt,$$

this gives

$$\int_I\left\{f(t,x,\dot{x})+x(t)^T z(t)+\sum_{j=1}^m \lambda^j(t)(g^j(t,x,\dot{x})+x(t)^T w^j(t))\right\}dt$$
$$\geq \int_I\left\{f(t,u,\dot{u})+u(t)^T z(t)+\sum_{j=1}^m \lambda^j(t)(g^j(t,u,\dot{u})+u(t)^T \bar{w}^j(t))\right\}dt.$$

In view of $x(t)^T z(t)\leq s(x(t)|K)$ and $x(t)^T w^j(t)\leq s(x(t)|C^j)$, $j=1,2,\ldots,m$, from this, we have

$$\int_I\left\{f(t,x,\dot{x})+s(x(t)I\ K)+\sum_{j=1}^m \lambda^j(t)(g^j(t,x,\dot{x})+s(x(t)|C^j)\right\}dt$$

$$\geq \int_I\left\{f(t,u,\dot{u})+u(t)^T z(t)+\sum_{j=1}^m \lambda^j(t)(g^j(t,u,\dot{u})+u(t)^T w^j(t))\right\}dt.$$

By feasibility of x for (CP) along with (31), this implies

$$\int_I\{f(t,x,\dot{x})+s(x(t)|K\}\ dt$$
$$\geq \int_I\left\{f(t,u,\dot{u})+u(t)^T z(t)+\sum_{j=1}^m \lambda^j(t)(g^j(t,u,\dot{u})+u(t)^T w^j(t))\right\}dt.$$

That is, $\inf(CP)\geq\sup(WCD)$. □

Now we further weaken the invexity requirements by formulating Mond-Weir type dual to the problem (CP).

Consider the following problem:

Dual (M-WCD) : $\underset{x\in X,\lambda,z,w}{\text{Minimize}}\ \psi(u,\lambda,z,w',\ldots,w^m)=\int_I[f(t,u,\dot{u})+u(t)^T z(t)]dt$

subject to

$$u(a)=\alpha,u(b)=\beta, \tag{4.34}$$

$$f_{\dot{x}}(t,u,\dot{u})+z(t)+\sum_{j=1}^m \lambda^I(t)(g_x^j(t,u,\dot{u})+w^j(t))$$

$$= D[(f_{\dot{x}}(t,u,\dot{u}) + \lambda(t)^T g_{\dot{x}}(t,u,\dot{u})], t \in I, \tag{4.35}$$

$$\sum_{j=1}^{m} \int_I \lambda^j(t)(g^I(t,u,\dot{u}) + u(t)^T w^j(t))dt \geq 0,\ t \in I, \tag{4.36}$$

$$z(t) \in K, w^j(t) \in C^j, (j = 1,2,\ldots,m), \tag{4.37}$$

$$\lambda(t) \geq 0,\ \ t \in I. \tag{4.38}$$

Theorem 4 (Weak duality). *Let x be feasible for* (CP) *and* $(u, \lambda, z, w^1, \ldots, w^m)$ *be feasible for the problem* (M-WCD). *If for all feasible* $(x, u, \lambda, z, w^1, \ldots, w^m)$, $\int_I \left[f(t,\cdot,\cdot) + (\cdot)^T z(t)\right] dt$ *is pseudoinvex and* $\sum_{j=1}^{m} \int_I \lambda^j(t)(g^j\left(t,\cdot,\cdot) + (\cdot)^T w^j(t)\right) dt$ *is quasi-invex both with respect to* $\eta \equiv \eta(t,x,u)$ *for* $z(t) \in R^n$, $\lambda(t) \in R^m$ *with* $\lambda(t) = (\lambda^1(t), \ldots, \lambda^m(t))^T$ *and,* $w(t) \in R^n$, $(j = 1,2,\ldots,m)$, *then* $\inf(CP) \geq \sup(M\text{-}WD)$.

Proof. From the feasibility of x for (CP) and the feasibility of $(u, \lambda, z, w^1, w^2, \ldots, w^m)$ for (M-WCD), we have

$$\sum_{j=1}^{m} \int_I \lambda^j(t) \left[g^j(t,x,\dot{x}) + s(x(t)|C^j)\right] dt \leq \sum_{j=1}^{m} \int_I \lambda^j(t) \left[g^j(t,u,\dot{u}) + x(t)^T w^j(t)\right] dt.$$

This, in view of $x(t)^T w^j(t) \leq s(x(t)|C^j)$, $j = 1,2,\ldots,m$, $t \in I$, yields

$$\begin{aligned} &\sum_{j=1}^{m} \int_I \lambda^j(t)[g^j(t,x,\dot{x}) + x(t)^T z(t)]dt \\ &\leq \ \sum_{j=1}^{m} \int_I \lambda^j(t)[g^j(t,u,\dot{u}) + u(t)^T w^j(t)]dt. \end{aligned}$$

By the quasi-invexity of $\sum_{j=1}^{m} \int_I \lambda^I(t)[g^I(t,\cdot,) + (\cdot)^T w^j(t)]dt$ with respect to η, this implies

$$\int_I \left[\eta^T \sum_{j=1}^{m} \lambda^j(t)(g^j_x(t,u,\dot{u}) + w^j(t)) + (D\eta)^T(\lambda(t)^T g_{\dot{x}}(t,u,\dot{u}))\right] dt \leq 0.$$

This, by integration by parts, gives,

$$\begin{aligned} &\int_I \left[\eta^T \sum_{j=1}^{m} \lambda^I(t)(g^j_x(t,u,\dot{u}) + w^j(t)) - D(\lambda(t)^T g_{\dot{x}}(t,u,\dot{u}))\right] dt \\ &\qquad + \eta^T(\lambda(t)^T g_{\dot{x}}(t,u,\dot{u}))\Big|_{t=a}^{t=b} \leq 0, \end{aligned}$$

from which, on using $\eta = 0$ at $t = a$, $t = b$, and (33), we get

$$\int_I \eta^T \left[f_{\dot{x}}(t, x, \dot{x}) + z(t) - Df_{\dot{x}}\Big(t, x, \dot{x}\Big) \right] dt \geq 0.$$

This, as earlier gives,

$$\int_I [\eta^T (f_{\dot{x}}(t, x, \dot{x}) + z(t)) + (D\eta)^T f_{\dot{x}}(t, x, \dot{x})] dt \geq 0,$$

which, by pseudo-invexity of $\int_I \{f(t, \cdot, \cdot) + (\cdot)^T z(t)\} dt$ implies,

$$\int_I [f(t, x, \dot{x}) + x(t)^T z(t)] dt \geq \int_I [f(t, u, \dot{u}) + u(t)^T z(t)] dt.$$

Since $x(t)^T z(t) \leq s(x(t)|K)$, therefore, this becomes

$$\int_I [f(t, x, \dot{x}) + s(\bar{x}(t)^T |K)] dt \geq \int_I [f(t, u, \dot{u}) + u(t)^T z(t)] dt.$$

That is, $\inf(P) \geq \sup(M\text{-}WD)$. □

A combined strong duality theorem for pairs of Wolfe type dual problems and Mond-Weir type dual problems is established in the next theorem.

Theorem 5 (Strong duality). *If $\bar{x}$ is an optimal solution of (CP) with the normality condition satisfied at $\bar{x}$, then there exist piecewise smooth $\bar{\lambda} : I \to R^m$ with $\bar{\lambda}^T(t) = (\bar{\lambda}^1(t), \ldots, \bar{\lambda}^m(t))$, $z : I \to R^n$ and $w^j : I \to R^n$, $(j = 1, 2, \ldots, m)$ such that $(\bar{x}, \bar{\lambda}, z, \bar{w}^1, \ldots, \bar{w}^m)$ is feasible for (WCD) as well as for (M-WCD) and in either case, the objective value of each of the duals is equal to that of the primal. If the hypotheses of Theorem 3 are satisfied then $(\bar{x}, \bar{\lambda}, \bar{z}, \bar{w}^1, \ldots, \bar{w}^m)$ is optimal for (WCD) and if the requirements of Theorem 4 are fulfilled, then $(\bar{x}, \bar{\lambda}, \bar{z}, \bar{w}^1, \ldots, \bar{w}^m)$ is an optimal solution of (M-WCD).*

Proof. Since $\bar{x}$ is optimal for (CP) and also normal, Theorem 1 implies that there exist Lagrange multipliers $\bar{\lambda} : I \to R^n$, $j = 1, 2, \ldots, m$ such that $(\bar{x}, \bar{\lambda}, z, \bar{w}^1, \ldots, \bar{w}^m)$ is feasible for both dual problems. Of course, for the problem (M-WCD),

$$\sum_{j=1}^{m} \lambda^I(t) g^I \Big((t, \bar{x}, \dot{x}) + \bar{x}(t)^T \bar{w}^j(t) \Big) = 0, \quad t \in I,$$

$$\Longrightarrow \quad \sum_{j=1}^{m} \int_I \lambda^j(t) \Big(g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t) \Big) dt = 0.$$

Consider

$$\psi(\bar{x}, \bar{\lambda}, z, \bar{w}^1, \ldots, \bar{w}^m)$$

$$= \int_I \left[f(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \bar{z}(t) + \sum_{j=1}^m \bar{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)) \right] dt$$

$$= \int_I \left[f(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \bar{z}(t) \right] dt,$$

$$\left(\text{since } \sum_{j=1}^m \bar{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)) = 0,\ t \in I\right)$$

$$= \int_I [f(t,\bar{x},\dot{\bar{x}}) + s(\bar{x}(t)IK)] dt,$$

$$\left(\text{since } \bar{x}(t)^T \bar{z}(t) = s(\bar{x}(t)IK),\ t \in I\right)$$

$$= \phi(\bar{x}).$$

Now, if invexity and pseudoinvexity hypotheses of Theorem 3 are satisfied, $(\bar{x}, \bar{\lambda}, z, \bar{w}^1, \ldots, \bar{w}^m)$ is optimal for (WCD) by weak duality, and, if pseudoinvexity and quasinvexity requirements of Theorem 4 are fufilled, then $(\bar{x}, \bar{\lambda}, z, \bar{w}^1, \ldots, \bar{w}^m)$ by weak duality theorem, is optimal for (M-WCD). □

5. Converse duality

In this section, we present a Mangasarian [13] type strict converse duality for each pair of the dual problems.

Theorem 6 (Strict converse duality). *Let $\bar{x}$ be an optimal solution of (CP) meeting normality condition. If $(\hat{u}, \hat{z}, \hat{\lambda}, \hat{w}^1, \ldots, \hat{w}^m)$ is an optimal solution of (WCD) and if with respect to $\eta \equiv \eta(t, \bar{x}, \hat{u})$*

(i) $\int_I \left[f(t,\cdot,\cdot) + (\cdot)^T \hat{z}(t) \right] dt$ *is strictly invex and* $\sum_{j=1}^m \int_I \hat{\lambda}^j(t)\left(g^j(t,\cdot,\cdot) + (\cdot)^T \hat{w}^j(t)\right) dt$ *is invex, or*

(ii) $\int_I \left[f(t,\cdot,\cdot) + (\cdot)^T \hat{z}(t) + \sum_{j=1}^m g^j(t,\cdot,\cdot) + (\cdot)^T \hat{w}^j(t) \right] dt$ *is strictly pseudoinvex,*

then $\bar{x} = \hat{u}$, i.e., $\hat{u}$ be an optimal solution of (CP).

Proof. (i) Assume that $\bar{x} \neq \hat{u}$. By Theorem 5, there exist piecewise smooth $\bar{\lambda} = I \to R^m$ with $\bar{\lambda}(t) = (\bar{\lambda}^1(t), \ldots, \bar{\lambda}^m(t))^T, \bar{z} : I \to R^n$ and $\bar{w}^j = I \to R^n, (j = 1, \ldots, m)$ such that $(\bar{x}, \bar{\lambda}, \bar{z}, \bar{w}^1, \ldots, \bar{w}^m)$ is optimal for (WCD). Hence

$$\int_I \left[f(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \bar{z}(t) + \sum_{j=1}^m \bar{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)) \right] dt$$

$$= \int_I f \left[(t,\hat{u},\dot{\hat{u}}) + \hat{u}(t)^T \hat{z}(t) + \sum_{j=1}^m \hat{\lambda}^j(t)(g^j(t,\hat{u},\dot{\hat{u}}) + \hat{u}(t)^T \hat{w}^j(t)) \right] dt \qquad (5.39)$$

By strict invexity of $\int_I [f(t,\cdot,\cdot)+(\cdot)^T\hat{z}(t)]dt$, we have

$$\int_I [f(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\hat{z}(t)]dt-\int_I (f(t,\hat{u},\dot{\hat{u}})+\hat{u}(t)^T\hat{z}^j(t))dt$$

$$> \int_I [\eta^T(f_x(t,\hat{u},\dot{\hat{u}})+\hat{z}(t))+(D\eta)^T f_{\dot{x}}(t,\hat{u},\dot{\hat{u}})]dt \tag{5.40}$$

Also by invexity of $\sum_{j=1}^m \int_I \hat{\lambda}^j(t)(g^j(t,\cdot,\cdot)+(\cdot)^T\hat{w}^j(t))dt$, we have

$$\begin{aligned}
&\sum_{j=1}^m \int_I \hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\hat{w}^j(t))dt \\
&-\sum_{j=1}^m \int_I \hat{\lambda}^j(g^j(t,\hat{u},\dot{\hat{u}})+\hat{u}(t)^T\hat{w}^j(t))dt \\
&\geq \sum_{j=1}^m \int_I [\eta^T\hat{\lambda}^j(t)(g_x^j(t,\hat{u},\hat{u}) \\
&\quad +\hat{w}^j(t))+(D\eta)^T(\hat{\lambda}(t)^T g_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))]dt
\end{aligned} \tag{5.41}$$

Addition of the inequalities (39) and (41), gives

$$\begin{aligned}
&\int_I \left[f(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\hat{z}(t)+\sum_{j=1}^m \hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\hat{w}^j(t))\right]dt \\
&\quad -\int_I \left[f(t,\hat{u},\dot{\hat{u}})+\hat{u}(t)^T\hat{z}(t)+\sum_{j=1}^m \hat{\lambda}^j(t)(g^j(t,\hat{u},\dot{\hat{u}})+\hat{u}(t)^T\hat{w}^j(t))\right]dt \\
&> \int_I [\eta^T\left\{f_x(t,\hat{u},\dot{\hat{u}})+\hat{z}(t)+\sum_{j=1}^m \hat{\lambda}^j(t)(g_x^j(t,\hat{u},\dot{\hat{u}})+\hat{w}^j(t))\right\}dt \\
&\quad +(D\eta)^T\{f_{\dot{x}}(t,\hat{u},\dot{\hat{u}})+\hat{\lambda}(t)^T g_{\dot{x}}(t,\hat{u},\dot{\hat{u}})\}dt \\
&= \int_I \eta^T\left[\left\{f_x(t,\hat{u},\dot{\hat{u}})+\hat{z}(t)+\sum_{j=1}^m \hat{\lambda}^j(t)(g_x^j(t,\hat{u},\dot{\hat{u}})+\hat{w}^j(t))\right\}\right. \\
&\quad \left. -D\{f_{\dot{x}}(t,\hat{u},\dot{\hat{u}})+\hat{\lambda}(t)^T g_{\dot{x}}(t,\hat{u},\dot{\hat{u}})\}\right]dt \\
&\quad +\eta^T\{f_{\dot{x}}(t,\hat{u},\dot{\hat{u}})+\hat{\lambda}(t)^T g_{\dot{x}}(t,\hat{u},\dot{\hat{u}})\}\Big|_{t=b}^{t=a},
\end{aligned}$$

(by integration by parts)

$$= \int_I \eta^T \Big[\Big\{ f_x(t,\hat{u},\dot{\hat{u}}) + \hat{z}(t) + \sum_{j=1}^{m} \hat{\lambda}^j(t)(g_x^j(t,\hat{u},\dot{\hat{u}}) + \hat{w}^j(t)) \Big\}$$

$$-D\{f_{\dot{x}}(t,\hat{u},\dot{\hat{u}}) + \hat{\lambda}(t)^T g_{\dot{x}}(t,\hat{u},\dot{\hat{u}})\} \Big] dt, \text{(since } \eta = 0 \text{ at } t=a, t=b).$$

This, in view of equality constraint (39) of the dual problem, implies that

$$\int_I \Big\{ f(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \hat{z}(t) + \sum_{j=1}^{m} \hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \hat{w}^j(t)) \Big\} dt$$

$$> \int_I \Big\{ f(t,\hat{u},\dot{\hat{u}}) + \hat{u}(t)^T \hat{z}(t) + \sum_{j=1}^{m} \hat{\lambda}^j(t)(g^j(t,\hat{u},\dot{\hat{u}}) + \hat{u}(t)^T \hat{w}^j(t)) \Big\} dt$$

$$= \int_I \Big[f_x(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \bar{z}(t) + \sum_{j=1}^{m} \hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)) \Big] dt,$$

(by 39)).

Since $\sum_{j=1}^{m} \bar{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)) = 0, \quad t \in I,$ (by Theorem 1) and $\bar{x}(t)^T \hat{w}^j(t) \leq s(\bar{x}(t)|C^j) for \hat{w}^j(t) \in C^j, j = 1,2,\ldots,m$, this inequality reduces to

$$\int_I \Big\{ \bar{x}(t)^T \hat{z}(t) + \sum_{j=1}^{m} \hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + s(\bar{x}(t)|C^j)) \Big\} dt > \int_I \bar{x}(t)^T \bar{z}(t) dt.$$

This, by (2) and (35), yields $\int_I \bar{x}(t)^T \hat{z}(t) dt > \int_I \bar{x}(t)^T \bar{z}(t) dt$. Since for $\hat{z}(t) \in K, \bar{x}(t)^T \hat{z}(t) \leq s(\bar{x}(t)|K) for \hat{z}(t) \in K$ and $\bar{x}(t)^T \bar{z}(t) = s(\bar{x}(t)|K)$, this gives $\int_I s(\bar{x}(t)|K) dt > \int_I s(\bar{x}(t)|K) dt$. which cannot happen. Hence $\bar{x} = \hat{u}$.

(ii) Let $\bar{x} \neq \hat{u}$. Then from the feasibility of $(\bar{x}, \bar{\lambda}, z, \bar{w}^1, \ldots, \bar{w}^m)$ for (WCD), we have

$$0 = \int_I \eta^T \Big\{ f_x(t,\hat{u},\dot{\hat{u}}) + \hat{z}(t) + \sum_{j=1}^{m} \hat{\lambda}^j(t)(g_x^j(t,\hat{u},\dot{\hat{u}}) + \hat{w}^j(t))$$

$$-D(f_x(t,\hat{u},\dot{\hat{u}}) + \hat{\lambda}(t)^T g_{\dot{x}}(t,\hat{u},\dot{\hat{u}})) \Big\} dt$$

$$= \int_I \Big[\eta^T \Big\{ f_x(t,\hat{u},\dot{\hat{u}}) + \hat{z}(t) + \sum_{j=1}^{m} \hat{\lambda}^j(t)(g_x^j(t,\hat{u},\dot{\hat{u}}) + \hat{w}^j(t)) \Big\}$$

$$+(D\eta)^T (f_{\dot{x}}(t,\hat{u},\dot{\hat{u}}) + \hat{\lambda}(t)^T g_{\dot{x}}(t,\hat{u},\dot{\hat{u}})) \Big] dt$$

$$-\eta^T(f_{\dot{x}}(t,\hat{u},\dot{\hat{u}}) + \hat{\lambda}(t)^T g_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))\Big|_{t=a}^{t=b}, \quad \text{(by integrating by parts)}$$

This, on using $\eta = 0$, at $t = a$ and $t = b$, gives

$$= \int_I \Big[\eta^T\Big\{f_x(t,\hat{u},\dot{\hat{u}}) + \hat{z}(t) + \sum_{j=1}^{m}\hat{\lambda}^j(t)(g_x(t,\hat{u},\dot{\hat{u}}) + \hat{w}^j(t))\Big\}$$

$$+(D\eta)^T(f_{\dot{x}}(t,\hat{u},\dot{\hat{u}}) + \hat{\lambda}(t)^T g_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))\Big]dt = 0$$

which because of strict pseudoinvexity of

$$\int_I \Big\{f_x(t,\hat{u},\dot{\hat{u}}) + \bar{x}(t)^T\hat{z}(t) + \sum_{j=1}^{m}\hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T\hat{w}^j(t))\Big\}dt$$

$$> \int_I \Big\{f(t,\hat{u},\dot{\hat{u}}) + \hat{u}(t)^T\hat{z}(t) + \sum_{j=1}^{m}\hat{\lambda}^j(t)(g^j(t,\hat{u},\dot{\hat{u}}) + \hat{u}(t)^T\hat{w}^j(t))\Big\}dt.$$

From this, as in (i), we have $\int_I s(\bar{x}(t)|K)dt > \int_I s(\bar{x}(t)|K)dt$ that cannot happen. Hence $\bar{x} = \hat{u}$. □

We now give the strict converse duality for the Mond-Weir type dual.

Theorem 7 (Strict converse duality). *Let the problem (CP) have an optimal solution $\bar{x}$ that satisfies normality condition. Let $(\hat{u}, \hat{z}, \hat{\lambda}, \hat{w}^1, \dots, \hat{w}^m)$ be an optimal solution of (M-WCD) and if $\int_I \Big[f(t,\cdot,\cdot) + (\cdot)^T\hat{z}(t)\Big]dt$ is strictly pseudoinvex and $\sum_{j=1}^{m}\int_I \lambda^j(t)\Big(g^j(t,\cdot,\cdot) + (\cdot)^T\hat{w}^j(t)\Big)dt$ is quasi-invex for all $\hat{\lambda}^j(t) \in R$ and $\hat{w}^j(t) \in R^n$, $j = 1, 2, \dots, m$ both respect to $\eta = \eta(t, \bar{x}, \hat{u})$, then $\bar{x} = \hat{u}$, i.e., $\hat{u}$ is an optical solution of (CP).*

Proof. Assume that $\bar{x} \neq \hat{u}$. Since $\bar{x}$ is optimal for (CP), by Theorem 5, there exist piecewise smooth $\bar{\lambda}$, $\bar{z}$ and $\bar{w}^I$, $j = 1, 2, \dots, m$ such that $(\bar{x}, \bar{\lambda}, \bar{z}, \bar{w}^1, \dots, \bar{w}^m)$ is optimal for the problem (M-WD). Thus

$$\int_I [f(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T\bar{z}(t)]dt = \int_I [f(t,\hat{u},\dot{\hat{u}}) + \hat{u}(t)^T\hat{z}(t)]dt. \tag{5.42}$$

From (2), (36) and (38) along with fact that

$$\bar{x}(t)^T\hat{w}^j(t) = s(\bar{x}(t)|C^j), \quad \text{for } \hat{w}^i(t) \in C^j, j = 1, 2, \dots, m,$$

we have

$$\sum_{j=1}^{m}\int_I \underline{\hat{\lambda}}^j(t)(g^j(t,\bar{x},\dot{\bar{x}}) + \bar{x}(t)^T\hat{w}^j(t))dt$$

$$\leq \sum_{j=1}^{m}\int_I \hat{\lambda}^j(t)(g^j(t,\hat{u},\dot{\hat{u}}+\hat{u}(t)^T\hat{w}^j(t))dt.$$

This, because of quasi-invexity of $\sum_{j=1}^{m}\int_I \hat{\lambda}^j(t)(g^j(t,\cdot,\cdot)+(\cdot)^T\hat{w}^j(t))dt$ with respect to η implies

$$\int_I\left[\eta^T\sum_{j=1}^{m}\hat{\lambda}^j(t)(g^j_x(t,\hat{u},\dot{\hat{u}})+\hat{w}^j(t))+(D\eta)^T(\lambda(t)^Tg_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))\right]dt\leq 0.$$

From this, on using integration by parts and then $\eta=0$ at $t=a$ and $t=b$ that gets the integrated part vanished, we have

$$\int_I\eta^T\left[\sum_{j=1}^{m}\lambda^j(t)(g^j_x(t,\hat{u},\dot{\hat{u}})+\hat{w}^j(t))-D(\hat{\lambda}(t)^Tg_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))\right]dt\leq 0.$$

This, using (35), gives $\int_I\eta^T\left[(f_x(t,\hat{u},\dot{\hat{u}})+\hat{z}(t))+(D\eta)^Tf_{\dot{x}}(t,\hat{u},\dot{\hat{u}})\right]dt\geq 0$ which, as earlier, by integration by parts and then using $\eta=0$ at $t=a$ and $t=b$, give

$$\int_I[\eta^T(f_x(t,\hat{u},\dot{\hat{u}})+\hat{z}(t))-Df_{\dot{x}}(t,\hat{u},\dot{\hat{u}})]dt\geq 0,$$

By strict pseudoinvexity of $\int_I[f(t,\cdot,\cdot)+(\cdot)^T\hat{z}(t)]dt$, this implies

$$\begin{aligned}\int_I[f(t,\bar{x},\dot{\bar{x}}))+\bar{x}(t)^T\hat{z}(t)]dt &> \int_I[f(t,\hat{u},\dot{\hat{u}}))+\hat{u},(t)^T\hat{z}(t)]dt\\ &= \int_I[f(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\bar{z}(t)]dt,\quad \text{(by (42))}.\end{aligned}$$

This implies $\int_I\bar{x}(t)^T\hat{z}(t)dt>\int_I\bar{x}(t)^T\bar{z}(t)dt$. This, also, by the same arguments as in the proof of Theorem 6 yields a contradiction. Hence $\bar{x}=\hat{u}$, i.e., $\hat{u}$ is an optimal solution of (CP). □

6. Mixed type duality

Following the scheme of formulations in Bector et. al. [1] and Xu [18], we propose the following mixed type dual (Mix CD) to(CP):

$$\text{(MixCD)} \quad : \quad \text{Maximize}\int_I\left\{f(t,u,\dot{u})+u(t)^Tz(t)\right.$$
$$\left.+\sum_{j\in J_0}\lambda^j(t)(g(t,u,\dot{u}+u(t)^Tw^j(t))\right\}dt$$
$$u\in X,\lambda,z,w^1,\ldots,w^m$$

$$\text{subject to } u(a) = \alpha, u(b) = \beta, \tag{6.43}$$

$$f_x(t,u,\dot{u}) + z(t) + \sum_{j=1}^{m} \lambda^j(t)(g_x^j(t,u,\dot{u}) + w^j(t)),$$

$$= D(f_{\dot{x}}(t,u,\dot{u}) + \lambda(t)^T g_{\dot{x}}(t,u,\dot{u})), t \in I, \tag{6.44}$$

$$\sum_{j \in J_\alpha} \int_I \lambda^j(t)(g^j(t,u,\dot{u} + u(t)^T w^j(t))dt \geq 0, \ \alpha = 1,\dots,r, \tag{6.45}$$

$$z(t) \in K, w^j(t) \in C^j, j = 1,2,\dots,m, \ t \in I, \tag{6.46}$$

$$\lambda(t) \geq 0, \ \ t \in I, \tag{6.47}$$

where $J_\alpha \subseteq M = \{1,2,\dots,m\}, \alpha = 0,1,2,\dots r$ with

$$\bigcup_{\alpha=0}^{r} J_\alpha = M \text{ and } J_\alpha \bigcap J_\beta = \phi, \ \text{ if } \alpha \neq \beta.$$

If $J_0 = M$ and $J_\alpha = \phi$, for $\alpha = 1,2,\dots,r$, then (Mix CD) becomes (WCD). In case $J_0 = \phi$ and $J_\alpha = M$, for some $\alpha \in \{1,2,\dots,1,2\}$, then (Mix CD) becomes (M-WCD).

Theorem 8 (Weak duality). *Let x be a feasible for (CP) and $(u,z,\lambda,w^1,\dots w^m)$ feasible for (Mix CD). If, for all feasible $(x,u,z,\lambda,w^1,\dots,w^m)$ $\int_I \left\{ f(t,\cdot,\cdot) + (\cdot)^T z(t) + \sum_{j \in J_0} \lambda^j(t)\Big(g^j(t,\cdot,\cdot) + (\cdot)^T w^j(t)\Big)\right\}dt$ is pseudoinvex and $\sum_{j \in J_\alpha} \int_I \lambda^j(t)\Big(g^j(t,\cdot,\cdot) + (\cdot)^T w^j(t)\Big)dt$, $\alpha = 1,2,\dots,r$ is quasi-invex with respect to the same $\eta \equiv \eta(t,x,u)$, then* $\inf(CP) \geq \sup(MixCD)$.

Proof. Since x is feasible for (CP) and $(u,z,y,w^1,\dots,w^m)$ feasible for (Mix CD), we have, in view of $x(t)^T w^j(t) \leq s(x(t)|C^j)$, $j = 1,2,\dots,m$,

$$\sum_{j \in J_\alpha} \int_I \lambda^j(t)(g^j(t,x,\dot{x}) + x(t)^T w^j(t))dt$$

$$\leq \sum_{j \in J_\alpha} \int_I \lambda^j(t)(g^j(t,u,\dot{u}) + u(t)^T w^j(t))dt, \quad \alpha = 1,2,\dots,r$$

By the quasi-invexity of $\sum_{j \in J_\alpha} \int_I \lambda^j(t)(g^j(t,\cdot,\cdot) + (\cdot)^T w^j(t))dt, \alpha = 1,2,\dots,r$, this yields,

$$\sum_{j \in J_\alpha} \int_I \{\eta^T \lambda^j(t)(g_x^j(t,u,\dot{u}) + w^j(t)) + (D\eta)^T(\lambda^j(t) g_{\dot{x}}^j(t,u,\dot{u}))\}dt \leq 0,$$

$$\alpha = 1,2,\dots,r.$$

Hence

$$\sum_{j\in M-J_0}\int_I\{\eta^T\lambda^j(t)(g_x^j(t,u,\dot u)+w^j(t))+(D\eta)^T(\lambda^j(t)g_{\dot x}^j(t,u,\dot u))\}dt\le 0.$$

This, by integration by parts, gives

$$\sum_{j\in M-J_0}\int_I\eta^T\{(\lambda^j(t)g_x^j(t,x,\dot x)+w^j(t))-D(\lambda^j(t)g_{\dot x}(t,x,\dot x))\}dt$$
$$+\sum_{j\in M-J_0}^{m}\eta^T(\lambda^j(t)g_{\dot x}^j(t,u,\dot u))\Big|_{t=a}^{t=b}\le 0\,.$$

Using $\eta=0$ at $t=a$ and $t=b$, from this we have,

$$\sum_{j\in M-J_0}\int_I\eta^T\{(\lambda^j(t)g_x^j(t,x,\dot x)+w^j(t))-D(\lambda^j(t)g_{\dot x}^j(t,x,\dot x))\}dt\le 0.$$

From (44), it follows

$$\begin{aligned}0\ \le\ &\int_I\eta^T\Big\{f_x(t,u,\dot u)+z(t)+\sum_{j\in J_0}\lambda^j(t)(g_x^j(t,u,\dot u)+w^j(t))\\&\qquad-D(f_{\dot x}(t,u,\dot u)+\sum_{j\in J_0}\lambda^j(t)g_{\dot x}^j(t,u,\dot u))\Big\}dt\\=\ &\int_I\Big[\eta^T\Big\{f_x(t,u,\dot u)+z(t)+\sum_{j\in J_0}\lambda^j(t)(g_x^j(t,u,\dot u)+w^j(t))\Big\}\\&\qquad+(D\eta)^T\Big\{f_{\dot x}(t,u,\dot u)+\sum_{j\in J_0}\lambda^j(t)g_{\dot x}^j(t,u,\dot u)\Big\}\Big]\,dt\\&\qquad-\eta^T\Big\{f_x(t,u,\dot u)+\sum_{j\in J_0}\lambda^j(t)(g_x^j(t,u,\dot u))\Big\}\Big|_{t=a}^{t=b},\end{aligned}$$

(by integrating by parts).

Using $\eta=0$ at $t=a$, $t=b$, we have

$$\int_I\Big[\eta^T\Big\{(f_x(t,u,\dot u)+z(t)+\sum_{j\in J_0}\lambda^j(t)(g_x^j(t,u,\dot u)+w^j(t))\Big\}$$
$$+(D\eta)^T\Big\{f_{\dot x}(t,x,\dot x)+\sum_{j\in J_0}\lambda^j(t)(g_x^j(t,x,\dot x))\Big\}\Big]dt\ge 0$$

This, because of pseudoinvexity of

$$\int_I\Big\{f(t,\cdot,\cdot)+(\cdot)^Tz(t)+\sum_{j\in J_0}\lambda^j(t)(g^j(t,\cdot,\cdot)+(\cdot)^Tw^j(t))\Big\}dt,$$

yields

$$\int_I \left\{ f(t,x,\dot{x}) + x(t)^T z(t) + \sum_{j\in J_0} \lambda^j(t)(g^j(t,x,\dot{x}) + x(t)^T w^j(t)) \right\} dt$$
$$\geq \int_I \left\{ f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{j\in J_0} \lambda^j(t)(g^j(t,u,\dot{u}) + u(t)^T w^j(t)) \right\} dt.$$

This, from $y^j(t) \geq 0$ and $g^j(t,x,\dot{x}) + s(x(t)|C^j) \leq 0$ along with $x(t)^T w^j(t) \leq s(x(t)|C^j)$, for $w^j(t) \in C^j$, $j = 1,2,\ldots,m$, implies

$$\int_I \{f(t,x,\dot{x}) + x(t)^T z(t)\} dt$$
$$\geq \int_I \left\{ f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{j\in J_0} \lambda^j(t)(g^j_{\dot{x}}(t,u,\dot{u}) + u(t)^T w^j(t)) \right\} dt.$$

This, due to $x(t)^T z(t) \leq s(x(t)|K)$, for $z(t) \in K$, $t \in I$, gives

$$\int_I \{(t,x,\dot{x}) + s(x(t)|K)\} dt$$
$$\geq \int_I \left\{ f(t,u,\dot{u}) + u(t)^T z(t) + \sum_{j\in J_0} \lambda^j(t)(g^j(t,u,\dot{u}) + u(t)^T w^j(t)) \right\} dt.$$

That is, $\inf(CP) \geq \sup(MixCD)$.

Corollary 1. *Let $\bar{x}$ be feasible for (CP) and $(\bar{x},\bar{z},\bar{y},\bar{w}^1,\ldots\bar{w}^m)$ be feasible for (Mix CD) with corresponding objective values are equal. Let the hypotheses of Theorem 8 hold. Then $\bar{x}$ is optimal for (CP) and $(\bar{x},\bar{z},\bar{y},\bar{w},\ldots\bar{w}_m)$ is optimal for (Mix CD).*

Theorem 9 (Strong duality). *If $\bar{x}$ is an optimal solution of (CP) and normal, then there exist piecewise smooth $\bar{z} : I \to R^n$, $\bar{\lambda} : I \to R^m$ with $\bar{\lambda}(t) = (\lambda^1(t)\ldots,\bar{\lambda}^m(t))$, and $\bar{w}^j : I \to R^n$, $j = 1,2,\ldots,m$ such that $(\bar{x},\bar{z},\bar{y},\bar{w}^1,\ldots,\bar{w}^m)$ is feasible for (Mix CD) and the corresponding objective values of (CP) and (Mix CD) are equal. If, also, $\int_I \left\{ f(t,\cdot,\cdot) + (\cdot)^T z(t) + \sum_{j\in J_o}^{m} \lambda^j((t)g^j(t,\cdot,\cdot) + (\cdot)^T w^j(t)) \right\} dt$ is pseudoinvex for and $\sum_{j\in J_\alpha} \int_I \lambda^j(t)(g^j(t,\cdot,\cdot) + (\cdot)^T w^j(t))dt$, $j \in J_\alpha$, $\alpha = 1,2,\ldots,rt \in I$ is qausinvex with respect to the same $\eta{=}\eta\ (t,x,\bar{x})$, then $(\bar{x},\bar{z},\bar{y},\bar{w}^1,\ldots,\bar{w}^m)$ is an optimal solution of (Mix CD).*

Proof. Since $\bar{x}$ is an optimal solution of (Mix CD) from Theorem 1, there exist piecewise smooth $\bar{\lambda} : I \to R^m, \bar{z} : I \to R^n$ and $\bar{w}^j : I \to R^n$, $(j = 1, 2, \ldots, m)$ such that

$$f_x(t, \bar{x}, \dot{\bar{x}}) + \bar{z}(t) + \sum_{j=1}^{m} \bar{\lambda}^j(t)(g_x^j(t, \bar{x}, \dot{\bar{x}}) + \bar{w}^j(t))$$

$$= D(f_{\dot{x}}(t, \bar{x}, \dot{\bar{x}}) + \bar{\lambda}(t)^T g_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})),\ t \in I \tag{6.48}$$

$$\sum_{j=1}^{m} \bar{\lambda}^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)) = 0,\ t \in I, \tag{6.49}$$

$$\bar{x}(t)^T \bar{z}(t) = s(\bar{x}(t)|K) \tag{6.50}$$

$$\bar{x}(t)^T \bar{w}^j(t) = s(\bar{x}(t)|C^j), j = 1, 2, \ldots, m. \tag{6.51}$$

$$\bar{z}(t) \in z,\ \bar{w}^j(t) \in C^j,\ j = 1, 2, \ldots, m \tag{6.52}$$

$$\lambda(t) \geq 0. \tag{6.53}$$

From (49), it implies

$$\sum_{j \in J_0} \bar{\lambda}^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)) = 0, t \in I,$$

and

$$\sum_{j \in J\alpha} \lambda^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)) = 0, \alpha = 1, 2, \ldots, r.$$

This implies

$$\sum_{j \in J_\alpha}^{m} \int_I \lambda^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t))dt = 0, \alpha = 1, 2, \ldots, r \tag{6.54}$$

This, together with (48) (52) and (53) implies $(\bar{x}, \bar{z}, \bar{y}, \bar{w}^1, \ldots, \bar{w}^m)$ is feasible for (Mix CD). Also

$$\int_I \left\{ f(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t) + \sum_{j \in J_0} \lambda^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{w}^j(t)) \right\} dt$$

$$= \int_I \left\{ f(t, \bar{x}, \dot{\bar{x}}) + \bar{x}(t)^T \bar{z}(t) \right\} dt$$

$$= \int_I \{ f(t, \bar{x}, \dot{\bar{x}}) + s(\bar{x}(t)|K) \} dt, \quad \text{(from (50))}$$

This shows that the objective values of (CP) and (Mix CD) are equal. If $\int_I \left\{ f(t, \cdot, \cdot) + (\cdot)^T z(t) + \sum_{j \in J_o} \lambda^j(t) g^j(t, \cdot, \cdot) + (\cdot)^T w^j(t) \right\} dt$ is psedudoinvex and

$\sum_{j\in J_\alpha}\int_I \lambda^j(t)(g^j(t,\cdot,\cdot)+(\cdot)^T w^j(t))dt, \alpha=1,2,\dots,r$ is quasiinvex with respect to the same η, then from Corollary 1 it implies that $(\bar{x},\bar{z},\bar{y},\bar{w}^1,\dots,\bar{w}^m)$ must be an optimal solution of (Mix CD). □

We now present Mangasarian type [12] strict converse duality theorem.

Theorem 10 (Strict converse duality). *Let $\bar{x}$ be an optimal solution of (CP) and normal. If $(\hat{u},\hat{z},\hat{\lambda},\hat{w}^1,\dots,\hat{w}^m)$ is an optimal solution of (Mix CD) and if $\int_I\left\{f(t,\cdot,\cdot)+(\cdot)^T\hat{z}(t)+\sum_{j\in J_o}\hat{y}(t)g^j(t,\cdot,\cdot)+(\cdot)^T\hat{w}^j(t)\right\}dt$ is strictly pseudoinvex and $\sum_{j\in J_\alpha}\int_I\lambda^j(t)\Big(g^j(t,\cdot,\cdot)+(\cdot)^T\hat{w}^j(t)\Big)dt$, $\alpha=1,2,\dots,r$ is quasi-invex with respect to $\eta\equiv\eta(t,\bar{x},\hat{u})$, then $\bar{x}=\hat{u}$, i.e. $\hat{u}$ is an optimal solution of (CP).*

Proof. We assume that $\bar{x}\neq\hat{u}$ and show that a contradiction occurs. Since $\bar{x}$ is an optimal solution of (CP) and normal, it follows form Theorem 2 that there exist piecewise smooth $\bar{\lambda}:I\to R^m$ with $\bar{\lambda}(t)=(\bar{\lambda}^1(t),\dots\bar{\lambda}^m(t))^T$ $\bar{z}:I\to R^n$ and $\bar{w}^j:I\to R^n$, $(j=1,2,\dots,m)$ such that $(\bar{x},\bar{z},\bar{\lambda},\bar{w}^1,\dots,\bar{w}^m)$ is an optimal solution for (Mix CD) and

$$\int_I\{f(t,\bar{x},\dot{\bar{x}})+s(\bar{x}(t)|K)\}dt$$
$$=\int_I\left\{f(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\bar{z}(t)+\sum_{j\in J_o}\bar{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\bar{w}^j(t))\right\}dt$$

$$=\int_I\left\{f(t_1,\hat{u},\dot{\hat{u}})+\hat{u}(t)^T\hat{z}(t)+\sum_{j\in J_0}\hat{\lambda}^j(t)(g^j(t,\hat{u},\dot{\hat{u}})+\hat{u}(t)^T\hat{w}^j(t))\right\}dt, \quad (6.55)$$

Also, since $\bar{x}$ is feasible for (CP) and $(\hat{u},\hat{z},\hat{\lambda},\hat{w}^1,\dots,\hat{w}^m)$ feasible for (Mix CD), we have

$$\sum_{j\in J_\alpha}\hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}})+s(\bar{x}(t)|C^j))\ \le 0,\ \ \alpha=1,2,\dots,r.$$

This, in view of $\bar{x}(t)^T\hat{w}^j(t)\le s(\bar{x}(t)|C^j), \hat{w}^j(t)\in C^j$, $j=1,2,\dots,m$,

$$\Longrightarrow \sum_{j\in J_\alpha}\int_I\hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\hat{w}^j(t))dt\ \le 0,\ \alpha=1,2,\dots,r. \quad (6.56)$$

Also,

$$\sum_{j\in J_\alpha}\int_I\hat{\lambda}^j(t)(g^j(t,\hat{u},\dot{\hat{u}})+\hat{u}(t)^T\hat{w}^j(t))dt\ge 0,\ \alpha=1,2,\dots,r. \quad (6.57)$$

Combining (56) and (57), we have

$$\sum_{j\in J_\alpha}\int_I \hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\hat{w}^j(t))dt$$
$$\leq \sum_{j\in J_\alpha}\int_I \hat{\lambda}^j(t)(g^j(t,\hat{u},\dot{\hat{u}})+\hat{u}(t)^T\hat{w}^j(t))dt \leq 0,\ \alpha=1,2,\ldots,r.$$

This, in view of quasi-invexity of $\sum_{j\in J_\alpha}\int_I \hat{\lambda}^j(t)(g^j(t,\cdot,\cdot)+(\cdot)^T\hat{w}^j(t))dt$, $\alpha = 1,2,\ldots,r$ with respect to $\eta{=}\eta(t,\bar{x},\hat{u})$ gives,

$$0 \geq \sum_{j\in J_\alpha}\int_I \{\eta^T\hat{\lambda}^j(t)(g^j(t,\hat{u},\dot{\hat{u}})+\hat{w}^j(t))+(D\eta)^T(\hat{\lambda}^j(t)^Tg^j_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))\}dt,$$
$$\alpha=1,2,\ldots,r.$$
$$= \sum_{j\in J_\alpha}\int_I \eta^T\{\hat{\lambda}^j(t)(g^j_x(t,\hat{u},\dot{\hat{u}})+\hat{w}^j(t))-D(\hat{\lambda}(t)^Tg_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))\}dt$$
$$+\sum_{j\in J_\alpha}\eta^T(\hat{\lambda}^j(t)(t)g^j_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))\Big|_{t=a}^{t=b},\ \alpha=1,2,\ldots,r$$

(by integration by parts)

This, on using $\eta=0$ at $t=b$, implies

$$\sum_{j\in J_\alpha}\int_I \eta^T\{\hat{\lambda}^j(t)(g^j_x(t,\hat{u},\dot{\hat{u}})+\hat{w}^j(t))-D(\hat{\lambda}^j(t)g^j_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))\}dt \leq 0,$$
$$\alpha=1,2,\ldots,r.$$

i.e.,

$$\sum_{J\in M-J_0}\int_I \eta^T\Big\{\hat{\lambda}^j(t)(g^j_x(t,\hat{u},\dot{\hat{u}})+\hat{w}^j(t))-D(\hat{\lambda}^j(t)g^j_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))\Big\}dt \leq 0.$$

This, along with (44) implies

$$0 \leq \int_I \eta^T\Bigg[\Big\{(f_x(t,\dot{u},\dot{\hat{u}})+\hat{z}(t))+\sum_{j\in J_0}\hat{\lambda}^j(t)(g^j_{\dot{x}}(t),\hat{u},\dot{\hat{u}})+\hat{w}^j(t))\Big\}$$
$$-D\Big\{f_{\dot{x}}(t,\dot{u},\dot{\hat{u}})+\sum_{j\in J_0}(\hat{\lambda}^j(t)g^j_{\dot{x}}(t,\hat{u},\dot{\hat{u}}))\Big\}\Bigg]dt$$

$$= \int_I \Bigg[\eta^T\Big\{(f_x(t,\hat{u},\dot{\hat{u}})+\hat{z}(t))+\sum_{j\in J_0}\hat{\lambda}^j(t)(g^j_x(t,\hat{u},\dot{\hat{u}})+\hat{w}^j(t))\Big\}$$

$$+(D\eta)^T\left\{f_{\dot{x}}(t,\dot{u},\dot{u})+\sum_{j\in J_0}(\hat{\lambda}^j(t)g^i_{\dot{x}}(t,\hat{u},\dot{u}))\right\}$$

$$-\eta^T\left\{f_{\dot{x}}(t,\dot{u},\dot{u})+\sum_{j\in J_0}(\hat{\lambda}^j(t)g^j_{\dot{x}}(t,\hat{u},\dot{u}))\right\}\Bigg|_{t=a}^{t=b}$$

(by integration by parts)

Using $\eta = 0$, at $t = a$ and $t = b$, this gives

$$\int_I\left[\eta^T\left\{(f_x(t,\hat{u},\dot{u})+\hat{z}(t))+\sum_{j\in J_0}\hat{\lambda}^j(t)(g^j_{\dot{x}}(t,\hat{u},\dot{u})+\hat{w}^j(t))\right\}\right.$$

$$\left.+(D\eta)^T\left\{f_{\dot{x}}(t,\dot{u},\dot{u})+\sum_{j\in J_0}(\hat{\lambda}^j(t)g^i_{\dot{x}}(t,\hat{u},\dot{u}))\right\}dt\geq 0.\right.$$

By strict pseudoinvexity of

$$\int_I\left\{f(t,\cdot,\cdot)+(\cdot)^T\hat{z}(t)+\sum_{j\in J_o}\hat{\lambda}^j(t)(g^j(t,\cdot,\cdot)+(\cdot)^T\hat{w}^j(t))\right\}dt$$

with respect to $\eta = \eta(t,\bar{x},,\hat{u},)$ for $\hat{z}(t)\in C$ and $\hat{w}^j(t)\in D_j$, $j = 1,2,\ldots,m$, this implies

$$\int_I\left\{f(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\hat{z}(t)+\sum_{j\in J_o}\hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\hat{w}_j(t))\right\}dt$$

$$>\int_I\left\{f(t,\hat{u},\dot{\hat{u}})+\hat{u}(t)^T\hat{z}(t)+\sum_{j\in J_0}\hat{\lambda}(t)(g^j(t,\hat{u},\dot{\hat{u}})+\hat{u}(t)^T\hat{w}^j(t))\right\}dt$$

$$=\int_I\left\{f(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\bar{z}(t)+\sum_{j\in J_0}\bar{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\bar{w}^j(t))\right\}dt,$$

(by (55)).

This, because of $\bar{x}(t)^T\bar{z}(t) = s(\bar{x}(t)|K)$ and $\sum_{j=1}^{m}\bar{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\bar{w}^j(t)) = 0$, $t\in I$, yields

$$\int_I\left\{\bar{x}(t)^T\hat{z}(t)+\sum_{j=1}\hat{\lambda}^j(t)(g^j(t,\bar{x},\dot{\bar{x}})+\bar{x}(t)^T\hat{w}^j(t))\right\}dt>\int_I s(\bar{x}(t)|K)dt$$

Because of $\bar{x}(t)^T\hat{z}(t)\leq s(\bar{x}(t)|K)$, $\hat{z}(t)\in K$, $t\in I$ and $\bar{x}(t)^T\hat{w}^j(t)\leq s(\bar{x}(t)|C^j)$, for $\hat{w}^j(t)\in C^j$ $(j = 1,2,\ldots,m)$, $t\in I$, from this we have,

$$\int_I\left\{s(\bar{x}(t)|K)+\sum_{j\in 1}\hat{\lambda}^j(g^j(t,\bar{x},\dot{\bar{x}})+s(\bar{x}(t)|C^j)\right\}dt>\int_I s(\bar{x}(t)|K)dt.$$

By $\hat{y}^j(t) \geq 0$ and $g^j(t, \bar{x}, \dot{\bar{x}}) + s(\bar{x}(t)|C^j) \leq 0$, $t \in I$, $j = 1, 2, \ldots, m$, this gives,

$$\int_I s(\bar{x}(t)|C)dt > \int_I s(\bar{x}(t)|C)dt.$$

This is not possible. Hence $\bar{x} = \hat{u}$ i.e. $\hat{u}$ is an optimal solution of (CP). □

7. Special cases

Let for $t \in I$, $B(t)$ and $D^j(t)$, $(j = 1, 2, \ldots, m)$ be positive semidefinite matrices and continuous on I. Then $(x(t)^T B(t)x(t))^{1/2} = s(x(t)|K)$, $t \in I$ where $K = \{B(t)z(t)|z(t)^T B(t)z(t) \leq 1, t \in I\}$ and $(x(t)^T D^j(t)x(t))^{1/2} = s(x(t)|C^j)$, $j = 1, 2, \ldots, m$, $t \in I$ where $C^j = \{D^j(t)w^j(t)|w^j(t)^T D^j(t)w^j(t) \leq 1, t \in I\}$. This is possible from [14].

Replacing the support functions by their corresponding square root of quadratic forms, we have

Primal (CP_0) :
$$\underset{x \in X}{\text{Minimize}} \int_I [f(t, x, \dot{x}) + (x(t)^T B(t)x(t))^{1/2}]dt$$
subject to
$$x(a) = \alpha, x(b) = \beta,$$
$$g^j(t, x, \dot{x}) + (x(t)^T D^j(t)x(t))^{1/2} \leq 0, t \in I, j = 1, 2, \ldots, m.$$

Dual (WCD_0) :
$$\underset{u \in X, \lambda, z, w^1, \ldots, w^m}{\text{Maximize}} \int_I \Big[f(t, u, \dot{u}) + u(t)^T B(t)z(t) + \sum_{j=1}^{m} \lambda^j(t)(g^j(t, u, \dot{u}) + u(t)^T D^j(t)w^j(t)) \Big] dt$$
subject to
$$u(a) = \alpha, u(b) = \beta,$$
$$f_{\dot{x}}(t, u, \dot{u}) + B(t)z(t) + \sum_{j=1}^{m} \lambda^I(t)(g_x^j(t, u, \dot{u}) + D^j(t)w^j(t)) = D[f_{\dot{x}}(t, u, \dot{u}) + \lambda(t)^T g_{\dot{x}}(t, u, \dot{u})], t \in I,$$
$$z(t)^T B(t)z(t) \leq 1, \ t \in I,$$
$$w^j(t)D^j(t)w^j(t) \leq 1, \ t \in I, \ (j = 1, 2, \ldots m),$$
$$\lambda(t) \geq 0, \ t \in I.$$

Dual$(M - WCD_0)$:
$$\underset{u \in X, \lambda, z, w^1, \ldots, w^m}{\text{Maximize}} \int_I [f(t, u, \dot{u}) + u(t)^T B(t)z(t)]dt$$
subject to
$$x(a) = \alpha, x(b) = \beta$$

$$
\begin{aligned}
&f_{\dot{x}}(t,u,\dot{u}) + B(t)z(t) + \sum_{j=1}^{m} \lambda^{I}(t)(g_x^j(t,u,\dot{u}) + D^j(t)w^j(t)) \\
&= D[f_{\dot{x}}(t,u,\dot{u}) + \lambda(t)^T g_{\dot{x}}(t,u,\dot{u})], t \in I, \\
&\sum_{j=1}^{m} \int_I \lambda^j(t)(g_x^j(t,u,\dot{u}) + u(t)^T D^j(t)w^j(t))dt \geq 0,\ t \in I, \\
&z(t)^T B(t)z(t) \leq 1,\ t \in I, \\
&w^j(t)^T D^j(t)w^j(t) \leq 1,\ t \in I,\ (j = 1,2,\dots m), \\
&\lambda(t) \geq 0,\ \ t \in I.
\end{aligned}
$$

The above Wolfe type dual problem (WCD_0) and Mond-Weir type dual problem (M-WCD_0) have not been reported in the literature of continuous programming but if $D^j(t) = 0, t \in I$ and $j \in \{1,2,\dots,m\}$, then the Wolfe dual problem (WCD_0) studied by Chandra, Craven and Husain [4] under convexity while the pair of Mond-Weir type dual problem (M-WCD_0), treated by Bector, Chandra and Husain [1]. The mixed dual (Mix CD) reduces to

$$
(\text{MixCD}_0) \quad : \quad \underset{u \in X, \lambda, z, w^1, \dots, w^m}{\text{Maximize}} \int_I \Big[f(t,u,\dot{u}) + u(t)^T B(t)z(t) + \sum_{j \in J_0} \lambda^j(t)(g^j(t,u,\dot{u}) + u(t)^T D^j(t)w^j(t)) \Big] dt
$$

subject to

$$
\begin{aligned}
&u(a) = \alpha, u(b) = \beta, \\
&f_x(t,u,\dot{u}) + B(t)z(t) + \sum_{j=1}^{m} \lambda^j(t)(g_x^j(t,u,\dot{u}) + D^j(t)w^j(t)) \\
&\quad = D[f_{\dot{x}}(t,u,\dot{u}) + \lambda(t)^T g_{\dot{x}}(t,u,\dot{u})],\ t \in I, \\
&\sum_{j \in J_\alpha} \int_I \lambda^{jI}(t)(g^j(t,u,\dot{u}) + u(t)^T D^j(t)w^j(t))dt \geq 0, \alpha = 1,2,\dots,r, \\
&z(t)^T B(t)z(t) \leq 1,\ t \in I \\
&w^j(t)^T D^j(t)w^j(t) \leq 1,\ t \in I,\ (j = 1,2,\dots m) \\
&\lambda(t) \geq 0,\ \ t \in I.
\end{aligned}
$$

The duality amongst (CP_0) and (Mix CD_0) can be investigated analogously to that amongst (CP) and (M-WCD), presented in the preceding section.

8.Related non-linear programming problems

If the time dependency of the problems (CP), (WCD), (M-WCD) and (Mix CD) is removed, then these problems reduce to the following problem (P_1) with its duals (WD_1) and (M-WD_1) recently studied by Husain, Abha and Jabeen [11], and mixed type dual (Mix D_1) studied by Husain and Jabeen [12].

$$\text{Primal}(P_1) \;:\; \text{Minimize} f(x) + s(x|K)$$
subject to
$$g^j(x) + s(x|C^j) \leq 0, j = 1, 2, \ldots, m.$$

$$\text{Dual}(WD_1) \;:\; \underset{u,\lambda,z,w^1,\ldots,w^m}{\text{Maximize}}\; f(u) + u^T z + \sum_{j=1}^{m} \lambda^j (g^j(u) + u^T w^j)$$
subject to
$$f_x(u) + z + \sum_{j=1}^{m} \lambda^j(t)(g_x^j(u) + w^j) = 0,$$
$$z \in K, w^j \in C^j, j = 1, 2, \ldots, m,$$
$$\lambda \geq 0.$$

$$\text{Dual}(M - WD_1) \;:\; \underset{u,\lambda,z,w^1,\ldots,w^m}{\text{Maximize}}\; f(u) + u^T z$$
subject to
$$f_x(u) + z + \sum_{j=1}^{m} \lambda^j (g_x(u) + w^j) = 0,$$
$$\sum_{j=1}^{m} \lambda^j (g^j(u) + u^T w^j) \geq 0,$$
$$z \in K, w^j \in C^j, j = 1, 2, \ldots, m,$$
$$\lambda \geq 0.$$

$$\text{Dual}(MixD_1) \;:\; \underset{u\lambda,z,w^1,\ldots,w^m}{\text{Maximize}}\; f(u) + u^T z + \sum_{jj \in J_0} \lambda^j (g(u) + u^T w^j)$$
subject to
$$f_x(u) + z + \sum_{j=1}^{m} \lambda^j (g_x(u) + w^j) = 0$$
$$\sum_{jj \in J_\alpha} \lambda^j (g(u) + u^T w^j) \geq 0, \alpha = 1, 2, \ldots, r$$
$$z \in K, w^j \in C^j, j = 1, 2, \ldots, m$$
$$\lambda \geq 0.$$

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I. Husain
Department of Mathematics, Jaypee Institute of Engineering & Technology
AB-Road, Raghogarh, Guna (M.P.)-473 226, India
Email: iqbal.husain@jiet.ac.in, or ihusain11@yahoo.com

I. Husain and **Z. Jabeen**
Department of Mathematics, Regional Engineering College, Hazratbal, Srinagar 190 006, Kashmir, India
e-mail: ihusain11@yahoo.com