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AN ALGORITHM FOR THE NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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ABSTRACT. We present and discuss an algorithm for the numerical solution of initial value problems of the form $D_*^\alpha y(t) = f(t, y(t))$, $y(0) = y_0$, where $D_*^\alpha y$ is the derivative of y of order α in the sense of Caputo and $0 < \alpha \leq 1$. The algorithm is based on the fractional Euler's method which can be seen as a generalization of the classical Euler's method. Numerical examples are given and the results show that the present algorithm is very effective and convenient.

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1. Introduction

In this paper we introduce an algorithm for the numerical solution of initial value problems of the form

$$D_*^\alpha y(t) = f(t, y(t)), \quad y(0) = y_0, \quad 0 < \alpha \leq 1, \tag{1}$$

where D_*^α denotes the Caputo fractional differential operator [1]. The development of this algorithm is motivated by a few classical and many recent applications of fractional differential equations. Among the classical problems we mention areas like the modelling of the behavior of viscoelastic materials in mechanics [2]. More recently fractional calculus has been applied to continuum and statistical mechanics for viscoelasticity problems, Brownian motion and fractional diffusion-wave equations and many physical phenomena [1-12].

Most nonlinear fractional differential equations do not have analytic solutions, so approximations and numerical techniques must be used [13-19]. The decomposition method [20-28] and the variational iteration method [29-45] are relatively new approaches to provide an analytical approximate solution to linear

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and nonlinear problems, and they are particulary valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations. The application of the two methods is successfully extended to obtain an analytical approximate solutions to linear and nonlinear differential equations of fractional order [24-28,32,45-47]. A comparison between the variational iteration method and Adomian decomposition method for solving fractional differential equations is given in [46,47]. The fact that the variational iteration method solves nonlinear equations without using Adomian polynomials can be considered as an advantage of this method over Adomian decomposition method.

A few numerical methods for fractional differential equations have been presented in the literature [13-19]. However many of these methods are used for very specific types of differential equations, often just linear equations or even smaller classes. Recently, Diethelm et al. [19] introduced a method for the numerical solution of the nonlinear fractional differential equation (1) which can be seen as a generalization of the classical Adams-Bashforth-Moulton scheme for first order differential equations.

The structure of this paper is as follows. We begin by introducing some necessary definitions and mathematical preliminaries of the fractional calculus theory which are required for establishing our results. In sections 3 and 4, we introduce the modified trapezoidal rule and a new generalization of Taylor's formula that involves Caputo derivatives, respectively. In section 5, we derive the fractional Euler's method that is a generalization of the classical Euler's method for the numerical solution of ordinary differential equations. The algorithm itself is presented in details in section 6. In section 7, we present three examples to show the efficiency and the simplicity of the algorithm. Finally, in section 8, we give the conclusions.

2. Basic definitions

For the concept of fractional derivative we will adopt Caputo's definition which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes.

Definition 1. *A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$* if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and *it is said to be in the space C_μ^m* iff $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$,* is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \ x > 0,$$

$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in [3,4,8], we mention only the following:

For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

(1) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$, (2) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,

(3) $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^α proposed by M. Caputo in his work on the theory of viscoelasticity [1]. Caputo's definition, which is a modification of the Riemann-Liouville definition, has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical processes.

Definition 3. *The fractional derivative of $f(x)$ in the Caputo sense* is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt,$$

for $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$, $f \in C_{-1}^m$.

Also, we need here two of its basic properties:

Lemma 1. *If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ and $f \in C_\mu^m$, $\mu \geq -1$, then*

$$D_*^\alpha J^\alpha f(x) = f(x), \tag{2}$$

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{3}$$

3. Modified trapezoidal rule

n this section, we present a review of the modified trapezoidal rule, which is introduced in [48]. This rule is used to approximate the fractional integral $J^\alpha f(t)$ by a weighted sum of function values at specified points. Suppose that the interval $[0, a]$ is subdivided into k subintervals $[t_j, t_{j+1}]$ of equal width $h = a/k$ by using the nodes $t_j = jh$, for $j = 0, 1, ..., k$. The modified trapezoidal rule

$$\begin{aligned} T(f,h,\alpha) &= ((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha) \frac{h^\alpha f(0)}{\Gamma(\alpha+2)} + \frac{h^\alpha f(a)}{\Gamma(\alpha+2)} \\ &+ \textstyle\sum_{j=1}^{k-1} ((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1}) \dfrac{h^\alpha f(t_j)}{\Gamma(\alpha+2)}, \end{aligned} \tag{4}$$

is an approximation to the fractional integral

$$(J^\alpha f(t))(a) = T(f,h,\alpha) - E_T(f,h,\alpha), \quad a > 0, \quad \alpha > 0. \tag{5}$$

Furthermore, if $f(t) \in \mathbf{C}^2[0,a]$, there is a constant C_α depending only on α so that the error term $E_T(f,h,\alpha)$ has the form

$$|E_T(f,h,\alpha)| \leq C_\alpha \|f''\|_\infty a^\alpha h^2 = O(h^2). \tag{6}$$

It is clear that if $\alpha = 1$, then the modified trapezoidal rule (4) reduces to the classical trapezoidal rule. This rule is simple for computational performance for all values of α and h. For more details about the rule and its applications, we refer the reader to [48].

4. Generalized Taylor's formula

In this section we introduce a new generalization of Taylor's formula that involves Caputo fractional derivatives. This generalization is presented in [49]. We begin by introducing the generalized mean value theorem.

Theorem 1. (Generalized mean value theorem) *Suppose that* $f(x) \in C[0,a]$ *and* $D_*^\alpha f(x) \in C(0,a]$, *for* $0 < \alpha \leq 1$. *Then we have*

$$f(x) = f(0+) + \frac{1}{\Gamma(\alpha)}(D_*^\alpha f)(\xi) \cdot x^\alpha, \tag{7}$$

with $0 \leq \xi \leq x$, $\forall x \in (0,a]$.

Proof. From the definitions of the Riemann-Liouville fractional integral operator and the Caputo fractional derivative operator, we have

$$(J^\alpha D_*^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} (D_*^\alpha f)(t) dt. \tag{8}$$

Using the integral mean value theorem, we get

$$(J^\alpha D_*^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}(D_*^\alpha f)(\xi) \int_0^x (x-t)^{\alpha-1} = \frac{1}{\Gamma(\alpha)}(D_*^\alpha f)(\xi) \cdot x^\alpha, \tag{9}$$

for $0 \leq \xi \leq x$. On the other hand, from equation (3), we have

$$(J^\alpha D_*^\alpha f)(x) = f(x) - f(0+). \tag{10}$$

So, from (9) and (10), the generalized mean value theorem (7) is obtained. □

In case of $\alpha = 1$, the generalized mean value theorem reduces to the classical mean value theorem. Before we present the generalized Taylor's formula in the Caputo sense, we need the following relation.

Theorem 2. *Suppose that* $D_*^{n\alpha} f(x)$, $D_*^{(n+1)\alpha} f(x) \in C(0,a]$, *for* $0 < \alpha \leq 1$. *Then we have*

$$(J^{n\alpha} D_*^{n\alpha} f)(x) - (J^{(n+1)\alpha} D_*^{(n+1)\alpha} f)(x) = \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}(D_*^{n\alpha} f)(0+), \tag{11}$$

where

$$D_*^{n\alpha} = D_*^{\alpha} D_*^{\alpha} \dots D_*^{\alpha} \quad (n - times).$$

Proof. The proof can be obtained using the properties of the Riemann-Liouville fractional integral operator and the Caputo fractional derivative operator and the relation:

$$(J^{n\alpha} D_*^{n\alpha} f)(x) - (J^{(n+1)\alpha} D_*^{(n+1)\alpha} f)(x) = J^{n\alpha}\Big((D_*^{n\alpha} f)(x) - (J^{\alpha} D_*^{\alpha})(D_*^{n\alpha} f)(x)\Big)$$
$$= J^{n\alpha}(D_*^{n\alpha} f)(0+). \quad \square$$

Theorem 3. (Generalized Taylor's formula) *Suppose that* $D_*^{k\alpha} f(x) \in C(0, a]$ *for* $k = 0, 1, \dots, n+1$, *where* $0 < \alpha \leq 1$. *Then we have*

$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha + 1)} (D_*^{i\alpha})(0+) + \frac{(D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha + 1)} x^{(n+1)\alpha}, \tag{12}$$

with $0 \leq \xi \leq x$, $\forall x \in (0, a]$.

Proof. From (11), we have

$$\sum_{i=0}^{n} (J^{i\alpha} D_*^{i\alpha} f)(x) - (J^{(i+1)\alpha} D_*^{(i+1)\alpha} f)(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha + 1)} (D_*^{i\alpha} f)(0+), \tag{13}$$

that is,

$$f(x) - (J^{(n+1)\alpha} D_*^{(n+1)\alpha} f)(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha + 1)} (D_*^{i\alpha} f)(0+). \tag{14}$$

Applying the integral mean value theorem to (14) yields

$$(J^{(n+1)\alpha} D_*^{(n+1)\alpha} f)(x) = \frac{(D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha + 1)} \int_0^x (x - t)^{(n+1)\alpha} dt$$
$$= \frac{(D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha + 1)} x^{(n+1)\alpha}. \tag{15}$$

Now, if we Substitute (15) into (14), then the generalized Taylor's formula (12) is obtained. $\square$

In case of $\alpha = 1$, the generalized Taylor's formula (12) reduces to the classical Taylor's formula.

5. Fractional Euler's method

In this section we shall derive the fractional Euler's method that we have developed for the numerical solution of initial value problems with Caputo derivatives. The method is a generalization of the classical Euler's method. Consider the initial value problem

$$D_*^\alpha y(t) = f(t, y(t)), \quad y(0) = y_0, \quad 0 < \alpha \leq 1, \quad t > 0. \tag{16}$$

Let $[0, a]$ be the interval over which we want to find the solution of the problem (16). In actuality, we will not find a function $y(t)$ that satisfies the initial value problem (16). Instead, a set of points $\{(t_j, y(t_j))\}$ is generated, and the points are used for our approximation.

For convenience we subdivide the interval $[0, a]$ into k subintervals $[t_j, t_{j+1}]$ of equal width $h = a/k$ by using the nodes $t_j = jh$, for $j = 0, 1, ..., k$. Assume that $y(t)$, $D_*^\alpha y(t)$ and $D_*^{2\alpha} y(t)$ are continuous on $[0, a]$ and use the generalized Taylor's formula (12) to expand $y(t)$ about $t = t_0 = 0$. For each value t there is a value c_1 so that

$$y(t) = y(t_0) + (D_*^\alpha y(t))(t_0)\frac{t^\alpha}{\Gamma(\alpha+1)} + (D_*^{2\alpha} y(t))(c_1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{17}$$

When $(D_*^\alpha y(t))(t_0) = f(t_0, y(t_0))$ and $h = t_1$ are substituted into equation (17), the result is an expression for $y(t_1)$:

$$y(t_1) = y(t_0) + f(t_0, y(t_0))\frac{h^\alpha}{\Gamma(\alpha+1)} + (D_*^{2\alpha} y(t))(c_1)\frac{h^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{18}$$

If the step size h is chosen small enough, then we may neglect the second-order term (involving $h^{2\alpha}$) and get

$$y(t_1) = y(t_0) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_0, y(t_0)). \tag{19}$$

The process is repeated and generates a sequence of points that approximates the solution $y(t)$. The general formula for fractional Euler's method is

$$\begin{aligned} t_{j+1} &= t_j + h, \\ y(t_{j+1}) &= y(t_j) + \tfrac{h^\alpha}{\Gamma(\alpha+1)} f(t_j, y(t_j)), \end{aligned} \tag{20}$$

for $j = 0, 1, ..., k-1$. It is clear that if $\alpha = 1$, then the fractional Euler's method (20) reduces to the classical Euler's method.

6. The algorithm

In this section we shall derive the fundamental algorithm for the numerical solution of the initial value problem (16). The new algorithm is based on the modified trapezoidal rule and the fractional Euler's method. Our approach depends on the analytical property that the initial value problem (16) is equivalent to the integral equation

$$y(t) = J^\alpha f(t, y(t)) + y(0). \tag{21}$$

Let $[0, a]$ be the interval over which we want to find the approximate the solution. Suppose that the interval $[0, a]$ is subdivided into k subintervals $[t_j, t_{j+1}]$ of equal

width $h = a/k$ by using the nodes $t_j = jh$, for $j = 0, 1, ..., k$. To obtain the solution point $(t_1, y(t_1))$, we substitute $t = t_1$ into (21) and we get

$$y(t_1) = (J^\alpha f(t, y(t)))(t_1) + y(0). \tag{22}$$

Now if the modified trapezoidal rule (4) is used to approximate $(J^\alpha f(t, y(t)))(t_1)$ with step size $h = t_1 - t_0$, then the result is

$$y(t_1) = \alpha \frac{h^\alpha f(t_0, y(t_0))}{\Gamma(\alpha + 2)} + \frac{h^\alpha f(t_1, y(t_1))}{\Gamma(\alpha + 2)} + y(0). \tag{23}$$

Notice that the formula on the right-hand side of (23) involves the term $y(t_1)$. So, we use an estimate for $y(t_1)$. Fractional Euler's method will suffice for this purpose. Substituting (19) into (23), yields

$$y(t_1) = \alpha \frac{h^\alpha f(t_0, y(t_0))}{\Gamma(\alpha + 2)} + \frac{h^\alpha f(t_1, y(t_0) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_0, y(t_0)))}{\Gamma(\alpha + 2)} + y(0). \tag{24}$$

The process is repeated to generate a sequence of points that approximate the solution $y(t)$. At each step, the fractional Euler's method is used as a prediction, and then the modified trapezoidal rule is used to make a correction to obtain the finite value. The general formula for our algorithm is :

$$\begin{aligned} y(t_j) &= \frac{h^\alpha}{\Gamma(\alpha + 2)} \Big((j-1)^{\alpha+1} - (j - \alpha - 1) j^\alpha \Big) f(t_0, y(t_0)) + y(0) \\ &+ \frac{h^\alpha}{\Gamma(\alpha + 2)} \Sigma_{i=1}^{j-1} \Big((j-i+1)^{\alpha+1} - 2(j-i)^{\alpha+1} + (j-i-1)^{\alpha+1} \Big) f(t_i, y(t_i)) \\ &+ \frac{h^\alpha}{\Gamma(\alpha + 2)} f\Big(t_j, y(t_{j-1}) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_{j-1}, y(t_{j-1})) \Big). \end{aligned} \tag{25}$$

The new algorithm is simple for computational performance for all values of α and h. It is clear that the behavior of the method is independent of the parameter α and, as we will see in the next section, the accuracy of the approximation depends on the step size h.

7. Numerical examples

To give a clear overview of the methodology as a numerical tool, we apply the proposed algorithm on three different examples of differential equations of fractional order.

Example 1. Our first example deals with the homogeneous linear equation

$$D_*^\alpha y(t) = -y(t), \quad y(0) = 1, \quad t > 0, \tag{26}$$

where $0 < \alpha \leq 1$.

The exact solution of equation (26) is given by

$$y(t) = E_\alpha(-t^\alpha), \tag{27}$$

where

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{28}$$

is the Mittag-Leffler function of order α. In view of the algorithm (25), we obtain the following iteration formula:

$$\begin{aligned} y(t_j) &= 1 - \frac{h^\alpha}{\Gamma(\alpha+2)}\Big((j-1)^{\alpha+1} - (j-\alpha-1)j^\alpha\Big) \\ &- \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=1}^{j-1} \Big((j-i+1)^{\alpha+1} - 2(j-i)^{\alpha+1} + (j-i-1)^{\alpha+1}\Big) y(t_i) \\ &+ \frac{h^\alpha}{\Gamma(\alpha+2)}\Big(\frac{h^\alpha}{\Gamma(\alpha+1)} - 1\Big) y(t_{j-1}). \end{aligned} \tag{29}$$

Table 1. Numerical values for Example 1 when $\alpha = 0.5$, 0.75 and 1 with $h = 0.001$.

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	$u_{Appro.}$	u_{Exact}	$u_{Appro.}$	u_{Exact}	$u_{Appro.}$	u_{Exact}
0.0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.723955	0.723578	0.828254	0.828251	0.904837	0.904837
0.2	0.644094	0.643788	0.732588	0.732585	0.818730	0.818731
0.3	0.592277	0.592018	0.660341	0.660337	0.740817	0.740818
0.4	0.553831	0.553606	0.602124	0.602121	0.670319	0.670320
0.5	0.523355	0.523157	0.553605	0.553603	0.606530	0.606531
0.6	0.498203	0.498025	0.512287	0.512285	0.548811	0.548812
0.7	0.476864	0.476703	0.476556	0.476555	0.496585	0.496585
0.8	0.458394	0.458246	0.445293	0.445292	0.449328	0.449329
0.9	0.442157	0.442021	0.417683	0.417682	0.406569	0.406570
1.0	0.427709	0.427584	0.393109	0.393108	0.367079	0.367079

First we investigate the effect of allowing the value of α to vary in the interval $(0, 1]$ with fixed step size $h = 0.001$. The results are given in Table 1. It is clear that the approximate solutions are in high agreement with the exact solutions and the solution continuously depends on the time-fractional derivative. The absolute errors when $\alpha = 0.5$ and different values of the step size h are given in Table 2. The results show that the accuracy can be improved if we use much smaller values of h.

Example 2. Our second example covers the inhomogeneous linear equation

$$D_*^\alpha y(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} - y(t) + t^2 - t, \quad y(0) = 0, \quad t > 0, \tag{30}$$

where $0 < \alpha \leq 1$.

Table 2. Absolute errors for Example 1 when $\alpha = 0.5$ and with varying step sizes h.

t	$h = 0.1$	$h = 0.01$	$h = 0.004$	$h = 0.0001$
0.0	0.0000e-00	0.0000e-00	0.0000e-00	0.0000e-00
0.1	2.7642e-02	2.5504e-03	1.2841e-03	4.1574e-05
0.2	6.3532e-02	2.4222e-03	1.1042e-03	3.2673e-05
0.3	2.1765e-02	2.1647e-03	9.5509e-04	2.7271e-05
0.4	6.6783e-03	1.9368e-03	8.4009e-04	2.3510e-05
0.5	1.0867e-02	1.7463e-03	7.9939e-04	2.0697e-05
0.6	1.0016e-02	1.5871e-03	6.7599e-04	1.8494e-05
0.7	9.8945e-03	1.4526e-03	6.1530e-04	1.6713e-05
0.8	9.4682e-03	1.3377e-03	5.6423e-04	1.5231e-05
0.9	9.0545e-03	1.2381e-03	5.2061e-04	1.3995e-05
1.0	8.6295e-03	1.1521e-03	4.8291e-04	1.2932e-05

The exact solution of equation (30) is given by

$$y(t) = t^2 - t, \tag{31}$$

In view of the algorithm (25), we obtain the following iteration formula:

$$\begin{aligned} y(t_j) = {} & \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{i=1}^{j-1} \Big((j-i+1)^{\alpha+1} - 2(j-i)^{\alpha+1} + (j-i-1)^{\alpha+1} \Big) \\ & \times \Big(\frac{2}{\Gamma(3-\alpha)} t_i^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t_i^{1-\alpha} - y(t_i) + t_i^2 - t_i \Big) \\ & + \frac{h^\alpha}{\Gamma(\alpha+2)} \Big(\frac{2}{\Gamma(3-\alpha)} t_j^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t_j^{1-\alpha} - y(t_{j-1}) \\ & + t_j^2 - t_j - \frac{h^\alpha}{\Gamma(\alpha+1)} \Big(\frac{2}{\Gamma(3-\alpha)} t_{j-1}^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t_{j-1}^{1-\alpha} - y(t_{j-1}) \\ & + t_{j-1}^2 - t_{j-1} \Big) \Big). \end{aligned} \tag{32}$$

The linear equation (30) is solved by Diethelm et al. [19] using the fractional Adams-Bashforth-Moulton method. Tables 3 shows the exact solution and the approximate solution for equation (30) using the algorithm (25) for different values of α. The results compare well with those obtained by the Adams-Bashforth-Moulton method [19].

Example 3. *The third example deals with the nonlinear equation*

$$D_*^\alpha y(t) = y(t)^2 - \frac{2}{(t+1)^2}, \quad y(0) = -2, \quad t > 0, \tag{33}$$

where $0 < \alpha \leq 1$.

Table 3. Numerical values for Example 2 when $\alpha = 0.5$, 0.75 and 1 with $h = 0.001$.

	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$	
t	$u_{Appro.}$	u_{Exact}	$u_{Appro.}$	u_{Exact}	$u_{Appro.}$	u_{Exact}
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	-0.089804	-0.090000	-0.089921	-0.090000	-0.089555	-0.090000
0.2	-0.159788	-0.160000	-0.159942	-0.160000	-0.159590	-0.160000
0.3	-0.209807	-0.210000	-0.209954	-0.210000	-0.209629	-0.210000
0.4	-0.239842	-0.240000	-0.239963	-0.240000	-0.239664	-0.240000
0.5	-0.249893	-0.250000	-0.249971	-0.250000	-0.249696	-0.250000
0.6	-0.239953	-0.240000	-0.239977	-0.240000	-0.239725	-0.240000
0.7	-0.210019	-0.210000	-0.209982	-0.210000	-0.209751	-0.210000
0.8	-0.160091	-0.160000	-0.159987	-0.160000	-0.159775	-0.160000
0.9	-0.090168	-0.090000	-0.089991	-0.090000	-0.089796	-0.090000
1.0	-0.000249	0.000000	-0.000005	0.000000	-0.000184	0.000000

The exact solution of equation (33) in case of $\alpha = 1$ is given by

$$y(t) = -\frac{2}{(t+1)}. \tag{34}$$

In view of the algorithm (25), we obtain the following iteration formula:

$$\begin{aligned} y(t_j) &= 2\tfrac{h^\alpha}{\Gamma(\alpha+2)}\Big((j-1)^{\alpha+1} - (j-\alpha-1)j^\alpha\Big) - 2 \\ &+ \tfrac{h^\alpha}{\Gamma(\alpha+2)}\Sigma_{i=1}^{j-1}\Big((j-i+1)^{\alpha+1} - 2(j-i)^{\alpha+1} + (j-i-1)^{\alpha+1}\Big)\Big(y^2(t_i) - \tfrac{2}{(t_i+1)^2}\Big) \\ &+ \tfrac{h^\alpha}{\Gamma(\alpha+2)}\Big(\Big(y(t_{j-1}) + \tfrac{h^\alpha}{\Gamma(\alpha+1)}\Big(y^2(t_{j-1}) - \tfrac{2}{(t_{j-1}+1)^2}\Big)\Big)^2 - \tfrac{2}{(t_j+1)^2}\Big) \end{aligned} \tag{35}$$

Table 4 shows the approximate solutions for equation (33) for different values of α using the algorithm (25). The value of $\alpha = 1$ is the only case for which we know the exact solution $y(t) = -\frac{2}{(t+1)}$ and our approximate solution in this case is in high agreement with the exact solution. Of course the accuracy can be improved if we use much smaller values of h.

8. Conclusions

The fundamental goal of this work has been to construct a numerical scheme for the numerical solution of linear and nonlinear differential equations of fractional order. The goal has been achieved by using the proposed algorithm (25). This algorithm is based on the modified trapezoidal rule and the fractional Euler's method.

There are few important points to make here. First, the algorithm were used in a direct way without using linearization, perturbation or restrictive assumptions. Second, the algorithm is a reliable and very useful one for the numerical

evaluation of the Mittag-Leffler function and the other functions arising in fractional calculus. Finally, the recent appearance of fractional differential equations as models in some fields of applied mathematics makes it necessary to investigate methods of solution for such equations (analytical and numerical) and we hope that this work is a step in this direction.

Table 4. Numerical values for Example 3 with varying order α and $h = 0.001$.

t	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
0.0	-2.00000	-2.00000	-2.00000	-2.00000	-2.00000
0.1	-1.57191	-1.58885	-1.56451	-1.73541	-1.81818
0.2	-1.48646	-1.48991	-1.53362	-1.59189	-1.66666
0.3	-1.42059	-1.41460	-1.44369	-1.48262	-1.53846
0.4	-1.36655	-1.35254	-1.36994	-1.39284	-1.42857
0.5	-1.32108	-1.29978	-1.30711	-1.31637	-1.33333
0.6	-1.28222	-1.25413	-1.25246	-1.24987	-1.25000
0.7	-1.24861	-1.21415	-1.20427	-1.19122	-1.17647
0.8	-1.21926	-1.17879	-1.16136	-1.13896	-1.11111
0.9	-1.19341	-1.14725	-1.12283	-1.09202	-1.05263
1.0	-1.17049	-1.11893	-1.08802	-1.04957	-0.99999

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