

SOME COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GAMMA AND POLYGAMMA FUNCTIONS

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ABSTRACT. In this paper, some logarithmically completely monotonic, strongly completely monotonic and completely monotonic functions related to the gamma, digamma and polygamma functions are established. Several inequalities, whose bounds are best possible, are obtained.

1. Introduction

Recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders and $0 \leq (-1)^n f^{(n)}(x) < \infty$ for all $n \geq 0$ on I . For our convenience, the class of completely monotonic functions on I is denoted by $\mathcal{C}[I]$. The well known Bernstein's Theorem [37] states that $f \in \mathcal{C}[(0, \infty)]$ if and only if $f(x) = \int_0^\infty e^{-xt} d\mu(t)$, where $\mu(t)$ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. A completely monotonic function on $(0, \infty)$ which is non-identically zero cannot vanish at any point on $(0, \infty)$. Completely monotonic functions appear naturally in various fields, for example, probability theory, numerical analysis, physics and potential theory. The main properties of these functions are given in [37, Chapter IV].

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies $(-1)^k [\ln f(x)]^{(k)} \geq 0$ for $k \in \mathbb{N}$ on I . The set of logarithmically completely monotonic functions on an interval I is denoted by $\mathcal{L}[I]$.

A function f on $(0, \infty)$ is called a Stieltjes transform if it can be written in the form

$$f(x) = a + \int_0^\infty \frac{d\mu(s)}{s+x},$$

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where a is a nonnegative number and μ a nonnegative measure on $[0, \infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s} d\mu(s).$$

The set of Stieltjes transform is denoted by \mathcal{S} .

A much useful and meaningful relation $\mathcal{L} \subset \mathcal{C}$ between the completely monotonic functions and the logarithmically completely monotonic functions was proved in [31, 32]. Among other things, it is further revealed in [13] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L} \subset \mathcal{C}$. The class of logarithmically completely monotonic functions can be characterized as the infinitely divisible completely monotonic functions which are established by Horn in [23, Theorem 4.4] and restated in [13, Theorem 1.1]. There has been a lot of literature about the (logarithmically) completely monotonic functions, for example, [8, 7, 12, 13, 14, 15, 22, 25, 31, 32, 33] and the references therein.

It is the aim of this paper to provide some new classes of (logarithmically) completely monotonic functions. The functions we study have in common that they are defined in terms of gamma, digamma, and polygamma functions. In the next section we establish a logarithmically completely monotonic function related to the volume of the unit ball. And, some completely monotonic functions are proved in section 4. In section 5 we show that two inequalities, whose bounds are best possible, follow from the monotonic functions. Subsequently, we present a strongly completely monotonic function. Finally, we prove that a ratio of the gamma function is logarithmically completely monotonic in section 7.

2. Lemmas

Lemma 1 ([1, 36]). *For $x > 0$ and $r > 0$,*

$$(2.1) \quad \frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} dt.$$

Lemma 2 ([26, p. 16]). *The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be defined as*

$$(2.2) \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt,$$

$$(2.3) \quad \psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt} dt$$

for $x > 0$ and $n \in \mathbb{N} := \{1, 2, \dots\}$, where $\gamma = 0.5772\dots$ is Euler-Mascheroni constant.

Lemma 3 ([1, p. 259]). For $x > 0$, the duplication and asymptotic formulas of ψ are expressed as

$$(2.4) \quad \psi(2x) = \frac{1}{2}\psi(x) + \frac{1}{2}\psi\left(x + \frac{1}{2}\right) + \ln 2,$$

$$(2.5) \quad \psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \dots \quad (x \rightarrow \infty).$$

Lemma 4 ([1, p. 260]). The recurrence and asymptotic formulas of $\psi^{(n)}(x)$:

$$(2.6) \quad \psi^{(n)}(x+1) = \psi^{(n)}(x) + (-1)^n \frac{n!}{x^{n+1}} \quad (x > 0; n = 0, 1, 2, \dots),$$

$$(2.7) \quad |\psi^{(n)}(x)| \sim \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \frac{(n+1)!}{12x^{n+2}} + \dots \quad (x \rightarrow \infty; n \in \mathbb{N}).$$

Lemma 5 ([37]). Let $f_i(t)$ for $i = 1, 2$ be piecewise continuous in arbitrary finite intervals included in $(0, \infty)$. Suppose there exist some constants $M_i > 0$ and $c_i \geq 0$ such that $|f_i(t)| \leq M_i e^{c_i t}$ for $i = 1, 2$, then

$$(2.8) \quad \int_0^\infty \left[\int_0^t f_1(u) f_2(t-u) du \right] e^{-st} dt = \int_0^\infty f_1(u) e^{-su} du \int_0^\infty f_2(v) e^{-sv} dv.$$

Remark. Lemma 5 is the convolution theorem of Laplace transforms. It can be found in standard textbooks of integral transform.

Lemma 6 ([6, Theorem 4.14]). Let $n \geq 1$ be an integer and let c be a real number. The function $x^c |\psi^{(n)}(x)|$ is strictly convex on $(0, \infty)$ if and only if $c \leq n$, or $c = n + 1$, or $c \geq n + 2$.

3. A function related to the volume of the unit ball

In the recent past, several sequences related to the volume of the unit ball in \mathbb{R}^k ,

$$\Omega_k = \frac{\pi^{k/2}}{\Gamma(1 + k/2)}, \quad k = 0, 1, 2, \dots,$$

have been a focus of research regarding their monotonicity and inequalities. We refer to [3, 29, 34] and the references therein. In 1987, K. H. Borgwardt [16, p. 253] provided elegant upper and lower bounds for the ratio Ω_{k-1}/Ω_k :

$$(3.1) \quad \frac{k}{2\pi} \leq \left(\frac{\Omega_{k-1}}{\Omega_k} \right)^2 \leq \frac{k+1}{2\pi}, \quad k = 2, 3, \dots$$

The following refinement of (3.1) was published in [3]:

$$(3.2) \quad \frac{k+a}{2\pi} \leq \left(\frac{\Omega_{k-1}}{\Omega_k} \right)^2 \leq \frac{k+b}{2\pi}, \quad k = 2, 3, \dots,$$

with the best possible constants $a = 1/2$ and $b = \pi/2 - 1 = 0.57079 \dots$. In 2004, H. Alzer [6] provided another inequalities, which sharpen inequalities (3.2):

$$(3.3) \quad \frac{1}{2\pi} \exp\left(\frac{\alpha}{k} + \psi(k)\right) < \left(\frac{\Omega_{k-1}}{\Omega_k}\right)^2 \leq \frac{1}{2\pi} \exp\left(\frac{\beta}{k} + \psi(k)\right), \quad k = 2, 3, \dots,$$

with the best possible constants $\alpha = 1$ and $\beta = 2(\ln(8/\pi) + \gamma - 1) = 1.02385 \dots$

By rearranging the right hand of inequalities (3.3), we can get a logarithmically completely monotonic function as follows.

Theorem 1. *The function*

$$f_\beta(x) = \frac{1}{2} \left(\frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)}\right)^2 \exp\left(\frac{\beta}{2x} + \psi(2x)\right)$$

is strictly logarithmically completely monotonic on $(0, \infty)$ if $\beta \geq 13/12$, that is, the function $f_\beta(x) \in \mathcal{L}[(0, \infty)]$ if $\beta \geq 13/12$.

Proof. Taking logarithm of $f_\beta(x)$ leads to

$$(3.4) \quad \ln f_\beta(x) = 2 \ln \Gamma\left(x + \frac{1}{2}\right) - 2 \ln \Gamma(x + 1) + \frac{\beta}{2x} + \psi(2x) - \ln 2.$$

Applying duplication formula of ψ (2.4) to (3.4), we obtain

$$(3.5) \quad \ln f_\beta(x) = 2 \ln \Gamma\left(x + \frac{1}{2}\right) - 2 \ln \Gamma(x + 1) + \frac{\beta}{2x} + \frac{1}{2}\psi(x) + \frac{1}{2}\psi\left(x + \frac{1}{2}\right).$$

Differentiating $\ln f_\beta(x)$ and applying (2.1), (2.2) and (2.3), for $n \geq 1$, we have

$$\begin{aligned} & (-1)^n (\ln f(x))^{(n)} \\ &= (-1)^n \left[2\psi^{(n-1)}\left(x + \frac{1}{2}\right) - 2\psi^{(n-1)}(x + 1) + (-1)^n \frac{n!\beta}{2x^{n+1}} + \frac{1}{2}\psi^{(n)}(x) \right. \\ & \quad \left. + \frac{1}{2}\psi^{(n)}\left(x + \frac{1}{2}\right) \right] \\ &= \frac{1}{2} \int_0^\infty e^{-xt} \left(\frac{4e^{-\frac{t}{2}}t^{n-1}}{1 - e^{-t}} - \frac{4e^{-t}t^{n-1}}{1 - e^{-t}} + \beta t^n - \frac{t^n}{1 - e^{-t}} - \frac{e^{-\frac{t}{2}}t^n}{1 - e^{-t}} \right) dt \\ &= \frac{1}{2} \int_0^\infty \frac{e^{-xt}t^{n-1}}{e^t - 1} \phi_\beta(t) dt, \end{aligned}$$

where

$$\begin{aligned} \phi_\beta(t) &= (\beta t - t)e^t + (4 - t)e^{\frac{t}{2}} - \beta t - 4 \\ &= \sum_{k=1}^\infty \frac{t^{k+1}}{k!} \left[\beta - 1 - \left(1 - \frac{2}{k+1}\right) \frac{1}{2^k} \right]. \end{aligned}$$

It is easy to see that $\max\left(\left(1 - \frac{2}{k+1}\right) \frac{1}{2^k}\right) = \frac{1}{12}$ when $k = 2$. So we conclude that $\phi_\beta(t) > 0$ for $\beta \geq \frac{13}{12}$. This implies

$$(-1)^n (\ln f(x))^{(n)} > 0$$

for $\beta \geq \frac{13}{12}$. The proof is completed. □

4. Several completely monotonic functions

In view of the importance of ψ function in the Theory of special Functions and in Mathematical Physics, they have been intensively investigated by various authors. In particular, many inequalities and monotonic properties for ψ and its derivatives have been published in the recent past; see [4, 5, 9, 10, 11, 18, 21, 24, 28].

For example, in [5], the author proved that the function $x^c|\psi^{(n)}(x)|$ is strictly decreasing (increasing) on $(0, \infty)$ for $n \in \mathbb{N}$ and $c \in \mathbb{R}$ if and only if $c \leq n$ ($c \geq n + 1$), respectively; Later, the author [6] further proved Lemma 6; In [4], Alzer obtained that the function $\frac{d^2(x\psi(x))}{dx^2}$ is strictly completely monotonic on $(0, \infty)$ and $\frac{d^2(x^2\psi'(x))}{dx^2}$ is strictly completely monotonic on $(0, \infty)$.

Motivated by the results mentioned above, we will establish some completely monotonic functions in the following theorems.

Theorem 2. *Let $m \geq 1$ be integer and $p_m(x) = x|\psi^{(m)}(x)|$. The function $p_m(x)$ is strictly completely monotonic on $(0, \infty)$. Moreover, the function $\frac{d^3(-x \ln \Gamma(x))}{dx^3}$ is strictly completely monotonic on $(0, \infty)$.*

Proof. If $m \geq 1$ is an odd, using (2.1), (2.3) and the convolution theorem for Laplace transforms (2.8) we have

$$\begin{aligned} \frac{1}{x}(-1)^n p_m^{(n)}(x) &= \frac{1}{x}n(-1)^n \psi^{(n+m-1)}(x) + (-1)^n \psi^{(n+m)}(x) \\ &= -n \int_0^\infty e^{-xt} dt \int_0^\infty \frac{e^{-xt} t^{n+m-1}}{1-e^{-t}} dt + \int_0^\infty \frac{e^{-xt} t^{n+m}}{1-e^{-t}} dt \\ (4.1) \quad &= \int_0^\infty e^{-xt} \lambda_{m,n}(t) dt, \end{aligned}$$

where

$$\lambda_{m,n}(t) = -n \int_0^t \frac{s^{n+m-1}}{1-e^{-s}} ds + \frac{t^{n+m}}{1-e^{-t}}.$$

Similarly, if $m \geq 1$ is an even, we can also get the formula (4.1). Differentiating $\lambda_{m,n}(t)$ yields

$$\begin{aligned} e^t t^{-n-m+1} (1-e^{-t})^2 \lambda'_{m,n}(t) &= m(e^t - 1) - t \\ &= (m-1)t + \sum_{k=2}^{\infty} \frac{mt^k}{k!} > 0. \end{aligned}$$

Moreover, we have $\lambda_{m,n}(0^+) = 0$. This implies $\lambda_{m,n}(t) > 0$ on $(0, \infty)$. Therefore, we obtain

$$(-1)^n p_m^{(n)}(x) > 0$$

for $m \geq 1$ and $x > 0$.

Using the same method, we can show that the function $\frac{d^3(-x \ln \Gamma(x))}{dx^3}$ is strictly completely monotonic on $(0, \infty)$. The proof is complete. \square

Theorem 3. Let $g(x) = -x^2\psi(x)$. The function $g'''(x)$ is strictly completely monotonic on $(0, \infty)$, that is, $g'''(x) \in \mathcal{C}[(0, \infty)]$.

Proof. A simple calculation yields

$$g^{(n)}(x) = -n(n-1)\psi^{(n-2)}(x) - 2nx\psi^{(n-1)}(x) - x^2\psi^{(n)}(x)$$

for $n \geq 3$ and $x > 0$. Using (2.1), (2.3) and (2.8) we get

$$(4.2) \quad \frac{1}{x^2}(-1)^n g^{(n)}(x) = \int_0^\infty e^{-xt} \delta_n(t) dt,$$

where

$$\delta_n(t) = n(n-1)t \int_0^t \frac{s^{n-2}}{1-e^{-s}} ds - n(n+1) \int_0^t \frac{s^{n-1}}{1-e^{-s}} ds + \frac{t^n}{1-e^{-t}}.$$

Differentiation yields

$$\delta'_n(t) = n(n-1) \int_0^t \frac{s^{n-2}}{1-e^{-s}} ds - \frac{nt^{n-1}(1-e^{-t}) + t^n e^{-t}}{(1-e^{-t})^2}$$

and

$$e^{2t} t^{-n+1} (1-e^{-t})^3 \delta''_n(t) = t(e^t + 1) > 0.$$

Moreover, we have $\delta_n(0^+) = \delta'_n(0^+) = 0$. This implies that $\delta_n(t) > 0$ on $(0, \infty)$, so that (4.2) leads to $(-1)^n g^{(n)}(x) > 0$ for $n \geq 3$ and $x > 0$, that is, $g'''(x)$ is strictly completely monotonic on $(0, \infty)$. \square

Theorem 4. Let $h(x) = x^2 \ln \Gamma(x)$. The function $h^{(4)}(x)$ is strictly completely monotonic on $(0, \infty)$, that is, $h^{(4)}(x) \in \mathcal{C}[(0, \infty)]$.

Proof. It is easy to see that

$$h^{(n)}(x) = n(n-1)\psi^{(n-3)}(x) + 2nx\psi^{(n-2)}(x) + x^2\psi^{(n-1)}(x)$$

for $n \geq 4$ and $x > 0$. Using (2.1), (2.3) and (2.8) we get

$$(4.3) \quad \frac{1}{x^2}(-1)^n h^{(n)}(x) = \int_0^\infty e^{-xt} \tau_n(t) dt,$$

where

$$\tau_n(t) = n(n-1)t \int_0^t \frac{s^{n-3}}{1-e^{-s}} ds - n(n+1) \int_0^t \frac{s^{n-2}}{1-e^{-s}} ds + \frac{t^{n-1}}{1-e^{-t}}.$$

Differentiation yields

$$\tau'_n(t) = n(n-1) \int_0^t \frac{s^{n-3}}{1-e^{-s}} ds - \frac{(n+1)t^{n-2}(1-e^{-t}) + t^{n-1}e^{-t}}{(1-e^{-t})^2}$$

and

$$\begin{aligned} e^{2t}t^{-n+3}(1 - e^{-t})^3\tau_n''(t) &= 2e^{2t} + e^t(t^2 + 2t - 4) + t^2 - 2t + 2 \\ &= (2e^{2t} - 4e^t + 2) + t^2(e^t + 1) + 2t(e^t - 1) \\ &= 2(e^t - 1)^2 + t^2(e^t + 1) + 2t(e^t - 1) \\ &> 0. \end{aligned}$$

Moreover, we have $\tau_n(0^+) = \tau_n'(0^+) = 0$. This implies that $\tau_n(t) > 0$ on $(0, \infty)$, so that (4.3) leads to $(-1)^n h^{(n)}(x) > 0$ for $n \geq 4$ and $x > 0$, that is, $h^{(4)}(x) \in \mathcal{C}[(0, \infty)]$. □

5. Functions and inequalities related to $\psi^{(m)}$

In 1995, G. D. Anderson et al. [10] proved that the function $f(x) = x(\ln x - \psi(x))$ is strictly decreasing and strictly convex on $(0, \infty)$. Moreover, we obtained a double inequality for $x > 0$

$$(5.1) \quad \frac{1}{2x} < \ln x - \psi(x) < \frac{1}{x}.$$

This extends a result of H. Minc and L. Sathre [30], who established (5.1) for $x > 0$, and employed it to prove several discrete inequalities involving the geometric mean of the first n positive integers.

Later, H. Alzer [2] provided an extension of the result given by Anderson et al. The author proved that the function $f_\alpha(x) = x^\alpha(\ln x - \psi(x))$ is strictly completely monotonic on $(0, \infty)$ if and only if $\alpha \leq 1$, and established a double inequality, whose bounds are best possible:

$$0 < (-1)^n x^{n-1} [x\psi(x)]^{(n)} < (n - 2)! \quad (x > 0, n = 2, 3, \dots).$$

For $x > 0$ and $m = 1, 2, \dots$, let

$$(5.2) \quad z_m(x) = x \left(|\psi^{(m)}(x)| - \frac{(m - 1)!}{x^m} \right)$$

and

$$(5.3) \quad q_m(x) = x^{m+1} \left(|\psi^{(m)}(x)| - \frac{(m - 1)!}{x^m} \right).$$

Another aim of this paper is to prove the completely monotonicity of $z_m(x)$ and the monotonicity of $q_m(x)$, which extends the results given by Alzer [2], and to provide several inequalities.

Theorem 5. *Let $z_m(x)$ be the function defined by (5.2). Then $z_m(x)$ is strictly completely monotonic on $(0, \infty)$.*

Proof. Using (2.1) and (2.3), we get

$$\begin{aligned} z_m(x) &= x \left(\int_0^\infty \frac{e^{-xt}t^m}{1-e^{-t}} dt - \int_0^\infty e^{-xt}t^{m-1} dt \right) \\ &= x \int_0^\infty e^{-xt}t^m \varphi(t) dt, \end{aligned}$$

where

$$\varphi(t) = \frac{1}{1-e^{-t}} - \frac{1}{t}.$$

It is easy to see that the function φ is strictly increasing on $(0, \infty)$ with $\lim_{t \rightarrow 0^+} \varphi(t) = 1/2$ and $\lim_{t \rightarrow \infty} \varphi(t) = 1$. Therefore, for $x > 0$ and $m, n = 1, 2, \dots$, we have

$$\begin{aligned} &(-1)^n z_m^{(n)}(x) \\ &= x(-1)^n \frac{d^n}{dx^n} \int_0^\infty e^{-xt}t^m \varphi(t) dt - (-1)^{n-1} n \frac{d^{n-1}}{dx^{n-1}} \int_0^\infty e^{-xt}t^m \varphi(t) dt \\ &= x \int_0^\infty e^{-xt}t^{m+n} \varphi(t) dt - n \int_0^\infty e^{-xt}t^{m+n-1} \varphi(t) dt \\ &= \int_0^{n/x} e^{-xt}t^{m+n-1} \varphi(t)(tx-n) dt + \int_{n/x}^\infty e^{-xt}t^{m+n-1} \varphi(t)(tx-n) dt \\ &> \varphi(n/x) \int_0^{n/x} e^{-xt}t^{m+n-1}(tx-n) dt + \varphi(n/x) \int_{n/x}^\infty e^{-xt}t^{m+n-1}(tx-n) dt \\ &= \varphi(n/x) \int_0^\infty e^{-xt}t^{m+n-1}(tx-n) dt \\ &= \varphi(n/x) \left(x \int_0^\infty e^{-xt}t^{m+n} dt - n \int_0^\infty e^{-xt}t^{m+n-1} dt \right) \\ &= \varphi(n/x) \left(x \frac{(m+n)!}{x^{m+n+1}} - n \frac{(m+n-1)!}{x^{m+n}} \right) \\ &> 0. \end{aligned}$$

So we obtain that the function $z_m(x)$ is strictly completely monotonic on $(0, \infty)$. □

Theorem 6. *Let $m, n \geq 1$ be integers. Then we have*

$$(5.4) \quad (m-1)(m+n-2)! < (-1)^n x^{m+n-1} (x|\psi^{(m)}(x)|)^{(n)}$$

for $x > 0$. The lower bound is best possible.

Proof. From Theorem 5 we obtain

$$\begin{aligned} 0 &< (-1)^n z_m^{(n)}(x) = (-1)^n (x|\psi^{(m)}(x)|)^{(n)} - (-1)^n \left[\frac{(m-1)!}{x^{m-1}} \right]^{(n)} \\ &= (-1)^n (x|\psi^{(m)}(x)|)^{(n)} - \frac{(m-1)(m+n-2)!}{x^{m+n-1}}. \end{aligned}$$

Since $(-1)^n(x|\psi^{(m)}(x)|)^{(n)} > 0$, see Theorem 2, then we get the inequality (5.4).

It remains to show that the bound in (5.4) cannot be more refined. For the functions $\psi^{(n)}(x)$, $\psi^{(n)}(x+1)$ and $(-1)^{n+1}n!/x^{n+1}$ keep the same sign, the recurrence formula (2.6) can be expressed as $|\psi^{(n)}(x)| = |\psi^{(n)}(x+1)| + n!/x^{n+1}$. Consequently, we get

$$\begin{aligned}
 & (-1)^n x^{m+n-1} (x|\psi^{(m)}(x)|)^{(n)} \\
 (5.5) \quad & = (-1)^n x^{m+n-1} (x|\psi^{(m+n)}(x)| - n|\psi^{(m+n-1)}(x)|) \\
 & = (-1)^n \left(x^{m+n} |\psi^{(m+n)}(x+1)| - n x^{m+n-1} |\psi^{(m+n-1)}(x+1)| \right. \\
 (5.6) \quad & \left. + \frac{m(m+n-1)!}{x} \right).
 \end{aligned}$$

Since $(-1)^n x^{m+n-1} (x|\psi^{(m)}(x)|)^{(n)} > 0$, from (5.6) we obtain

$$\lim_{x \rightarrow 0^+} (-1)^n x^{m+n-1} (x|\psi^{(m)}(x)|)^{(n)} = \infty.$$

By asymptotic formula of $\psi^{(n)}(x)$ (2.7) and (5.5), we get

$$\lim_{x \rightarrow \infty} (-1)^n x^{m+n-1} (x|\psi^{(m)}(x)|)^{(n)} = (m-1)(m+n-2)!.$$

Hence, the lower bound in (5.4) is best possible. □

Theorem 7. Let $q_m(x)$ be the function defined by (5.3).

- (1) The function $q_m(x)$ is strictly decreasing and strictly convex on $(0, \infty)$.
- (2)

$$(5.7) \quad \frac{m!}{2} < x^{m+1} \left(|\psi^{(m)}(x)| - \frac{(m-1)!}{x^m} \right) < m!.$$

Both bounds are best possible.

Proof. For $x > 0$ and $m = 1, 2, \dots$, we differentiate $q_m(x)$ and apply (2.1), (2.3) and (2.8). This leads to

$$\begin{aligned}
 & \frac{1}{x^{m+1}} q'_m(x) \\
 & = \frac{m+1}{x} |\psi^{(m)}(x)| - |\psi^{(m+1)}(x)| - \frac{(m-1)!}{x^{m+1}} \\
 & = (m+1) \int_0^\infty e^{-xt} dt \int_0^\infty \frac{e^{-xt} t^m}{1-e^{-t}} dt - \int_0^\infty \frac{e^{-xt} t^{m+1}}{1-e^{-t}} dt - \frac{1}{m} \int_0^\infty e^{-xt} t^m dt \\
 & = \int_0^\infty e^{-xt} \mu_m(t) dt,
 \end{aligned}$$

where

$$\mu_m(t) = (m+1) \int_0^t \frac{s^m}{1-e^{-s}} ds - \frac{t^{m+1}}{1-e^{-t}} - \frac{t^m}{m}.$$

A simple calculation yields

$$\begin{aligned} e^{2t}(1 - e^{-t})^2 t^{1-m} \mu'_m(t) &= (t^2 + 2)e^t - e^{2t} - 1 \\ &= \sum_{k=3}^{\infty} \frac{(2 - 2^k)t^k}{k!} + t^2 \sum_{k=1}^{\infty} \frac{t^k}{k!} \\ &= \sum_{k=3}^{\infty} \frac{[2 + k(k - 1) - 2^k]t^k}{k!}. \end{aligned}$$

It is easy to see that $2 + k(k - 1) - 2^k \leq 0$ for $k \geq 3$ and $\mu_m(0+) = 0$. These imply $\mu_m(t) < 0$ for $t > 0$. Therefore, we obtain that $q_m(x)$ is strictly decreasing on $(0, \infty)$.

From Lemma 6, we can easily conclude that $q_m(x)$ is strictly convex on $(0, \infty)$.

Next, using the recurrence and asymptotic formula of $\psi^{(n)}(x)$ (see Lemma 4), we obtain

$$\lim_{x \rightarrow 0+} x^{m+1} \left(|\psi^{(m)}(x)| - \frac{(m - 1)!}{x^m} \right) = m!$$

and

$$\lim_{x \rightarrow \infty} x^{m+1} \left(|\psi^{(m)}(x)| - \frac{(m - 1)!}{x^m} \right) = \frac{m!}{2}.$$

Since $q_m(x)$ is strictly decreasing on $(0, \infty)$, then the double inequality (5.7) holds for $x > 0$ and $m = 1, 2, \dots$ □

6. Strongly completely monotonicity

In 1989, S. Y. Trimble et al. [35] introduced an interesting subclass of the completely monotonic functions. A function f is called strongly completely monotonic on $(0, \infty)$ if $(-1)^n x^{n+1} f^{(n)}(x)$ is nonnegative and decreasing on $(0, \infty)$ for $n = 0, 1, 2, \dots$. The authors showed that if f is strongly completely monotonic on $(0, \infty)$, then $1/f$ is star-shaped and therefore super-additive.

Recall that a function f is said to be super-additive on $(0, \infty)$ if

$$f(x) + f(y) \leq f(x + y)$$

for all $x, y > 0$. A function f is said to be star-shaped on $(0, \infty)$ if

$$f(ax) \leq af(x)$$

is valid for all $x > 0$ and for all $a \in (0, 1)$. It is well known that star-shaped functions are super-additive. For more information about these functions, please see [17, 25].

Theorem 8. Define for $x > 0$

$$s(x) = \ln x - \psi(x).$$

The function $s(x)$ is strongly completely monotonic on $(0, \infty)$. The function $\frac{1}{s(x)}$ is star-shaped and therefore super-additive on $(0, \infty)$.

Proof. From Theorem 7, we obtain that

$$q_m(x) = x^{m+1} \left(|\psi^{(m)}(x)| - \frac{(m-1)!}{x^m} \right) = (-1)^m x^{m+1} (\ln x - \psi(x))^{(m)}$$

is nonnegative and decreasing on $(0, \infty)$ for $m = 1, 2, \dots$. Moreover, the function $x(\ln x - \psi(x))$ is strictly decreasing and satisfies $1/2 < x(\ln x - \psi(x)) < 1$. By the definition and properties of strongly completely monotonicity, the theorem follows. \square

7. A ratio of the gamma function

In 1997, Merkle [27] showed that the function $G(x) = \frac{\Gamma(2x)}{\Gamma^2(x)}$ is strictly log-concave on $(0, \infty)$. Later, Chao-Ping Chen [19] proved that the function $1/G$ is strictly logarithmically completely monotonic on $(0, \infty)$. In this section, we will generalize the above result as follows.

Theorem 9. *Let α be real number. The function $G_\alpha(x) = \frac{\Gamma^\alpha(x)}{\Gamma(\alpha x)}$ is strictly logarithmically completely monotonic with $x \in (0, \infty)$ for $\alpha > 1$; The function $[G_\alpha(x)]^{-1}$ is strictly logarithmically completely monotonic with $x \in (0, \infty)$ for $0 < \alpha < 1$.*

Proof. Taking logarithm and differentiation yields

$$\begin{aligned} (\ln G_\alpha(x))' &= \alpha[\psi(x) - \psi(\alpha x)] \triangleq \alpha g_\alpha(x), \\ \left(\ln \frac{1}{G_\alpha(x)} \right)' &= \alpha[\psi(\alpha x) - \psi(x)] = -\alpha g_\alpha(x). \end{aligned}$$

By (2.3), for $n \geq 2$, we have

$$\begin{aligned} (-1)^n g_\alpha^{(n-1)}(x) &= (-1)^n [\psi^{(n-1)}(x) - \alpha^{n-1} \psi^{(n-1)}(\alpha x)] \\ &= \int_0^\infty \frac{u^{n-1}}{1-e^{-u}} e^{-xu} du - \alpha^{n-1} \int_0^\infty \frac{t^{n-1}}{1-e^{-t}} e^{-\alpha xt} dt \\ &= \alpha^n \int_0^\infty \frac{t^{n-1}}{1-e^{-\alpha t}} e^{-\alpha xt} dt - \alpha^{n-1} \int_0^\infty \frac{t^{n-1}}{1-e^{-t}} e^{-\alpha xt} dt \\ &= \alpha^{n-1} \int_0^\infty \left(\frac{\alpha e^{\alpha t}}{e^{\alpha t} - 1} - \frac{e^t}{e^t - 1} \right) t^{n-1} e^{-\alpha xt} dt \\ (7.1) \quad &= \alpha^{n-1} \int_0^\infty \left[e^t - (1-\alpha)e^{(\alpha+1)t} - \alpha e^{\alpha t} \right] \frac{t^{n-1} e^{-\alpha xt}}{(e^{\alpha t} - 1)(e^t - 1)} dt. \end{aligned}$$

Easy computations reveal that

$$\begin{aligned}
 & e^t - (1 - \alpha)e^{(\alpha+1)t} - \alpha e^{\alpha t} \\
 &= \sum_{k=2}^{\infty} [1 - \alpha^{k+1} - (1 - \alpha)(\alpha + 1)^k] \frac{t^k}{k!} \\
 &= \sum_{k=2}^{\infty} (1 - \alpha) [(\alpha^k + \alpha^{k-1} + \cdots + 1) - (\alpha + 1)^k] \frac{t^k}{k!} \\
 (7.2) \quad &= \sum_{k=2}^{\infty} (1 - \alpha) \sum_{i=0}^k \left[\left(1 - \binom{k}{i} \right) \alpha^i \right] \frac{t^k}{k!} \geq 0 \quad (\alpha \geq 1).
 \end{aligned}$$

Combining (7.1) with (7.2), we have

$$(7.3) \quad (-1)^n g_{\alpha}^{(n-1)}(x) \geq 0 \quad (\alpha \geq 1).$$

Clearly,

$$(7.4) \quad g_{\alpha}'(x) \geq 0 \quad (\alpha \geq 1).$$

Applying asymptotic expansion for $\psi(x)$ (2.5) to $g_{\alpha}(x)$ and taking limit $x \rightarrow \infty$, we conclude

$$\begin{aligned}
 \lim_{x \rightarrow \infty} g_{\alpha}(x) &= \lim_{x \rightarrow \infty} [\psi(x) - \psi(\alpha x)] \\
 &= \lim_{x \rightarrow \infty} \left[\left(\ln x - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4}) \right) \right. \\
 &\quad \left. - \left(\ln \alpha x - \frac{1}{2\alpha x} - \frac{1}{12(\alpha x)^2} + O((\alpha x)^{-4}) \right) \right] \\
 &= -\ln \alpha \leq 0 \quad (\alpha \geq 1).
 \end{aligned}$$

Since $g_{\alpha}(x)$ is increasing (decreasing) on $(0, \infty)$ according as $\alpha \geq 1$, then we obtain

$$(\ln G_{\alpha}(x))' = \alpha g_{\alpha}(x) < 0 \quad (\alpha > 1)$$

and

$$\left(\ln \frac{1}{G_{\alpha}(x)} \right)' = -\alpha g_{\alpha}(x) < 0 \quad (0 < \alpha < 1).$$

Therefore, we obtain

$$(-1)^n (\ln G_{\alpha}(x))^{(n)} = (-1)^n \alpha g_{\alpha}^{(n-1)}(x) > 0 \quad (\alpha > 1)$$

and

$$(-1)^n \left(\ln \frac{1}{G_{\alpha}(x)} \right)^{(n)} = -(-1)^n \alpha g_{\alpha}^{(n-1)}(x) > 0 \quad (0 < \alpha < 1)$$

for $x \in (0, \infty)$ and $n = 1, 2, \dots$

The proof is complete. □

8. Appendix

Recall that Bernstein's Theorem [37] states that a function f is completely monotonic on $[0, \infty)$ if and only if $f(x) = \int_0^\infty e^{-xt} d\mu(t)$, where $\mu(t)$ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x \geq 0$. Motivated by Widder's result, we propose a necessary and sufficient condition for logarithmically completely monotonic function in the following theorem.

Theorem 10. *A necessary and sufficient condition that a positive function $f(x)$ should be logarithmically completely monotonic for $0 < x < \infty$ is that*

$$(8.1) \quad -(\ln f(x))' = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$.

Proof. If (8.1) is valid, then we have

$$\begin{aligned} (-1)^n [\ln f(x)]^{(n)} &= (-1)^{n-1} [-(\ln f(x))']^{(n-1)} \\ &= (-1)^{n-1} \left[\int_0^\infty e^{-xt} d\alpha(t) \right]^{(n-1)} \\ &= \int_0^\infty e^{-xt} t^{n-1} d\alpha(t) \\ &\geq 0. \end{aligned}$$

Conversely, let $f(x)$ be a logarithmically completely monotonic function for $0 < x < \infty$, we have

$$(-1)^{n+1} [\ln f(x)]^{(n+1)} = (-1)^n [-(\ln f(x))']^{(n)} = (-1)^n g(x)^{(n)} \geq 0$$

for $n = 0, 1, 2, \dots$, where $g(x) = -(\ln f(x))'$. So we get that $g(x)$ is completely monotonic. Therefore, we have

$$g(x) = -(\ln f(x))' = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. \square

Corollary 1. *A necessary and sufficient condition that a positive function $f(x)$ should be logarithmically completely monotonic on an interval I is that*

$$-(\ln f(x))' = \int_0^\infty e^{-xt} P(t) dt,$$

where $P(t) \geq 0$ and the integral converges for $x \in I$.

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