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NUMERICAL SIMULATION OF THE RIESZ FRACTIONAL DIFFUSION EQUATION WITH A NONLINEAR SOURCE TERM

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ABSTRACT. In this paper, A Riesz fractional diffusion equation with a nonlinear source term (RFDE-NST) is considered. This equation is commonly used to model the growth and spreading of biological species. According to the equivalent of the Riemann-Liouville(R-L) and Grünwald-Letnikov(G-L) fractional derivative definitions, an implicit difference approximation (IFDA) for the RFDE-NST is derived. We prove the IFDA is unconditionally stable and convergent. In order to evaluate the efficiency of the IFDA, a comparison with a fractional method of lines (FMOL) is used. Finally, two numerical examples are presented to show that the numerical results are in good agreement with our theoretical analysis.

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1. Introduction and notations

It is well known that the fundamental solution (or Green function) for the classical diffusion equation

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

is provided by the normal or Gaussian probability density functions of a Brownian motion [17,36,38], which describes the motion of small macroscopic particles in a liquid or a gas which experience unbalanced bombardments due to surrounding atoms, and hence reveals the atomistic structure of the medium in which the motion occurs. But the anomalous diffusion (subdiffusion or superdiffusion) phenomena is almost ubiquitous in nature [10,14,19,28,35]. Many academicians

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find that evolution equations containing fractional derivatives can provide suitable mathematical models for describing phenomena of anomalous diffusion and transport dynamics in complex systems [6,9,11,12]. Here we especially refer to Metzler and Klafter [27] for a comprehensive review.

It is regret that it is difficult to solve the fractional order differential equation in that the fractional derivative operators are quasi-differential operators with singularity. Until recently, by various integral transforms, separation of variables, various finite difference discretization, variational iteration, Adomian decomposition and finite element methods have proposed to obtain the analysis or numerical solutions of linear fractional order differential equations [8,15,16,20,22-26,31,33,37]. But many phenomena in the nature must be simulated by nonlinear differential equations, such as a hyperchaotic autonomous nonlinear system [1], large uni-axial deformation behavior of poly-urethane foam [4], the wave propagation of an advantageous gene in a population [2], and so on. To the best of our knowledge, while the studied of analysis or numerical solutions of nonlinear fractional differential equations (NFDEs) are relatively limited. Diethelm [5], Zhang [39] and Bai [3] discussed the existence and uniqueness of solution or existence of positive solution of NFDEs. The variational iteration and Adomian decomposition techniques were used to get the approximate or analysis solutions of NFDEs by many academicians [7,15,18,29,30,32,34], but the above techniques are only suit to solve nonlinear fractional ordinary differential equations and nonlinear fractional partial differential equations with initial value or boundary value conditions, as for the mixed problems, the two techniques are inefficient. Lin and Liu [21] considered a fractional nonlinear ordinary differential equation, they got high order convergent discrete schemes. Guy [13]gave the series form of solution of a class of nonlinear partial differential equations of fractional order by Lagrange characteristic method. Baeumer, et.al. [2] got numerical solution of the initial problem of nonlinear fractional reaction-diffusion equation by time discretization and operator splitting. But the numerical approximation and analysis of the Riesz fractional diffusion equation with a nonlinear source term (RFDE-NST) have not been studied previously.

In the paper, we consider the numerical simulation of the RFDE-NST:

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^\alpha u}{\partial |x|^\alpha} + f(x,t,u), \quad 1 < \alpha \leq 2,$$

where $\lambda (> 0)$ is diffusion coefficient and $\frac{\partial^\alpha}{\partial |x|^\alpha}$ is a Riesz fractional derivative operator defined by

$$\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = \begin{cases} -\frac{1}{2cos(\frac{\alpha\pi}{2})}\{{}_aD_x^\alpha u(x,t) + {}_xD_b^\alpha u(x,t)\}, 1 < \alpha < 2, \\ \frac{\partial^2 u(x,t)}{\partial x^2}, \alpha = 2 \end{cases}$$

and

$$\begin{cases} {}_aD_x^\alpha u(x,t) = \dfrac{1}{\Gamma(2-\alpha)} \dfrac{d^2}{dx^2} \displaystyle\int_a^x (x-\xi)^{1-\alpha} u(\xi,t) d\xi, \\ {}_xD_b^\alpha u(x,t) = \dfrac{1}{\Gamma(2-\alpha)} \dfrac{d^2}{dx^2} \displaystyle\int_x^b (\xi-x)^{1-\alpha} u(\xi,t) d\xi \end{cases}$$

(see [9] or [33]). Solutions to considered fractional diffusion equation is provided by the α-stable probability density functions [25] of a Lévy motion [28]. The RFDE-NST is commonly used to model the growth and spreading of biological species [2], where the classical second derivative diffusion term is replaced by a Riesz fractional derivative of order less than two. The resulting model captures the faster spreading rate and power law invasion profiles observed in many applications and is strongly motivated by a generalised central limit theorem for random movements with power-law probability tails.

The structure of the paper is as follows. In Section 2, an implicit finite difference approximation (IFDA) for the RFDE-NST is proposed. The stability and convergence of the IFDA are discussed in Section 3. A fractional method of lines for the RFDE-NST is introduced in Section 4. In Section 5, some numerical examples are given. Theoretical results are in excellent agreement with numerical testing.

2. An implicit finite difference approximation for the RFDE-NST

In the section, we consider the following RFDE-NST:

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^\alpha u}{\partial |x|^\alpha} + f(x,t,u), \quad 1 < \alpha \le 2, \quad 0 < x < l, \ 0 < t \le T, \tag{1}$$

$$u(0,t) = u(l,t) = 0, \quad 0 \le t \le T, \tag{2}$$

$$u(x,0) = g(x), \quad 0 < x < l, \tag{3}$$

where the nonlinear term $f(x,t,u)$ satisfies a Lipschitz condition about u, i.e., $\exists L$, s.t. $|f(x,t,u) - f(x,t,v)| \le L|u-v|$ for $u, v \in (-\infty, +\infty)$.

Let $x_j = jh (j = 0, 1, \cdots, N)$ and $t_k = k\tau (k = 0, 1, \cdots, K)$, where $h = \frac{l}{N}$ and $\tau = \frac{T}{K}$ are space and time steps, respectively. Assume that $u(x,t) \in C^2([0,l] \times [0,T])$. Using the relationship between the Grünwald-Letnikov derivative and Riemann-Liouville derivative [31], we discrete the Riesz fractional derivative $\dfrac{\partial^\alpha u}{\partial |x|^\alpha}$ by the shifted Grünwald-Letnikov formulae [26]

$$\begin{aligned} {}_0D_x^\alpha u(x_i, t_{k-1}) &= {}_0D_x^\alpha u(x_i, t_k) + O(\tau) \\ &= \frac{1}{h^\alpha} \sum_{j=0}^{i+1} \omega_j^\alpha u(x_{i+1-j}, t_k) + O(h+\tau), \end{aligned} \tag{4}$$

$${}_xD_l^\alpha u(x_i, t_{k-1}) = {}_xD_l^\alpha u(x_i, t_k) + O(\tau) \tag{5}$$

$$= \frac{1}{h^\alpha}\sum_{j=0}^{N-i+1}\omega_j^\alpha u(x_{i-1+j},t_k)+O(h+\tau),$$

where the coefficients $\omega_j^\alpha = (-1)^j\frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$. For the left term of equation (1), the following forward difference scheme can be used:

$$\frac{\partial u(x_i,t_{k-1})}{\partial t} = \frac{u(x_i,t_k)-u(x_i,t_{k-1})}{\tau}+O(\tau). \tag{6}$$

Let u_i^k be the numerical approximation to $u(x_i,t_k)$, then we obtain the implicit difference approximation (IFDA) of equations (1)-(3)

$$\frac{u_i^k-u_i^{k-1}}{\tau} = \frac{-\lambda}{2h^\alpha cos\frac{\alpha\pi}{2}}\left\{\sum_{j=0}^{i+1}\omega_j^\alpha u_{i-j+1}^k+\sum_{j=0}^{N-i+1}\omega_j^\alpha u_{i+j-1}^k\right\}$$
$$+f(x_i,t_{k-1},u_i^{k-1}),\quad i=1,2,\cdots,N-1, t=1,2,\cdots,K, \tag{7}$$
$$u_0^k=u_N^k=0, k=0,1,\cdots,K, \tag{8}$$
$$u_i^0=g_i=g(x_i), i=0,1,\cdots,N \tag{9}$$

Let $r=\frac{-\tau\lambda}{2h^\alpha cos\frac{\alpha\pi}{2}}$ and taking the boundary value conditions (8) into account, then the discrete schemes (7)-(9) can be rewritten as

$$u_i^k-r\left\{\sum_{j=1}^{i+1}\omega_{i-j+1}^\alpha u_j^k+\sum_{j=i-1}^{N-1}\omega_{j-i+1}^\alpha u_j^k\right\}=u_i^{k-1}+\tau f(x_i,t_{k-1},u_i^{k-1}),$$
$$i=1,2,\cdots,N-1, k=1,2,\cdots,K, \tag{10}$$
$$u_i^0=g_i=g(x_i), i=0,1,\cdots,N. \tag{11}$$

Its matrix form is

$$AU^k=U^{k-1}+\tau F^{k-1},\quad k=1,2,\cdots,K, \tag{12}$$

where the vectors

$$U^k=(u_1^k,u_2^k,\cdots,u_{N-1}^k)^T,$$
$$F^{k-1}=\Big(f(x_1,t_{k-1},u_1^{k-1}),\ f(x_2,t_{k-1},u_2^{k-1}),\cdots,f(x_{N-1},t_{k-1},u_{N-1}^{k-1})\Big)^T$$

and the coefficient matrix $A=(a_{ij})_{(N-1)\times(N-1)}$

$$a_{ij}=\begin{cases}-r\omega_{i-j+1}^\alpha, & j<i-1,\\ -r(\omega_0^\alpha+\omega_2^\alpha), & j=i\pm1\\ 1-2r\omega_1^\alpha, & j=i\\ -r\omega_{j-i+1}^\alpha, & j>i+1\end{cases} \tag{13}$$

3. Stability and convergence of the IFDA for the RFDE-NST

According to the literatures [26], it gets

Proposition 1. *The coefficients* ω_j^α $(j = 0, 1, \cdots)$ *satisfy*

(i) $\omega_0^\alpha = 1,\ \omega_1^\alpha = -\alpha < 0,\ \omega_j^\alpha > 0,\ j = 2, 3, \cdots$.

(ii) $\sum\limits_{j=0}^{\infty} \omega_j^\alpha = 0,\ \sum\limits_{j=0}^{n} \omega_j^\alpha < 0,\ n = 1, 2, \cdots$.

Assume the initial data has error ε_i^0. Let $\tilde{g}_i^0 = g_i^0 + \varepsilon_i^0 (i = 1, \cdots, N-1)$, u_i^k and $\tilde{u}_i^k (i = 1, \cdots, N-1)$ be the numerical solutions of equation (10) corresponding for the initial data g_i^0 and $\tilde{g}_i^0$ $(i = 1, \cdots, N-1)$, respectively, then $\varepsilon_i^k = u_i^k - \tilde{u}_i^k$ satisfies

$$\begin{aligned}
&\varepsilon_i^k - r\left\{\sum_{j=1}^{i+1} \omega_{i-j+1}^\alpha \varepsilon_j^k + \sum_{j=i-1}^{N-1} \omega_{j-i+1}^\alpha \varepsilon_j^k\right\} \\
&= \varepsilon_i^{k-1} + \tau f(x_i, t_{k-1}, u_i^{k-1}) - \tau f(x_i, t_{k-1}, \tilde{u}_i^{k-1}). \qquad (14)
\end{aligned}$$

Let $\varepsilon^k = (\varepsilon_1^k, \varepsilon_2^k, \cdots, \varepsilon_{N-1}^k)^t$. Then we obtain

Theorem 1. *The implicit finite difference scheme (10)-(11) is stable unconditionally.*

Proof It is easy to see $r = \dfrac{-\tau\lambda}{2h^\alpha cos\frac{\alpha\pi}{2}} > 0 (1 < \alpha \le 2)$. Therefore, according to the proposition 1, we have $a_{ij} < 0, (i, j = 1, \cdots, N-1, i \neq j)$, $a_{ii} > 1 (i = 1, \cdots, N-1)$ and $\sum\limits_{j=1}^{N-1} a_{ij} > 1 (i = 1, 2, \cdots, N-1)$. Let $\| \varepsilon^k \|_\infty = \max\limits_{0<i<N} | \varepsilon_i^k | = | \varepsilon_m^k |$ $(0 < m < N)$. Due to (14) and the Lipschitz condition of nonlinear function f, it gets

$$\begin{aligned}
| \varepsilon_m^k | &\le \sum_{j=1}^{N-1} a_{ij} | \varepsilon_m^k | \\
&= (1 - 2r\omega_1^\alpha) | \varepsilon_m^k | - r\left\{\sum_{j=1, j\neq m}^{m+1} \omega_{m-j+1}^\alpha + \sum_{j=m-1, j\neq m}^{N-1} \omega_{j-m+1}^\alpha\right\} | \varepsilon_m^k | \\
&\le (1 - 2r\omega_1^\alpha) | \varepsilon_m^k | \\
&\quad - r\left\{\sum_{j=1, j\neq m}^{m+1} \omega_{m-j+1}^\alpha | \varepsilon_j^k | + \sum_{j=m-1, j\neq m}^{N-1} \omega_{j-m+1}^\alpha | \varepsilon_j^k |\right\} \\
&\le (1 - 2r\omega_1^\alpha) | \varepsilon_m^k | - r\left|\sum_{j=1, j\neq m}^{m+1} \omega_{m-j+1}^\alpha \varepsilon_j^k + \sum_{j=m-1, j\neq m}^{N-1} \omega_{j-m+1}^\alpha \varepsilon_j^k\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|(1-2r\omega_1^\alpha)\varepsilon_m^k - r\left\{\sum_{j=1,j\neq m}^{m+1}\omega_{m-j+1}^\alpha\varepsilon_j^k + \sum_{j=m-1,j\neq m}^{N-1}\omega_{j-m+1}^\alpha\varepsilon_j^k\right\}\right| \\
&= \left|\varepsilon_m^{k-1} + \tau[f(x_m,t_{k-1},u_m^{k-1}) - f(x_m,t_{k-1},\tilde{u}_m^{k-1})]\right| \\
&\leq \left|\varepsilon_m^{k-1}\right| + \tau L\left|\varepsilon_m^{k-1}\right|,
\end{aligned}
$$

and then

$$
\begin{aligned}
\| \varepsilon^K \|_\infty &\leq (1+\tau L)\| \varepsilon^{K-1} \|_\infty \leq (1+\tau L)^2 \| \varepsilon^{K-2} \|_\infty \\
&\leq \cdots \leq (1+\tau L)^K \| \varepsilon^0 \|_\infty \leq e^{LT} \| \varepsilon^0 \|_\infty .
\end{aligned}
$$

So the implicit difference scheme (10)-(11) is stable unconditionally. □

We now consider the convergence of the IFDA (10)-(11), we also have

Theorem 2. *The IFDA (10)-(11) is convergent unconditionally.*

Proof Assume that $u(x_i,t_k)$ is the exact solution of the RFDE-NST (1)-(3) and u_i^k is the numerical solution of the IFDA (10)-(11). Then according to formulae (4)-(6), $e_i^k = u_i^k - u(x_i,t_k)$ $(i=1,\cdots,N-1; k=1,\cdots K)$ satisfy

$$
\begin{aligned}
&e_i^k - r\left\{\sum_{j=1}^{i+1}\omega_{i-j+1}^\alpha e_j^k + \sum_{j=i-1}^{N-1}\omega_{j-i+1}^\alpha e_j^k\right\} \\
&= e_i^{k-1} + \tau f(x_i,t_{k-1},u_i^{k-1}) - \tau f(x_i,t_{k-1},u(x_i,t_{k-1})) + \tau\cdot O(\tau) + \tau\cdot O(h+\tau) \\
&= e_i^{k-1} + \tau f(x_i,t_{k-1},u_i^{k-1}) - \tau f(x_i,t_{k-1},u(x_i,t_{k-1})) + \tau\cdot O(h+\tau)
\end{aligned}
$$

Moreover, we suppose that $e_0^k = e_N^k = 0 (k=1,\cdots,K)$, $e_i^0 = 0 (i=0,1,\cdots,N)$ and $\|e^k\|_\infty = |e_m^k|$. Then

$$
\begin{aligned}
| e_m^k | &\leq (1-2r\omega_1^\alpha) | e_m^k | - r\left\{\sum_{j=1,j\neq m}^{m+1}\omega_{m-j+1}^\alpha + \sum_{j=m-1,j\neq m}^{N-1}\omega_{j-m+1}^\alpha\right\} | e_m^k | \\
&\leq (1-2r\omega_1^\alpha) | e_m^k | \\
&\qquad - r\left\{\sum_{j=1,j\neq m}^{m+1}\omega_{m-j+1}^\alpha | e_j^k | + \sum_{j=m-1,j\neq m}^{N-1}\omega_{j-m+1}^\alpha | e_j^k |\right\} \\
&\leq \left|(1-2r\omega_1^\alpha)e_m^k - r\left\{\sum_{j=1,j\neq m}^{m+1}\omega_{m-j+1}^\alpha e_j^k + \sum_{j=m-1,j\neq m}^{N-1}\omega_{j-m+1}^\alpha e_j^k\right\}\right| \\
&= \left|e_m^{k-1} + \tau[f(x_m,t_{k-1},u_m^{k-1}) - f(x_m,t_{k-1},u(x_m,t_{k-1}))] + \tau O(h+\tau)\right| \\
&\leq \left|\varepsilon_m^{k-1}\right| + \tau L\left|\varepsilon_m^{k-1}\right| + \tau\cdot C(h+\tau),
\end{aligned}
$$

where C is a positive constant. Therefore,

$$\begin{aligned}\| e^K \|_\infty &\le (1+\tau L) \| e^{K-1} \|_\infty + \tau \cdot C(h+\tau) \\ &\le \cdots \\ &\le (1+\tau L)^K \| e^0 \|_\infty + \tau(1+\tau L)^{K-1} C(h+\tau) + \cdots + \tau C(h+\tau) \\ &\le (1+\tau L)^K \| e^0 \|_\infty + K\tau(1+\tau L)^{K-1} C(h+\tau) \\ &\le \exp(LT) \| e^0 \|_\infty + \exp(LT) TC(h+\tau) \\ &= CTe^{LT}(h+\tau),\end{aligned}$$

and then

$$\lim_{\tau,h\to 0} \| e^K \|_\infty = 0.$$

This completes the proof. □

4. Fractional method of lines

It is very difficult to obtain the exact solution of the nonlinear partial differential equations. In order to demonstrate the efficiency of the IFDA, a fractional method of lines (FMoL) for RFDE-NST also is presented. The FMoL was firstly introduced by Liu et al. [22]-[23] to solve the space fractional Fokker-Planck equation and simulate the Lévy motion with α-stable densities successfully. The FMoL for the RFDE-NST can be written as the following form:

$$\begin{cases} \dfrac{du(x_i,t)}{dt} = \dfrac{-\lambda}{2h^\alpha cos\frac{\alpha\pi}{2}} \left\{ \displaystyle\sum_{j=0}^{i+1} \omega_j^\alpha u(x_{i-j+1},t) + \sum_{j=0}^{N-i+1} \omega_j^\alpha u(x_{i+j-1},t) \right\} \\ \qquad + f(x_i,t,u(x_i,t)),\ i=1,2,\cdots,N-1, 0<t\le T, \\ u(x_0,t) = u(x_N,t) = 0, 0 \le t \le T, \\ u(x_i,0) = g(x_i), i=0,1,\cdots,N. \end{cases}$$

5. Numerical examples

In this section, two numerical examples are given to demonstrate our theoretical analysis.

Example 1. Consider the following NFR-SubDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \lambda \dfrac{\partial^\alpha u}{\partial |x|^\alpha} + \sin u, \quad 1<\alpha\le 2, 0<x<\pi, 0<t\le T \\ u(0,t) = u(\pi,t) = 0, 0 \le t \le T, \\ u(x,0) = \sin x, 0 < x < \pi, \end{cases} \tag{15}$$

to show the preciseness of the above theoretic analysis.

Table 1 lists the numerical results using the IFDA (10)-(11) with $\alpha = 1.8$, $t = 1.0$, $h \approx \tau$, $\lambda = 0.1$ and the last column data are the numerical results by FMoL with $\alpha = 1.8$, $t = 1.0$, $h \approx \tau = 0.016$, $\lambda = 0.1$. From Table 1, it can

TABLE 1. Comparison of the results of IFDA (10)-(11) with different space and time step $h \approx \tau$ and the FMoL when $\alpha = 1.8$, $t = 1.0$, $\lambda = 0.1$

(X,1.0)	$h = \frac{\pi}{20} \approx 0.157$ τ=0.16(IFDA)	$h = \frac{\pi}{100} \approx 0.0314$ τ=0.03(IFDA)	$h = \frac{\pi}{200} \approx 0.0157$ τ=0.016(IFDA)	$h = \frac{\pi}{200} \approx 0.0157$ τ=0.016(FMoL)
0.3142	0.66664661	0.71766001	0.72323636	0.71312860
0.6283	1.18091679	1.24052658	1.24612454	1.24780762
0.9425	1.53124465	1.58383641	1.58819777	1.59121579
1.2566	1.72803276	1.77216149	1.77545588	1.77773300
1.5708	1.79085073	1.83164748	1.83457607	1.83648052
1.8850	1.72803397	1.77217864	1.77552773	1.77783878
2.1991	1.53124729	1.58387242	1.58831821	1.59176807
2.5133	1.18092060	1.24057971	1.24628205	1.25101528
2.8274	0.66665013	0.71771309	0.72338189	0.72768746

be seen that the numerical results using IFDA are in good agreement with the convergence analysis and close to the results of FMOL.

Furthermore, Figure 1 shows the characters of diffusion system response with nonlinear source term at different times t with $\alpha = 1.8$, $\tau = 0.01$, $N = 100$, $\lambda = 0.1$. Figure 2 shows the characters of diffusion about different diffusion coefficients λ at time $t = 1.0$ with $\alpha = 1.8, \tau = 0.01, N = 100$. From Figure 2, it can be seen that the larger the diffusion coefficient λ, the greater the velocity of diffusion.

Example 2. Considering the following nonlinear reaction diffusion equation using Fisher's growth equation and a symmetric (Riesz) fractional diffusion term of order $1 < \alpha \leq 2$ ([2]):

$$\begin{cases} \dfrac{\partial u}{\partial t} = \lambda \dfrac{\partial^\alpha u}{\partial |x|^\alpha} + \mu * u(1 - \dfrac{u}{K}), & 0 < x < l, \quad 0 < t \leq T, \\ u(0,t) = u(l,t) = 0, & 0 \leq t \leq T, \\ u(x,0) = f(x), & 0 < x < l, \end{cases} \tag{16}$$

where μ is the intrinsic growth rate and K is the carrying capacity.

We take $\lambda = 0.1, K = 1, l = 100$, and a smooth step-like initial function $f(x)$ which takes the constant value $f = 0.8$ around the point $x = 50$ and rapidly decays to 0 away from the point $x = 50$.

Figure 3 shows that the numerical simulations of the problem (16) using IFDA (10)-(11)with $1.5 \leq \alpha \leq 2.0$ at $t = 32$. The curve shows heavier tails and faster spreading in the fractional case $\alpha < 2$. Figure 4 shows the characters of the diffusion system response with Fisher's nonlinear source term at different times t, which is similar to the result (Figure 3) reported in Baeumer et al. [2]. Figure 5 shows the characters of the diffusion system response with different intrinsic growth rate μ at $t = 32$ with $\alpha = 1.8$.

6. Conclusions

In this paper, numerical simulation of the Riesz fractional diffusion equation with a nonlinear source term (RFDE-NST) has been described and demonstrated. The stability and the convergence of the IFDA have also been proved. Finally, some numerical results of IFDA and FMoL are presented. These numerical results are given to demonstrate that our IFDA is a computationally efficient method for REDE-NST. This method and analysis technique can be used to solve and analyze other kinds of fractional-order partial differential equations with nonlinear source term.

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LIST OF FIGURES

Figure 1. Comparison of the response of the diffusion system with nonlinear source term at different times t when $\alpha = 1.8, \tau = 0.01, h = \frac{\pi}{100}, \lambda = 0.1$

Figure 2. Comparison of the characters of the diffusion system with nonlinear source term for different diffusion coefficients λ when $\alpha = 1.8, \tau = 0.01, h = \frac{\pi}{100}, t = 1.0$.

Figure 3. Numerical solutions of the RFDE-NST in example 2 with $1.5 \leq \alpha \leq 2.0$ at t=32 with $\mu = 0.25$ showing heavier tails and faster spreading in the fractional case $1.5 < \alpha < 2$.

Figure 4. Comparison of the characters of the RFDE-NST in Example 2 at different time t with $\alpha = 1.8, \mu = 0.25$.

Figure 5. Comparison of the characters of the RFDE-NST in Example 2 at different intrinsic growth rate μ with $\alpha = 1.8, t = 32$.

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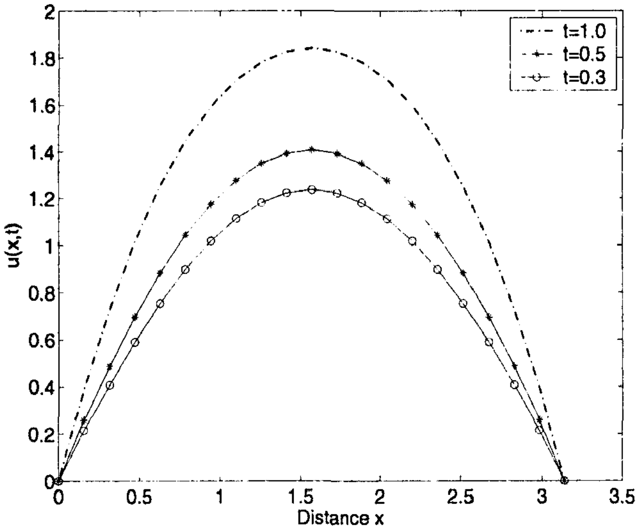

FIGURE 1. Comparison of the response of the diffusion system with nonlinear source term at different times t when $\alpha = 1.8, \tau = 0.01, h = \frac{\pi}{100}, \lambda = 0.1$

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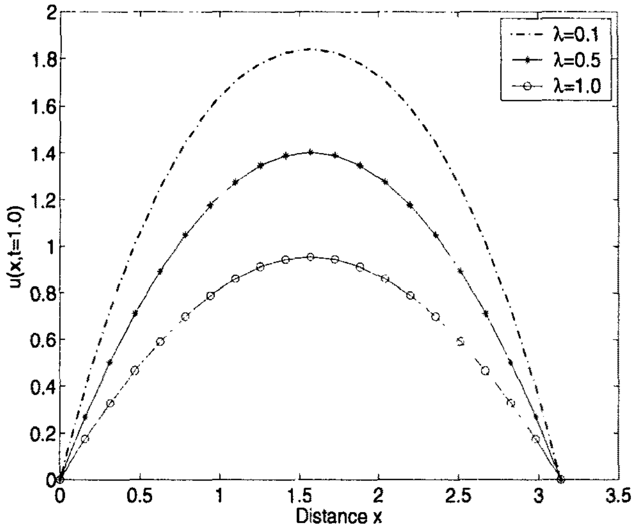

FIGURE 2. Comparison of the characters of the diffusion system with nonlinear source term for different diffusion coefficients λ when $\alpha = 1.8, \tau = 0.01, h = \frac{\pi}{100}, t = 1.0$

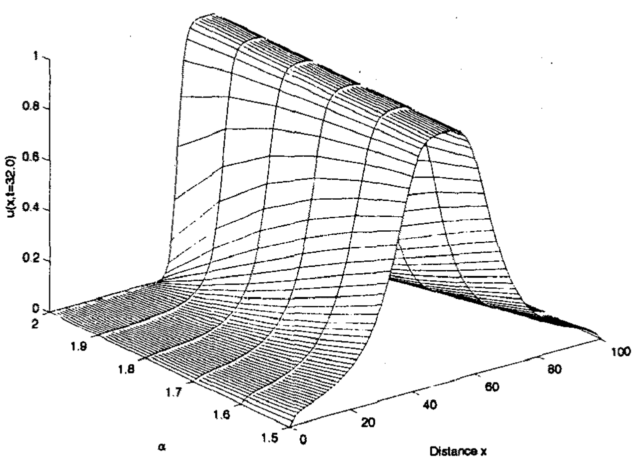

FIGURE 3. Numerical solutions of the RFDE-NST in example 2 with $1.5 \leq \alpha \leq 2.0$ at t=32 with $\mu = 0.25$ showing heavier tails and faster spreading in the fractional case $1.5 < \alpha < 2$

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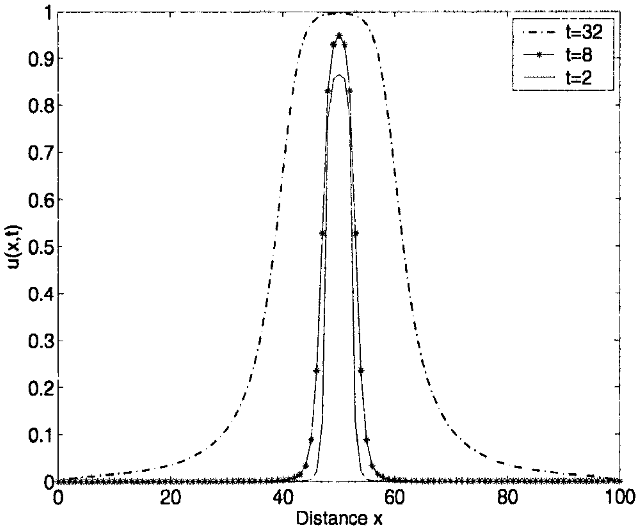

FIGURE 4. Comparison of the characters of the RFDE-NST in Example 2 at different time t with $\alpha = 1.8, \mu = 0.25$

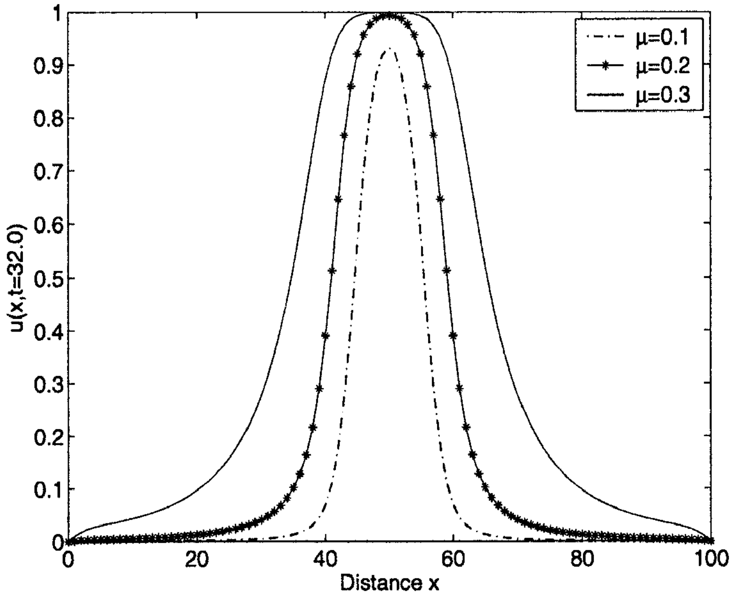

FIGURE 5. Comparison of the characters of the RFDE-NST in Example 2 at different intrinsic growth rate μ with $\alpha = 1.8, t = 32$

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