J. Appl. Math. & Informatics Vol. **26***(2008), No. 1 - 2, pp. 45 - 60*
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COMPUTATION OF THE DYNAMIC FORCE COMPONENT ON A VERTICAL CYLINDER DUE TO SECOND ORDER WAVE DIFFRACTION

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ABSTRACT. Here we consider the evaluation of the the dynamic component of the second order force due to wave diffraction by a circular cylinder analytically and numerically. The cylinder is fixed, vertical, surface piercing in water of finite uniform depth. The formulation of the wave-structure interaction is based on the assumption of a homogeneous, ideal, incompressible, and inviscid fluid. The nonlinearity in the wave-structure interaction problem arises from the free surface boundary conditions, namely, dynamic and kinematic free surface boundary conditions. We expand the velocity potential and free surface elevation functions in terms of a small parameter and then consider the second order diffraction problem. After deriving the pressure using Bernoulli's equation, we obtain the analytical expression for the dynamic component of the second order force on the cylinder by integrating the pressure over the wetted surface. The computation of the dynamic force component requires only the first order velocity potential. Numerical results for the dynamic force component are presented.

AMS Mathematics Subject Classification : 76B15, 35Q35, 35J05.
Key words and phrases : Dynamic free surface condition, kinematic free surface condition, water wave, second order diffraction, wave force components.

1. Introduction

Designing safe offshore structures is one of the aims of many companies throughout the world. The forces due to surface waves on offshore structures such as drilling rigs or submerged oil storgae tanks are important in designing large submerged or semi-merged structures. The accurate prediction of the water wave forces on offshore structures is one of the main interests in designing safe offshore structures. The structure may be fixed or floating as semi-merged

Received January 23, 2007. Revised July 24, 2007.

structure in sea. There is a large number of structures which are composed of tubular members like circular cylinders. When the structure spans a significant amount of wavelength, the incident waves undergo scattering or diffraction. Diffraction of waves needs to be considered while evaluating the wave forces.

Many scientific investigations have been perfomed by many scientists and engineers in the field of floating and fixed structures. The linear problem at the first order for a fixed cylinder (diffraction problem) was first solved by MacCamy and Fuchs [11]. The problem of scattering of surface waves was carried out by Miles and Gilbert [12] and then by Garrett [8]. Black, Mei and Bray [1] have calculated the wave forces on a truncated cylinder which either extends to the free surface or rests on the seabed. Garrison [9] presented a numerical method for the compuatations of wave excitation forces as well as added mass and damping coefficients for large objects in water of finite depth. The investigation of nonlinear diffraction theory was introduced by many researchers including Chakrabarti [2, 3] to correlate experimental data with the theory. Solutions for standing and progressive small amplitude water waves provide the basis for application to numerous problems of engineering interest. Dean and Dalrymple [5] presented the formulation of the linear water ware theory and development of the simplest two-dimensional solution for standing and progressive waves. Debnath [6] discussed theoretical studies of nonlinear water waves over the last few decades. He studied the theory of nonlinear shallow water waves and solitons, with emphasis on methods and solutions of several evolution equations that are originated in the theory of water waves. Rahman [13, 14] presented a nonlinear wave loads on a large circular cylinder using perturbation technique. A second order solution for the diffraction of nonlinear progressive wave in deep water was derived by Hunt and Baddour [10]. Rahman and Heaps [15] studied nonlinear theory of wave diffraction and derived the forces exerted on a cylinder of large diameter using perturbation technique. Analytical solution was expressed in the form an integral. Eatock Taylor and Hung [7] presented analytical expressions for the second order force on a circular cylinder. Chau and Eatock Taylor [4] provided a detailed analysis of the second order diffraction problem of a uniform vertical cylinder in regular waves.

In the present work, we consider the evaluation of the the dynamic component of the second order force due to wave diffraction by a circular cylinder analytically and numerically. The cylinder is fixed, vertical, surface piercing in water of finite uniform depth. The formulation of the wave-structure interaction is based on the assumption of a homogeneous, ideal, incompressible, and inviscid fluid. We expand the velocity potential and free surface elevation functions in terms of a small parameter and then consider the second order diffraction problem. After deriving the pressure using Bernoulli's equation, we obtain the analytical expression for the dynamic component of the second order force on the cylinder by integrating the pressure over the wetted surface. The computation of the dynamic force component requires only the first order velocity potential. Numerical results for the dynamic force component are presented.

2. Boundary value problem in terms of velocity potential

Here we consider a fixed, vertical, circular cylinder in water of finite uniform depth. The cylinder extends from sea floor ($z = -h$) to the free surface ($z = \eta(x, y, t)$). Water depth is h, the free surface elevation function is $\eta(x, y, t)$ and the radius of the cylinder is a. Incident wave is propagating along positive x-direction. The incoming wave incident upon the surface of the cylinder undergoes a scattering or diffraction. To evaluate the wave loads on the cylinder, we need to consider the effect due to the incident wave and the diffracted wave.

The cylindrical coordinate system (r, θ, z) with z vertically upwards from the still water level (SWL), r measured radially from the z-axis and θ from the positive $x-$axis is used. For Cartesian coordinates (x, y, z), $xy-$plane represents the still water level (SWL) and z-axis positive upward from the SWL. Cartesian and Cylindrical coordinates are related by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

The formulation is based on the assumptions of ideal, incompressible and inviscid fluid. We assume that sea floor is flat, horizontal and located at $z = -h$.

The equation of continuity for a fluid with the velocity $\mathbf{v}$ and density ρ is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{1}$$

For an incompressible fluid, the continuity equation is

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

where $\mathbf{v}$ is the velocity of the fluid. For an incompressible and inviscid fluid with irrotational motion, we can introduce a velocity potential $\phi(r, \theta, z, t)$ such that

$$\mathbf{v} = \nabla \phi(r, \theta, z, t) \tag{3}$$

Equations (2) and (3) yield that $\phi(r, \theta, z, t)$ satisfies the Laplace equation in the fluid domain, i.e.,

$$\nabla^2 \phi(r, \theta, z, t) = 0. \tag{4}$$

We will assume that ϕ is time harmonic.

The force components F_x, F_y along x, y directions are given by

$$F_x = \int_{\theta=0}^{2\pi} \int_{z=-h}^{\eta} P(a, \theta, z, t)(-\cos\theta)\, dA, \tag{5}$$

$$F_y = \int_{\theta=0}^{2\pi} \int_{z=-h}^{\eta} P(a, \theta, z, t)(-\sin\theta)\, dA \tag{6}$$

respectively. Here $P(a, \theta, z, t)$ is the pressure on the curved surface of the cylinder which can be computed from the velocity potential ϕ using Bernoulli's equation, η is the free surface elevation function and $dA = a d\theta dz$.

The boundary value problem in terms of $\phi(x, y, z, t)$ and $\eta(x, y, t)$ can be expressed as governing Equation :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \tag{7}$$

dynamic free surface boundary condition at $z = \eta$:

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = 0. \tag{8}$$

Here g is the acceleration due to gravity. Kinematic free surface boundary condition at $z = \eta$ is :

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} - \frac{\partial \phi}{\partial z} = 0. \tag{9}$$

Since the cylinder is fixed, the normal velocity is zero, so the body surface boundary condition on the curved surface is given by

$$\frac{\partial \phi}{\partial n} = 0, \qquad r = a \tag{10}$$

where n is the outward normal. Assuming that sea floor is flat and horizontal, the bottom boundary condition at $z = -h$ can be expressed as

$$\frac{\partial \phi}{\partial z} = 0, \qquad z = -h \tag{11}$$

The velocity potential ϕ can be expressed as $\phi = \phi_I + \phi_D$ where ϕ_I and ϕ_D represent the incident velocity potential and diffracted velocity potential respectively. The diffracted potential ϕ_D satisfies the radiation condition at infinity, i.e.,

$$\lim_{r \to \infty} \sqrt{r}\left(\frac{\partial \phi_D}{\partial r} - ik\phi_D\right) = 0$$

where $r = \sqrt{x^2 + y^2}$ and k is the wavenumber.

The velocity potential $\phi(x, y, z, t)$ and the surface elevation $\eta(x, y, t)$ can be written in terms of Stokes expansion as

$$\phi(x, y, z, t) = \epsilon\phi_1(x, y, z, t) + \epsilon^2\phi_2(x, y, z, t) + O(\epsilon^3) + \ldots \tag{12}$$

$$\eta(x, y, t) = \epsilon\eta_1(x, y, t) + \epsilon^2\eta_2(x, y, t) + O(\epsilon^3) + \ldots. \tag{13}$$

Here ϵ is the dimensionless small parameter defined by $(= kA)$ where k $(= 2\pi/L)$ the wavenumber, L the wavelength, A wave amplitude. ϕ_1 is the first expansion term corresponding to a linear approximation of the velocity potential, ϕ_2 is the second order approximation. η_1 and η_2 are the first order (linear) and second order approximations of the elevation function.

At the free surface, we have $z = \eta(x, y, t)$, so $\phi(x, y, z, t) = \phi(x, y, \eta, t)$. Expanding by Taylor's theorem about $z = 0$, we have

$$\begin{aligned}\phi(x,y,\eta,t) &= \phi(x,y,0,t) + \eta\left(\frac{\partial\phi}{\partial z}\right)_{z=0} + \frac{\eta^2}{2}\left(\frac{\partial^2\phi}{\partial z^2}\right)_{z=0} + \ldots \\ &= \epsilon\phi_1(x,y,0,t) + \epsilon^2\phi_2(x,y,0,t) + \ldots \\ &\quad + \left(\epsilon\eta_1 + \epsilon^2\eta_2 + \ldots\right)\left\{\frac{\partial}{\partial z}\left(\epsilon\phi_1 + \epsilon^2\phi_2 + \ldots\right)\right\}_{z=0} \\ &\quad + \frac{\left(\epsilon\eta_1 + \epsilon^2\eta_2 + \ldots\right)^2}{2}\left\{\frac{\partial^2}{\partial z^2}\left(\epsilon\phi_1 + \epsilon^2\phi_2 + \ldots\right)\right\}_{z=0}\end{aligned}$$

The modified velocity potential at the free surface is

$$\phi(x,y,\eta,t) = \epsilon\phi_1(x,y,0,t) + \epsilon^2\left[\phi_2(x,y,0,t) + \eta_1\left(\frac{\partial\phi_1}{\partial z}\right)_{z=0}\right] + O\left(\epsilon^3\right). \quad (14)$$

Dynamic free surface boundary condition now is given by

$$\begin{aligned}&\epsilon\left(\frac{\partial\phi_1}{\partial t} + g\eta_1\right) + \epsilon^2\left[\frac{\partial\phi_2}{\partial t} + g\eta_2 + \eta_1\frac{\partial^2\phi_1}{\partial t\partial z}\right. \\ &\left. + \frac{1}{2}\left\{\left(\frac{\partial\phi_1}{\partial x}\right)^2 + \left(\frac{\partial\phi_1}{\partial y}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2\right\}\right] + O\left(\epsilon^3\right) = 0,\ z = 0.\end{aligned}$$

Kinematic free surface boundary condition becomes

$$\begin{aligned}&\epsilon\left(\frac{\partial\eta_1}{\partial t} - \frac{\partial\phi_1}{\partial z}\right) + \epsilon^2\left(\frac{\partial\eta_2}{\partial t} + \frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x}\right. \\ &\left. + \frac{\partial\phi_1}{\partial y}\frac{\partial\eta_1}{\partial y} - \frac{\partial\phi_2}{\partial z} - \eta_1\frac{\partial^2\phi_1}{\partial z^2}\right) + O\left(\epsilon^3\right) = 0, \quad z = 0.\end{aligned}$$

Thus the free surface boundary conditions for the second order theory can be derived from the coefficients of ϵ^2.
The seond order dynamic free surface boundary condition is given by

$$\begin{aligned}&\frac{\partial\phi_2}{\partial t} + g\eta_2 + \eta_1\frac{\partial^2\phi_1}{\partial t\partial z} + \\ &\frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial x}\right)^2 + \left(\frac{\partial\phi_1}{\partial y}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2\right] = 0, \qquad z = 0. \quad (15)\end{aligned}$$

The second order kinematic free surface boundary condition can be written as

$$\frac{\partial\eta_2}{\partial t} + \frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x} + \frac{\partial\phi_1}{\partial y}\frac{\partial\eta_1}{\partial y} - \frac{\partial\phi_2}{\partial z} + \eta_1\frac{\partial^2\phi_1}{\partial z^2} = 0, \qquad z = 0. \quad (16)$$

Eliminating η_1 and η_2 from the equations (15) and (16), the combined free surface condition is obtained as

$$\frac{\partial^2 \phi_2}{\partial t^2} + g\frac{\partial \phi_2}{\partial z} = -\eta_1 \frac{\partial}{\partial z}\left(\frac{\partial^2 \phi_1}{\partial t^2} + g\frac{\partial \phi_1}{\partial z}\right)$$
$$-\frac{\partial}{\partial t}\left[\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial y}\right)^2 + \left(\frac{\partial \phi_1}{\partial z}\right)^2\right], \qquad z = 0. \tag{17}$$

Now the pressure $P(x, y, z, t)$ can be determined from Bernoulli's equation

$$\frac{P}{\rho} + gz + \frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = 0 \tag{18}$$

where $(\nabla\phi)^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{r\partial\theta}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2$.
Substituting power series expansion for ϕ, we have

$$P = -\rho g z - \epsilon\rho\frac{\partial \phi_1}{\partial t} - \epsilon^2\rho\left[\frac{1}{2}\left\{(\nabla\phi_1)^2\right\} + \frac{\partial \phi_2}{\partial t}\right] + O\left(\epsilon^3\right). \tag{19}$$

The total horizontal force can be obtained by integrating the wetted surface of the cylinder and can be expressed as

$$F_x = \int_0^{2\pi}\int_{-h}^{\eta} P(a, \theta, z, t)(-a\cos\theta)\, dz\, d\theta.$$

Since the incident wave is parallel to $x-$axis, $F_y = 0$. Writing the $z-$integral as the sum of $\int_{-h}^{0} + \int_0^{\eta}$, we obtain

$$F_x = a\rho\int_0^{2\pi}\left[\int_{-h}^{0}\left\{gz + \epsilon\frac{\partial \phi_1}{\partial t} + \epsilon^2\left(\frac{1}{2}(\nabla\phi_1)^2 + \frac{\partial \phi_2}{\partial t}\right)\right\}_{r=a} dz\right.$$
$$\left. + \int_0^{\epsilon\eta_1+\epsilon^2\eta_2}\left\{gz + \epsilon\frac{\partial \phi_1}{\partial t} + \epsilon^2\left(\frac{1}{2}(\nabla\phi_1)^2 + \frac{\partial \phi_2}{\partial t}\right)\right\}_{r=a} dz\right]\cos\theta d\theta. \tag{20}$$

Using the series expansion for force components, we have

$$F_x = \epsilon F_{x_1} + \epsilon^2 F_{x_2} + \epsilon^3 F_{x_3} + \ldots \tag{21}$$

where F_{x_i} stands for the i^{th}-order contribution.

The velocity potential function for the first order diffraction problem can be obtained as

$$\phi_1(r, \theta, z) = \frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh} Re\left[\sum_{m=0}^{\infty}\beta_m i^m\left\{J_m(kr)\right.\right.$$
$$\left.\left. - \frac{J_m'(ka)}{H_m^{(1)'}(ka)} H_m^{(1)}(kr)\right\} e^{-i\sigma t}\cos m\theta\right]. \tag{22}$$

Here $\beta_0 = 1$, $\beta_m = 2$ for $m \geq 1$, $J_m(kr)$ and $H_m^{(1)}(kr)$ $(= J_m(kr) + iY_m(kr))$ are Bessel and Hankel functions of first kind of order m respectively and σ is the frequency. We use the separation of variables method to obtain the solution by writing $\Phi_1(r,\theta,z) = R(r)\,\Theta(\theta)\,Z(z)$. This yields Bessel equation, for the incident potential we obtain the solution in terms of Bessel function of first kind and for the scattered potential we obtain the solution in terms of Hankel function of first kind. Also from the first order combined free surface condition, we have

$$\sigma^2 = gk\tanh kh. \tag{23}$$

This relation is known as the dispersion relation for the first order problem.

The second order contribution to the force on the cylinder is given by

$$\begin{aligned}\epsilon^2 F_{x_2} &= a\rho\int_0^{2\pi}\left[\int_0^{\epsilon\eta_1}\left\{gz+\epsilon\frac{\partial\phi_1}{\partial t}\right\}_{r=a}dz\right.\\ &\quad\left.+\,\epsilon^2\int_{-h}^{0}\left\{\frac{1}{2}(\nabla\phi_1)^2+\frac{\partial\phi_2}{\partial t}\right\}_{r=a}dz\right]\cos\theta\,d\theta \qquad (24)\\ &= \epsilon^2\left\{F_{x_2}^{(1)}+F_{x_2}^{(2)}+F_{x_2}^{(3)}+F_{x_2}^{(4)}\right\}. \qquad (25)\end{aligned}$$

The third component $F_{x_2}^{(3)}$ is known as the dynamic component of the second order force. We will concentrate on derivation and evaluation of this component which requires the first order velocity potential function.

Thus the dynamic component of the second order force is given by

$$F_{x_2}^{(3)} = a\rho\int_0^{2\pi}\left[\int_{-h}^{0}\frac{1}{2}(\nabla\phi_1)^2_{r=a}\,dz\right]\cos\theta\,d\theta \tag{26}$$

where $(\nabla\phi_1)^2 = \left(\frac{\partial\phi_1}{\partial r}\right)^2+\left(\frac{\partial\phi_1}{r\partial\theta}\right)^2+\left(\frac{\partial\phi_1}{\partial z}\right)^2$.

For two complex numbers U and V, we can write

$$Re\left[U\,e^{-i\sigma t}\right]Re\left[V\,e^{-i\sigma t}\right] = \frac{1}{2}Re\left[UV\,e^{-2i\sigma t}+U\,\overline{V}\right]. \tag{27}$$

Using the result mentioned in equation (27), we obtain

$$\begin{aligned}\left(\frac{\partial\phi_1}{\partial r}\right)^2 &= \frac{1}{2}\left[Re\left\{\left(\frac{\partial\Phi_1}{\partial r}\right)^2 e^{-2i\sigma t}\right\}+\frac{\partial\Phi_1}{\partial r}\overline{\frac{\partial\Phi_1}{\partial r}}\right]\\ \left(\frac{\partial\phi_1}{r\partial\theta}\right)^2 &= \frac{1}{2}\left[Re\left\{\left(\frac{\partial\Phi_1}{r\partial\theta}\right)^2 e^{-2i\sigma t}\right\}+\frac{\partial\Phi_1}{r\partial\theta}\overline{\frac{\partial\Phi_1}{r\partial\theta}}\right]\\ \left(\frac{\partial\phi_1}{\partial z}\right)^2 &= \frac{1}{2}\left[Re\left\{\left(\frac{\partial\Phi_1}{\partial z}\right)^2 e^{-2i\sigma t}\right\}+\frac{\partial\Phi_1}{\partial z}\overline{\frac{\partial\Phi_1}{\partial z}}\right].\end{aligned}$$

Thus we have

$$(\nabla\phi_1)^2_{r=a} = \frac{1}{2}\left[Re\left\{(\nabla\Phi_1)^2 e^{-2i\sigma t}\right\} + \frac{\partial\Phi_1}{\partial r}\overline{\frac{\partial\Phi_1}{\partial r}} + \frac{\partial\Phi_1}{r\partial\theta}\overline{\frac{\partial\Phi_1}{r\partial\theta}} + \frac{\partial\Phi_1}{\partial z}\overline{\frac{\partial\Phi_1}{\partial z}}\right]_{r=a}. \tag{28}$$

Now defining $B_n = \frac{\beta_n i^{n+1}}{H_n^{(1)'}(ka)}$, we obtain

$$\left(\frac{\partial\Phi_1}{\partial z}\right)^2_{r=a} = \frac{4g^2A^2}{\sigma^2\pi^2a^2}\frac{\sinh^2 k(z+h)}{\cosh^2 kh}\left(\sum_{n=0}^{\infty} B_n\cos n\theta\right)^2$$

$$\left(\frac{\partial\Phi_1}{r\partial\theta}\right)^2_{r=a} = \frac{4g^2A^2}{\sigma^2\pi^2k^2a^4}\frac{\cosh^2 k(z+h)}{\cosh^2 kh}\left(\sum_{n=0}^{\infty} nB_n\sin n\theta\right)^2$$

$$\left(\frac{\partial\Phi_1}{\partial r}\right)^2_{r=a} = 0.$$

Since

$$\int_{-h}^{0}\cosh^2 k(z+h)\,dz = \frac{1}{2}\int_{-h}^{0}[\cosh 2k(z+h)+1]\,dz = \frac{h}{2}\left(\frac{\sinh 2kh}{2kh}+1\right) \quad \text{and}$$

$$\int_{-h}^{0}\sinh^2 k(z+h)\,dz = \frac{h}{2}\left(\frac{\sinh 2kh}{2kh}-1\right),$$

we obtain

$$\int_{-h}^{0}\left(\frac{\partial\Phi_1}{\partial z}\right)^2_{r=a}dz = \frac{4g^2A^2}{\pi^2a^2\sigma^2}\frac{h}{2\cosh^2 kh}\left(\frac{\sinh 2kh}{2kh}-1\right)\left(\sum_{n=0}^{\infty} B_n\cos n\theta\right)^2 = \frac{2gA^2}{\pi^2k^2a^2}\left(1-\frac{2kh}{\sinh 2kh}\right)\left(\sum_{n=0}^{\infty} B_n\cos n\theta\right)^2$$

$$\int_{-h}^{0}\left(\frac{\partial\Phi_1}{r\partial\theta}\right)^2_{r=a}dz = \frac{2gA^2}{\pi^2k^4a^4}\left(1+\frac{2kh}{\sinh 2kh}\right)\left(\sum_{n=0}^{\infty} nB_n\sin n\theta\right)^2$$

and on $r=a$,

$$\int_{-h}^{0}\left(\frac{\partial\Phi_1}{\partial z}\right)\overline{\left(\frac{\partial\Phi_1}{\partial z}\right)}dz = \frac{2gA^2}{\pi^2k^2a^2}\left(1-\frac{2kh}{\sinh 2kh}\right) \times\left(\sum_{n=0}^{\infty} B_n\cos n\theta\right)\left(\sum_{m=0}^{\infty}\overline{B_m}\cos m\theta\right) \tag{29}$$

$$\int_{-h}^{0}\left(\frac{\partial\Phi_1}{r\partial\theta}\right)\overline{\left(\frac{\partial\Phi_1}{r\partial\theta}\right)}dz = \frac{2gA^2}{\pi^2k^4a^4}\left(1+\frac{2kh}{\sinh 2kh}\right) \times\left(\sum_{n=0}^{\infty}nB_n\sin n\theta\right)\left(\sum_{m=0}^{\infty}m\overline{B_m}\sin m\theta\right). \tag{30}$$

Multiplying the equations (29) and (30) by $\cos\theta$, and integrating with respect to θ from 0 to 2π, we have

$$\int_0^{2\pi}\int_{-h}^{0}\left(\frac{\partial\Phi_1}{\partial z}\right)\overline{\left(\frac{\partial\Phi_1}{\partial z}\right)}dz\,\cos\theta d\theta = -\frac{8gA^2}{\pi k^2a^2}\left(1-\frac{2kh}{\sinh 2kh}\right)\sum_{l=0}^{\infty}P_l(ka) \tag{31}$$

$$\int_0^{2\pi}\int_{-h}^{0}\left(\frac{\partial\Phi_1}{r\partial\theta}\right)\overline{\left(\frac{\partial\Phi_1}{r\partial\theta}\right)}dz\,\cos\theta d\theta = -\frac{8gA^2}{\pi^2k^4a^4}\left(1+\frac{2kh}{\sinh 2kh}\right) \times\sum_{l=0}^{\infty}l(l+1)P_l(ka) \tag{32}$$

where

$$P_l(ka) = \frac{J'_{l+1}(ka)Y'_l(ka)-J'_l(ka)Y'_{l+1}(ka)}{D_l(ka)D_{l+1}(ka)}, \tag{33}$$

$$D_l(ka) = \left(J'_l(ka)\right)^2+\left(Y'_l(ka)\right)^2 \tag{34}$$

and

$$\int_0^{2\pi}\int_{-h}^{0}Re\left[\left(\frac{\partial\Phi_1}{\partial z}\right)^2e^{-2i\sigma t}\right]dz\,\cos\theta d\theta = \frac{8gA^2}{\pi k^2a^2}\left(1-\frac{2kh}{\sinh 2kh}\right) \times\sum_{l=0}^{\infty}(-1)^{l+1}\left\{Q_l(ka)\cos 2\sigma t+R_l(ka)\sin 2\sigma t\right\} \tag{35}$$

and

$$\int_0^{2\pi}\int_{-h}^{0}Re\left[\left(\frac{\partial\Phi_1}{r\partial\theta}\right)^2e^{-2i\sigma t}\right]dz\,\cos\theta d\theta = \frac{8gA^2}{\pi^2k^4a^4}\left(1+\frac{2kh}{\sinh 2kh}\right) \times\sum_{l=0}^{\infty}(-1)^{l+1}l(l+1)\left\{Q_l(ka)\cos 2\sigma t+R_l(ka)\sin 2\sigma t\right\}. \tag{36}$$

Here

$$Q_l(ka) = -\frac{J'_l(ka)Y'_{l+1}(ka)+J'_{l+1}(ka)Y'_l(ka)}{D_l(ka)D_{l+1}(ka)}, \tag{37}$$

$$R_l(ka) = \frac{Y'_l(ka)Y'_{l+1}(ka)-J'_l(ka)J'_{l+1}(ka)}{D_l(ka)D_{l+1}(ka)}. \tag{38}$$

Hence dynamic component in the second order force is

$$\begin{aligned} F_{x_2}^{(3)} &= a\rho \int_0^{2\pi} \left[\int_{-h}^{0} \frac{1}{2} \left(\nabla\phi_1\right)^2_{r=a} dz \right] \cos\theta \, d\theta \\ &= \frac{a\rho}{4} \frac{8gA^2}{\pi k^2 a^2} \left[\left(1 - \frac{2kh}{\sinh 2kh}\right) \sum_{l=0}^{\infty} \{-P_l \right. \\ & \quad + (-1)^{l+1} \left(Q_l \cos 2\sigma t + R_l \sin 2\sigma t\right)\} \\ & \quad \left(1 + \frac{2kh}{\sinh 2kh}\right) \sum_{l=0}^{\infty} \frac{l(l+1)}{k^2a^2} \{-P_l \\ & \quad \left. + (-1)^{l+1} \left(Q_l \cos 2\sigma t + R_l \sin 2\sigma t\right)\} \right]. \end{aligned}$$

Nondimensional dynamic force component is

$$\begin{aligned} F_2 = \frac{F_{x_2}^{(3)}}{\rho g a A^2} = \frac{2}{\pi (ka)^2} \sum_{l=0}^{\infty} & \left[\left\{ \left(1 - \frac{2kh}{\sinh 2kh}\right) \right. \right. \\ & \left. + \frac{l(l+1)}{k^2a^2} \left(1 + \frac{2kh}{\sinh 2kh}\right) \right\} \\ & \times \left\{ -P_l + (-1)^{l+1} \left(Q_l \cos 2\sigma t + R_l \sin 2\sigma t\right) \right\} \Big] \end{aligned} \qquad (39)$$

where P_l, Q_l, R_l are given by equations (33), (37) and (38) respectively.

3. Numerical results

Table 1. Coefficient $P_l(ka)/(ka)^2$ for various ka.

$l \downarrow$	ka=0.5	ka=1.0	ka=2.0	ka=5.0
1	0.35515038796	0.91897207019	0.71798356187	0.31078056984
2	0.34172646214	0.80345154422	1.11564372289	0.61891433173
3	0.34172179716	0.80146131731	1.00703643158	0.91780493601
4	0.34172179691	0.80145958914	1.00008079423	1.18229031284
5	0.34172179691	0.80145958868	1.00004712071	1.31108863429
6	0.34172179691	0.80145958868	1.00004705866	1.24268190466
7	0.34172179691	0.80145958868	1.00004705861	1.22482706741
8	0.34172179691	0.80145958868	1.00004705861	1.22428938225
9	0.34172179691	0.80145958868	1.00004705861	1.22428209884
10	0.34172179691	0.80145958868	1.00004705861	1.2242820399
11	0.34172179691	0.80145958868	1.00004705861	1.2242820396
12	0.34172179691	0.80145958868	1.00004705861	1.2242820396
13	0.34172179691	0.80145958868	1.00004705861	1.2242820396
14	0.34172179691	0.80145958868	1.00004705861	1.2242820396
15	0.34172179691	0.80145958868	1.00004705861	1.2242820396
16	0.34172179691	0.80145958868	1.00004705861	1.2242820396

Table 2. Coefficient $Q_l(ka)/(ka)^2$ for various ka.

$l \downarrow$	ka=0.5	ka=1.0	ka=2.0	ka=5.0
1	-0.01747800339	0.1856387247	0.74911350851	0.22987741645
2	-0.03180394572	0.00377653753	0.5158831474	-0.07778580088
3	-0.03180870912	0.0016091573	0.29623077823	0.22195434804
4	-0.03180870937	0.00160735376	0.28784431802	0.37049930902
5	-0.03180870937	0.00160735328	0.28780682487	0.10452794738
6	-0.03180870937	0.00160735328	0.28780675828	-0.10985705815
7	-0.03180870937	0.00160735328	0.28780675822	-0.13711066689
8	-0.03180870937	0.00160735328	0.28780675822	-0.13781993899
9	-0.03180870937	0.00160735328	0.28780675822	-0.13782889759
10	-0.03180870937	0.00160735328	0.28780675822	-0.1378289668
11	-0.03180870937	0.00160735328	0.28780675822	-0.13782896714
12	-0.03180870937	0.00160735328	0.28780675822	-0.13782896714
13	-0.03180870937	0.00160735328	0.28780675822	-0.13782896714
14	-0.03180870937	0.00160735328	0.28780675822	-0.13782896714
15	-0.03180870937	0.00160735328	0.28780675822	-0.13782896714
16	-0.03180870937	0.00160735328	0.28780675822	-0.13782896714

Now we present the computational results for the dynamic component $F_{x_2}^{(3)}$ of the second order force. Since the contribution to the dynamic component of the second order force is based on the coefficients $P_l(ka)$, $Q_l(ka)$,$R_l(ka)$, first we compute the expressions for $P_l(ka)$, $Q_l(ka)$,$R_l(ka)$ given by the equations (33), (37) and (38) respectively.

The terms involved in the computation of the dynamic component of the second order force are computed as function of l, the number of terms in the summation for various ka. These computational results for $P_l(ka)/(ka)^2$, $Q_l(ka)/(ka)^2$ and $R_l(ka)/(ka)^2$ are presented in table 1, table 2 and table 3 for l = 1, 2, ..., 16. From these numerical results, it is obvious that we only need few terms in the infinite summation over l.

The term $P_l(ka)/(ka)^2$ given by equation (33) is shown in table 1 for ka=0.5, 1.0, 2.0, 5.0.

The term $Q_l(ka)/(ka)^2$ given by equation (37) is shown in table 2 for ka = 0.5, 1.0, 2.0, 5.0.

The term $R_l(ka)/(ka)^2$ given by equation (38) is presented in table 3 for ka=0.5, 1.0, 2.0, 5.0.

Numerical results for the term $P_l(ka)/(ka)^2$ given by equation (33) is displayed in figure 1 for ka= 1.0, 2.0.

Computational results for the term $Q_l(ka)/(ka)^2$ given by equation (37) is shown in figure 2 for ka= 1.0, 2.0.

Similar results for $R_l(ka)/(ka)^2$ given by equation (38) is presented in figure 3 for ka= 1.0, 2.0.

As a conclusion from the above results, we take l=15 for all further computations.

Now we present the numerical results for the nondimensional dynamic force component

$$F_2 = \frac{F_{x_2}^{(3)}}{\rho g a A^2} = \frac{2}{\pi (ka)^2} \sum_{l=0}^{\infty} \left[\left\{ \left(1 - \frac{2kh}{\sinh 2kh}\right) + \frac{l(l+1)}{k^2a^2}\left(1 + \frac{2kh}{\sinh 2kh}\right) \right\} \times \left\{ -P_l + (-1)^{l+1}\left(Q_l \cos 2\sigma t + R_l \sin 2\sigma t\right) \right\} \right] \tag{40}$$

where P_l, Q_l, R_l are given by equations (33), (37) and (38) respectively.

Table 3. Coefficient $R_l(ka)/(ka)^2$ for various ka.

l ↓	ka=0.5	ka=1.0	ka=2.0	ka=5.0
1	1.05614031698	1.1870367375	0.0522677932	0.21163876987
2	1.13242538234	1.57224494449	0.80741461263	0.10885706054
3	1.13322361276	1.59715579926	1.1572042222	-0.07473893988
4	1.13322771179	1.5976525683	1.20879045762	0.2963727395
5	1.13322772461	1.59765876017	1.21136152271	0.65910565091
6	1.13322772463	1.59765881214	1.21144810527	0.90708167604
7	1.13322772463	1.59765881246	1.21145022062	0.99371975271
8	1.13322772463	1.59765881246	1.21145025935	1.00536972057
9	1.13322772463	1.59765881246	1.2114502599	1.00647290806
10	1.13322772463	1.59765881246	1.2114502599	1.00655729852
11	1.13322772463	1.59765881246	1.2114502599	1.00656255935
12	1.13322772463	1.59765881246	1.2114502599	1.00656282986
13	1.13322772463	1.59765881246	1.2114502599	1.00656284151
14	1.13322772463	1.59765881246	1.2114502599	1.00656284194
15	1.13322772463	1.59765881246	1.2114502599	1.00656284195
16	1.13322772463	1.59765881246	1.2114502599	1.00656284195

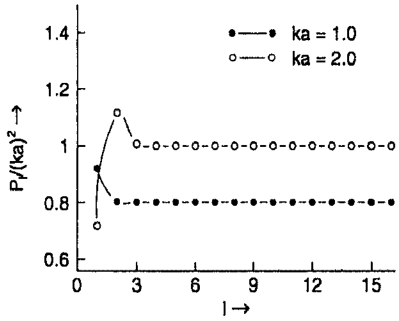

Figure 1. Coefficient $P_l(ka)/(ka)^2$ as function of l.

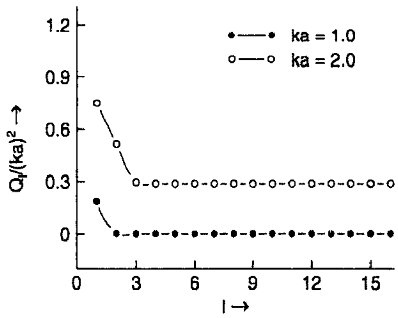

Figure 2. Coefficient $Q_l(ka)/(ka)^2$ as function of l.

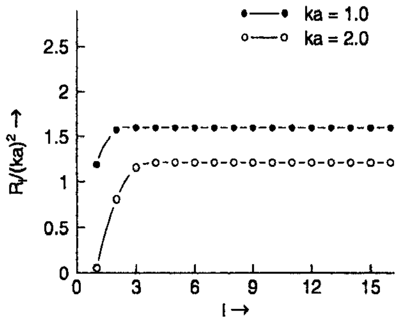

Figure 3. Coefficient $R_l(ka)/(ka)^2$ as function of l.

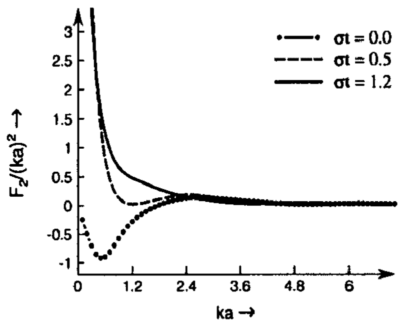

Figure 4. Dynamic force component $F_2/(ka)^2$ as a function of ka for $h/a = 1.0$.

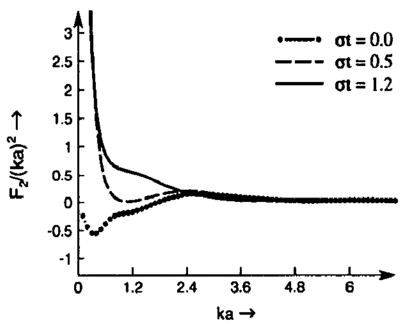

Figure 5. Dynamic force component $F_2/(ka)^2$ as a function of ka for $h/a = 2.0$.

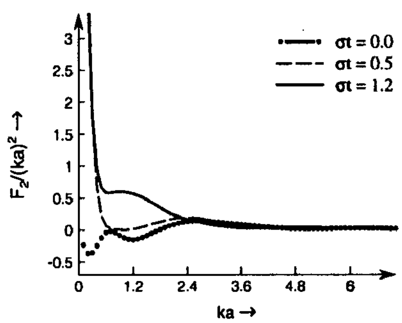

Figure 6. Dynamic force component $F_2/(ka)^2$ as a function of ka for $h/a = 3.0$.

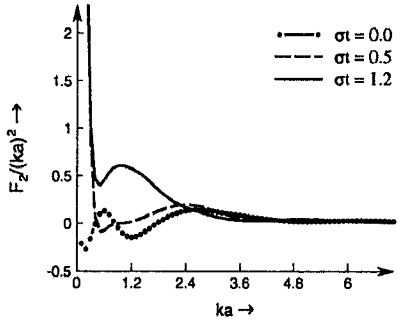

Figure 7. Dynamic force component $F_2/(ka)^2$ as a function of ka for $h/a = 4.0$.

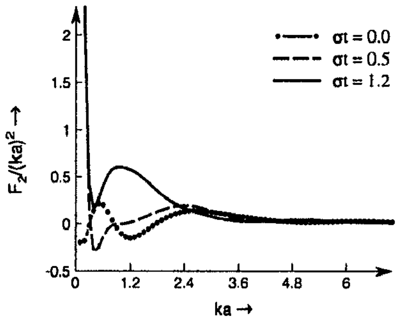

Figure 8. Dynamic force component $F_2/(ka)^2$ as a function of ka for $h/a = 5.0$.

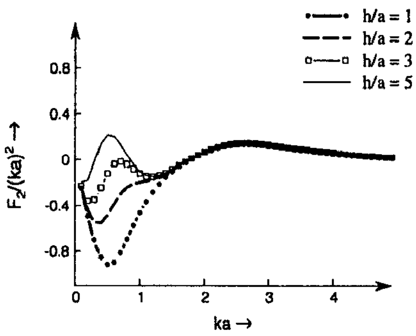

Figure 9. Dynamic force component $F_2/(ka)^2$ as a function of ka for $\sigma t = 0.0$.

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