J. Korean Math. Soc. **45** (2008), No. 1, pp. 229–248

VOLTERRA COMPOSITION OPERATORS BETWEEN WEIGHTED BERGMAN SPACES AND BLOCH TYPE SPACES

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ABSTRACT. The boundedness and compactness of the Volterra composition operators between weighted Bergman spaces and Bloch type spaces are discussed in this paper.

1. Introduction

Let D be the open unit disk in the complex plane $\mathbb{C}$ and let $H(D)$ be the space of analytic functions on D. An analytic function f on D is said to belong to the Bloch type space $\mathcal{B}^\beta$, if

$$B(f) = \sup_{z\in D}(1-|z|^2)^\beta|f'(z)| < \infty.$$

The expression $B(f)$ defines a seminorm while the natural norm is given by $\|f\|_{\mathcal{B}^\beta} = |f(0)| + B(f)$. Let $\mathcal{B}_0^\beta$ denote the subspace of $\mathcal{B}^\beta$ consisting of those $f \in \mathcal{B}^\beta$ for which $(1-|z|^2)^\beta|f'(z)| \to 0$ as $|z| \to 1$. This space is called the little Bloch type space.

Let dA denote the normalized Lebesgue area measure in the unit disk D such that $A(D) = 1$. For $0 < p < \infty$, $\alpha > -1$, the weighted Bergman space A^p_α, consists of those $f \in H(D)$ for which

$$\|f\|^p_{A^p_\alpha} = \int_D |f(z)|^p(1-|z|^2)^\alpha dA(z) < \infty.$$

Throughout the paper φ denotes a nonconstant analytic self map of the unit disk D. Associated with φ is the composition operator C_φ defined by

$$C_\varphi f = f \circ \varphi$$

for $f \in H(D)$. It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces (see, for example, [5, 14, 21]).

Received June 23, 2006; Revised September 8, 2006.
2000 *Mathematics Subject Classification.* Primary 47B35; Secondary 30H05.
Key words and phrases. Volterra composition operator, Bergman space, Bloch space.

Suppose that $g : D \to \mathbb{C}^1$ is a holomorphic map and $f \in H(D)$, the Volterra type operator J_g (see [15]) or the Riemann-Stieltjes operator (see [18]) is defined by

$$J_g f(z) = \int_0^z f(\xi)g'(\xi)d\xi, \qquad z \in D.$$

The companion operator I_g (see [19]) is defined as

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)d\xi, \qquad z \in D.$$

The importance of the operators J_g and I_g comes from the fact that

$$J_g f + I_g f = M_g f - f(0)g(0),$$

where the multiplication operator M_g is defined by

$$(M_g f)(z) = g(z)f(z), \quad f \in H(D), \quad z \in D.$$

In [13] Pommerenke introduced the operator J_g and showed that J_g is a bounded operator on the Hardy space H^2 if and only if $g \in$ BMOA. Aleman and Siskakis studied the integral operator J_g on the Hardy space H^p (see [2]) and then on the Bergman space (see [3]). Recently, the operators J_g and I_g acting on various function spaces, including Bloch spaces, weighted Bergman spaces, BMOA and VMOA spaces, have been studied. See [1, 2, 3, 7, 8, 15, 16, 18, 19] and the related references therein.

In this paper, we consider the Volterra composition operators which defined as

$$(J_{g,\varphi} f)(z) = \int_0^z (f \circ \varphi)(\xi)(g \circ \varphi)'(\xi)d\xi \tag{1}$$

and

$$(I_{g,\varphi} f)(z) = \int_0^z (f \circ \varphi)'(\xi)(g \circ \varphi)(\xi)d\xi. \tag{2}$$

When $\varphi(z) = z$, then $J_{g,\varphi} = J_g, I_{g,\varphi} = I_g$. When $g = 1$, then $I_{g,\varphi} = C_\varphi$. Therefore we can regard the operators $J_{g,\varphi}$ and $I_{g,\varphi}$ as the generalization of composition operator C_φ and J_g, I_g. In addition, these operators are closely related with the product of composition operator and Volterra type operator. Since the parameter g is free, if we replace $g \circ \varphi$ by g we obtain the products $J_g C_\varphi$ and $I_g C_\varphi$. Here

$$(J_g C_\varphi f)(z) = \int_0^z (f \circ \varphi)(\xi)g'(\xi)d\xi$$

and

$$(I_g C_\varphi f)(z) = \int_0^z (f \circ \varphi)'(\xi)g(\xi)d\xi.$$

Therefore we can obtain the characterizations of boundedness and compactness of the operators $J_g C_\varphi$ and $I_g C_\varphi$ by modifying all results stated for $J_{g,\varphi}$ and $I_{g,\varphi}$ respectively.

Moreover, $C_\varphi J_g - J_{g,\varphi}$ and $C_\varphi I_g - I_{g,\varphi}$ are constants. In fact, we see that the following two equalities hold.

$$J_{g,\varphi}f(z) = (C_\varphi J_g f)(z) - \int_0^{\varphi(0)} f(\xi)g'(\xi)d\xi$$

and

$$I_{g,\varphi}f(z) = (C_\varphi I_g f)(z) - \int_0^{\varphi(0)} f'(\xi)g(\xi)d\xi.$$

In this paper we combine the composition operators and Volterra type operators and characterize the boundedness and compactness of the operators $J_{g,\varphi}, I_{g,\varphi}$ between weighted Bergman spaces and Bloch type spaces. As some corollaries, we obtain the characterizations of composition operator and Volterra type operator between the weighted Bergman space and the Bloch type space. Moreover we provide a unified way of treating these operators.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $C^{-1}B \le A \le CB$.

2. The boundedness and compactness of $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$

In this section, we characterize the boundedness and compactness of the operator $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta(\mathcal{B}^\beta_0)$. First, we give some auxiliary results which are incorporated in the following lemmas.

Lemma 1. *Let* $0 < p < \infty$ *and* $\alpha > -1$. *If* $f \in A^p_\alpha$, *then*

$$|f(z)| \le C \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}.$$

Proof. Let $\beta(z,w)$ denote the Bergman metric between two points z and w in D. It is given by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}.$$

For $a \in D$ and $r > 0$, the set $D(a,r) = \{z \in D : \beta(a,z) < r\}$ is a Bergman metric disk with center a and radius r. It is well known that (see [21])

$$\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4} \asymp \frac{1}{(1-|z|^2)^2} \asymp \frac{1}{(1-|a|^2)^2} \asymp \frac{1}{|D(a,r)|},$$

when $z \in D(a,r)$. For $0<r<1$ and $z \in D$, by the subharmonicity of $|f(z)|^p$, we get

$$\begin{aligned} |f(z)|^p &\le \frac{C}{(1-|z|^2)^2}\int_{D(z,r)} |f(a)|^p dA(a) \\ &\le \frac{C}{(1-|z|^2)^{2+\alpha}}\int_{D(z,r)} (1-|a|^2)^\alpha |f(a)|^p dA(a) \\ &\le \frac{C\|f\|^p_{A^p_\alpha}}{(1-|z|^2)^{2+\alpha}}, \end{aligned}$$

from which we get the desired result. □

By the similar arguments and using the well known asymptotic formula (see [21, 22])

$$\int_D |f(z)|^p(1-|z|^2)^\alpha dA(z) \asymp |f(0)|^p + \int_D |f'(z)|^p (1-|z|^2)^{\alpha+p} dA(z),$$

we obtain the following lemma.

Lemma 2. *Assume that $p>0$ and $\alpha>-1$. Let $f \in A^p_\alpha$. Then there is a positive constant C independent of f such that*

$$|f'(z)| \le C\frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}+1}}.$$

The following lemma was proved in [11]. For the case $\beta=1$, the lemma was proved in [10].

Lemma 3. *Let $\beta>0$. A closed set K in $\mathcal{B}^\beta_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)^\beta |f'(z)| = 0.$$

The following criterion for compactness follows from standard arguments similarly, for example, to those outlined in Proposition 3.11 of [5].

Lemma 4. *Let φ be an analytic self-map of the unit disk and $g \in H(D)$. Assume that $0<p,\beta<\infty$ and $\alpha>-1$. Then the operator $I_{g,\varphi}$(or $J_{g,\varphi}$) $: A^p_\alpha \to \mathcal{B}^\beta(\mathcal{B}^\beta \to A^p_\alpha)$ is compact if and only if $I_{g,\varphi}$(or $J_{g,\varphi}$) $: A^p_\alpha \to \mathcal{B}^\beta(\mathcal{B}^\beta \to A^p_\alpha)$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $A^p_\alpha(\mathcal{B}^\beta)$ which converges to zero uniformly on compact subsets of D, $I_{g,\varphi}f_k \to 0$ $(J_{g,\varphi}f_k \to 0)$ in $\mathcal{B}^\beta(A^p_\alpha)$ as $k\to\infty$.*

Theorem 1. *Let φ be an analytic self-map of the unit disk and $g \in H(D)$. Assume that $0<p,\beta<\infty$ and $\alpha>-1$. Then,*

(a) *$J_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{z\in D}\frac{|\varphi'(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}|g'(\varphi(z))| < \infty; \tag{3}$$

(b) $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta_0$ *is bounded if and only if* $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ *is bounded and*

$$\lim_{|z|\to 1} |g'(\varphi(z))||\varphi'(z)|(1-|z|^2)^\beta = 0. \tag{4}$$

Proof. (a). First, assume that (3) holds. Let $f \in A^p_\alpha$. By Lemma 1 we have

$$\begin{aligned} |(J_{g,\varphi}f)'(z)|(1-|z|^2)^\beta &= |f(\varphi(z))||g'(\varphi(z))||\varphi'(z)|(1-|z|^2)^\beta \\ &\le C\frac{||f||_{A^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))||\varphi'(z)|(1-|z|^2)^\beta \\ &= C||f||_{A^p_\alpha}\frac{|\varphi'(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))|. \end{aligned} \tag{5}$$

From (5) and $(J_{g,\varphi}f)(0) = 0$, we see that $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded.

Conversely, assume that $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded. For $w \in D$, set

$$f_w(z) = \Big(\frac{1-|w|^2}{(1-\overline{w}z)^2}\Big)^{\frac{2+\alpha}{p}}. \tag{6}$$

It is easy to check that $f_w \in A^p_\alpha$, and moreover $\sup_{w\in D}||f_w||_{A^p_\alpha} \le C$. Hence we have

$$\begin{aligned} \frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))| &= |f_{\varphi(z)}(\varphi(z))|(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)| \\ &= |(J_{g,\varphi}f_{\varphi(z)})'(z)|(1-|z|^2)^\beta \\ &\le ||J_{g,\varphi}f_{\varphi(z)}||_{\mathcal{B}^\beta} \le C||J_{g,\varphi}||. \end{aligned} \tag{7}$$

Taking the supremum in (7) over $z \in D$, we obtain (3).

(b). If $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded, then it is clear that $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded. Taking $f(z) = 1$, we get (4).

Conversely, suppose that $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded and (4) holds. For each polynomial $p(z)$ the following inequality holds

$$\begin{aligned} (1-|z|^2)^\beta|(J_{g,\varphi}p)'(z)| &= |p(\varphi(z))|(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)| \\ &\le M_p(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|, \end{aligned}$$

where $M_p = \sup_{z\in D}|p(z)|$. Since $M_p < \infty$ and (4) holds, we obtain that for each polynomial p, $J_{g,\varphi}(p) \in \mathcal{B}^\beta_0$. The set of all polynomials is dense in A^p_α, thus for every $f \in A^p_\alpha$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $||p_k - f||_{A^p_\alpha} \to 0$ as $k \to \infty$. Hence

$$||J_{g,\varphi}p_k - J_{g,\varphi}f||_{\mathcal{B}^\beta} \le ||J_{g,\varphi}||_{A^p_\alpha\to\mathcal{B}^\beta}||p_k - f||_{A^p_\alpha} \to 0 \quad \text{as } k \to \infty,$$

since the operator $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded. Since $\mathcal{B}^\beta_0$ is the closed subset of $\mathcal{B}^\beta$, we see that $J_{g,\varphi}(A^p_\alpha) \subset \mathcal{B}^\beta_0$. This completes the proof of Theorem 1. □

Theorem 2. *Let* φ *be an analytic self-map of the unit disk and* $g \in H(D)$. *Assume that* $0 < p, \beta < \infty$ *and* $\alpha > -1$. *Then,*

(a) $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ *is compact if and only if*

$$M = \sup_{z\in D}(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| < \infty \tag{8}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{|\varphi'(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))| = 0; \tag{9}$$

(b) $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta_0$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))| = 0. \tag{10}$$

Proof. (a) Assume that the conditions (8) and (9) hold. We have that there is an $r_0 \in (0,1)$ such that

$$\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))| < \varepsilon$$

for every $|\varphi(z)| > r_0$. Moreover, it is easy to see that (3) holds. Hence $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be a norm bounded sequence in A^p_α such that $f_k \to 0$ on compact subset of D as $k \to \infty$. It follows that

$$\begin{aligned}
&|(J_{g,\varphi}f_k)'(z)|(1-|z|^2)^\beta \\
= \ & |f_k(\varphi(z))||g'(\varphi(z))||\varphi'(z)|(1-|z|^2)^\beta \\
\le \ & \sup_{|\varphi(z)|\le r_0}|f_k(\varphi(z))| \sup_{|\varphi(z)|\le r_0}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)| \\
& + C\|f_k\|_{A^p_\alpha} \sup_{|\varphi(z)|>r_0}\frac{|\varphi'(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))| \\
\le \ & M \sup_{|\varphi(z)|\le r_0}|f_k(\varphi(z))| + \varepsilon C\|f_k\|_{A^p_\alpha}
\end{aligned}$$

for sufficiently large k. Taking the supremum in the above inequality over $z \in D$ and letting $k \to \infty$ we obtain that $\|J_{g,\varphi}f_k\|_{\mathcal{B}^\beta} \to 0$ as $k \to \infty$. Hence, by Lemma 4, we see that the operator $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is compact.

Conversely, suppose $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is compact. Then it is clear that $J_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded. Taking $f = 1$, we get (8). Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in D such that $\lim_{k\to\infty}|\varphi(z_k)| = 1$. Let

$$f_k(z) = \Big(\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^2}\Big)^{\frac{2+\alpha}{p}}. \tag{11}$$

Then $f_k \in A^p_\alpha$, moreover $\sup_{k\in\mathbb{N}}\|f_k\|_{A^p_\alpha} \le C$ and f_k converges to 0 uniformly on compact subsets of D as $k \to \infty$. Since $J_{g,\varphi}$ is compact, by Lemma 4 we have

$$\|J_{g,\varphi}f_k\|_{\mathcal{B}^\beta} \to 0 \quad \text{as} \quad k \to \infty.$$

Similarly to the proof of Theorem 1, we have

$$\begin{aligned}\|J_{g,\varphi}f_k\|_{\mathcal{B}^\beta} &= \sup_{z\in D}(1-|z|^2)^\beta|(J_{g,\varphi}f_k)'(z)| \\ &\geq \frac{|\varphi'(z_k)|(1-|z_k|^2)^\beta}{(1-|\varphi(z_k)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z_k))|,\end{aligned}$$

i.e., we get

$$\lim_{k\to\infty}\frac{|\varphi'(z_k)|(1-|z_k|^2)^\beta}{(1-|\varphi(z_k)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z_k))|=0.$$

Then the result follows.

(b). Now assume that (10) holds. It follows from Lemma 3 that $J_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta_0$ is compact if and only if

$$\lim_{|z|\to 1}\sup_{\|f\|_{A^p_\alpha}\leq 1}(1-|z|^2)^\beta|(J_{g,\varphi}f)'(z)|=0.$$

By Lemma 1 we have

$$|(J_{g,\varphi}f)'(z)|(1-|z|^2)^\beta \leq C\|f\|_{A^p_\alpha}\frac{|\varphi'(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))|.$$

Therefore (10) implies that $J_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta_0$ is compact.

Conversely, suppose that $J_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta_0$ is compact. Then it is clear that $J_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded and $J_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta$ is compact. Hence by (a) and Theorem 1, we have

$$\lim_{|\varphi(z)|\to 1}\frac{|\varphi'(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))|=0 \tag{12}$$

and

$$\lim_{|z|\to 1}|\varphi'(z)|(1-|z|^2)^\beta|g'(\varphi(z))|=0. \tag{13}$$

By (12), for every $\varepsilon>0$, there exists an $r\in(0,1)$,

$$\frac{|\varphi'(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))|<\varepsilon$$

when $r<|\varphi(z)|<1$. By (13), there exists a $\sigma\in(0,1)$,

$$|g'(\varphi(z))||\varphi'(z)|(1-|z|^2)^\beta<(1-r^2)^{\frac{2+\alpha}{p}}\varepsilon$$

when $\sigma<|z|<1$.

Therefore, when $\sigma<|z|<1$ and $r<|\varphi(z)|<1$, we have that

$$\frac{|\varphi'(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))|<\varepsilon. \tag{14}$$

If $\sigma < |z| < 1$ and $|\varphi(z)| \le r$, then we obtain

$$\begin{aligned}\frac{|\varphi'(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))| &< \frac{1}{(1-r^2)^{\frac{2+\alpha}{p}}}|g'(\varphi(z))||\varphi'(z)|(1-|z|^2)^\beta \\ &< \varepsilon. \end{aligned} \tag{15}$$

Combining (14) with (15), we obtain (10). □

Corollary 1. *Let $g \in H(D)$. Assume that $0 < p, \beta < \infty$ and $\alpha > -1$. Then,*

(a) *$J_g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{z\in D}(1-|z|^2)^{\beta-\frac{2+\alpha}{p}}|g'(z)| < \infty;$$

(b) *$J_g : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded if and only if $J_g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded and $g \in \mathcal{B}^\beta_0$.*

Corollary 2. *Let $g \in H(D)$. Assume that $0 < p, \beta < \infty$ and $\alpha > -1$. Then the following statements are equivalent:*

(a) *$J_g : A^p_\alpha \to \mathcal{B}^\beta$ is compact;*

(b) *$J_g : A^p_\alpha \to \mathcal{B}^\beta_0$ is compact;*

(c)

$$\lim_{|z|\to 1}(1-|z|^2)^{\beta-\frac{2+\alpha}{p}}|g'(z)| = 0.$$

Remark 1. If $\beta < \frac{2+\alpha}{p}$, then $J_g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if $g \equiv const$ by the maximal module principle. Similarly, if $\beta \le \frac{2+\alpha}{p}$, then $J_g : A^p_\alpha \to \mathcal{B}^\beta$ is compact if and only if $g \equiv const$.

3. The boundedness and compactness of $I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta(\mathcal{B}^\beta_0)$

In this section, we characterize the boundedness and compactness of the operator $I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta(\mathcal{B}^\beta_0)$.

Theorem 3. *Let φ be an analytic self-map of the unit disk and $g \in H(D)$. Assume that $0 < p, \beta < \infty$ and $\alpha > -1$. Then,*

(a) *$I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$\sup_{z\in D}\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}}|g(\varphi(z))| < \infty; \tag{16}$$

(b) *$I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded if and only if $I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))| = 0. \tag{17}$$

Proof. (a) From (2) we see that

$$(I_{g,\varphi}f)'(z) = \varphi'(z)g(\varphi(z))f'(\varphi(z)).$$

Let $f \in A^p_\alpha$. We have

$$\begin{aligned} |(I_{g,\varphi}f)'(z)|(1-|z|^2)^\beta &= (1-|z|^2)^\beta |f'(\varphi(z))||\varphi'(z)||g(\varphi(z))| \\ &\le C\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}}|g(\varphi(z))|\|f\|_{A^p_\alpha}. \end{aligned}$$

From the above inequality, the condition (16) and $(I_{g,\varphi}f)(0)=0$ show that the operator $I_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta$ is bounded.

Suppose that $I_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta$ is bounded, i.e., there exists a constant C such that

$$\|I_{g,\varphi}f\|_{\mathcal{B}^\beta} \le C\|f\|_{A^p_\alpha}$$

for all $f \in A^p_\alpha$. Taking $f(z)=z$, we get

$$\sup_{z\in D}(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))| < \infty. \tag{18}$$

Let $f_w(z)$ be defined by (6). From the proof of Theorem 1, we see that $f_w(z) \in A^p_\alpha$. Hence we have

$$\frac{(1-|z|^2)^\beta|\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}}|g(\varphi(z))| \le C\|I_{g,\varphi}f_{\varphi(z)}\|_{\mathcal{B}^\beta} \le C\|I_{g,\varphi}\|\|f_{\varphi(z)}\|_{A^p_\alpha}. \tag{19}$$

For $r\in(0,1)$, when $|\varphi(z)|\le r$, then

$$\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}}|g(\varphi(z))| \le \frac{1}{(1-r^2)^{\frac{2+\alpha}{p}+1}}(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))|. \tag{20}$$

When $|\varphi(z)|>r$, then

$$\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}}|g(\varphi(z))| \le \frac{1}{r}\frac{(1-|z|^2)^\beta|\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}}|g(\varphi(z))|. \tag{21}$$

From (18), (19), (20) and (21) we get the desired result.

(b) Assume that $I_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded. It is clear that $I_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta$ is bounded. Taking $f(z)=z\in A^p_\alpha$, we obtain

$$\lim_{|z|\to 1}(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))| = 0.$$

Conversely, assume that $I_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta$ is bounded and (17) holds. For each polynomial $p(z)$, we have that

$$(1-|z|^2)^\beta|(I_{g,\varphi}p)'(z)| \le (1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))||p'(\varphi(z))|,$$

from which it follows that $I_{g,\varphi}p \in \mathcal{B}^\beta_0$. Since the set of all polynomials is dense in A^p_α, we have that for every $f\in A^p_\alpha$ there is a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that $\|f-p_n\|_{A^p_\alpha}\to 0$, as $n\to\infty$. Hence, by the boundedness of the operator $I_{g,\varphi}: A^p_\alpha \to \mathcal{B}^\beta$, we have

$$\|I_{g,\varphi}f - I_{g,\varphi}p_n\|_{\mathcal{B}^\beta} \le \|I_{g,\varphi}\|_{A^p_\alpha\to\mathcal{B}^\beta}\|f-p_n\|_{A^p_\alpha} \to 0$$

as $n \to \infty$. Since $\mathcal{B}_0^\beta$ is the closed subset of $\mathcal{B}^\beta$, we obtain

$$I_{g,\varphi}(A^p_\alpha) \subset \mathcal{B}_0^\beta.$$

Therefore $I_{g,\varphi} : A^p_\alpha \to \mathcal{B}_0^\beta$ is bounded. □

Theorem 4. *Let φ be an analytic self-map of the unit disk and $g \in H(D)$. Assume that $0 < p, \beta < \infty$ and $\alpha > -1$. Then,*

(a) *$I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is compact if and only if*

$$H = \sup_{z\in D}(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))| < \infty \tag{22}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}}|g(\varphi(z))| = 0; \tag{23}$$

(b) *$I_{g,\varphi} : A^p_\alpha \to \mathcal{B}_0^\beta$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}}|g(\varphi(z))| = 0. \tag{24}$$

Proof. (a) Suppose that the conditions (22) and (23) hold. Then it is clear that (16) holds. Hence $I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in A^p_α such that $\sup_{k\in\mathbb{N}} \|f_k\|_{A^p_\alpha} \le L$ and f_k converges to 0 uniformly on compact subsets of D as $k \to \infty$. By the assumption, for any $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}}|g(\varphi(z))| < \varepsilon.$$

Then, we have

$$\begin{aligned} \|I_{g,\varphi}f_k\|_{\mathcal{B}^\beta} &= \sup_{z\in D}|(I_{g,\varphi}f_k)'(z)|(1-|z|^2)^\beta \\ &= \sup_{z\in D}(1-|z|^2)^\beta|\varphi'(z)||f_k'(\varphi(z))||g(\varphi(z))| \\ &= \sup_{\{z\in D:|\varphi(z)|\le\delta\}}(1-|z|^2)^\beta|\varphi'(z)||f_k'(\varphi(z))||g(\varphi(z))| \\ &\quad + \sup_{\{z\in D:|\varphi(z)|>\delta\}}(1-|z|^2)^\beta|\varphi'(z)||g(\varphi(z))||f_k'(\varphi(z))| \\ &\le \sup_{\{z\in D:|\varphi(z)|\le\delta\}} H|f_k'(\varphi(z))| \\ &\quad + \sup_{\{z\in D:|\varphi(z)|>\delta\}} \frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}}|g(\varphi(z))|\|f_k\|_{A^p_\alpha}. \end{aligned} \tag{25}$$

Since f_k converges to 0 uniformly on compact subsets of D as $k \to \infty$, Cauchy's estimate gives that $f_k' \to 0$ as $k \to \infty$ on compact subsets of D. Hence, letting

$k \to \infty$ in (25) we obtain

$$\lim_{k\to\infty} \|I_{g,\varphi} f_k\|_{\mathcal{B}^\beta} = 0.$$

From this and applying Lemma 4 the result follows.

For the converse, assume that $I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is compact. Taking $f(z) = z$, we get (22). Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in D such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$. Let

$$f_k(z) = \Big(\frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^2}\Big)^{\frac{2+\alpha}{p}}. \tag{26}$$

Then $f_k \in A^p_\alpha$, moreover $\sup_{k\in\mathbb{N}} \|f_k\|_{A^p_\alpha} \le C$ and f_k converges to 0 uniformly on compact subsets of D as $k \to \infty$. Since $I_{g,\varphi}$ is compact, by Lemma 4, we have $\lim_{k\to\infty} \|I_{g,\varphi} f_k\|_{\mathcal{B}^\beta} = 0$. From this and since

$$\begin{aligned} \|I_{g,\varphi} f_k\|_{\mathcal{B}^\beta} &= \sup_{z\in D}(1 - |z|^2)^\beta |(I_{g,\varphi} f_k)'(z)| \\ &\ge (1 - |z_k|^2)^\beta |\varphi'(z_k)||f_k'(\varphi(z_k))||g(\varphi(z_k))| \\ &= \frac{2+\alpha}{p} \frac{(1 - |z_k|^2)^\beta |\varphi'(z_k)||\varphi(z_k)|}{(1 - |\varphi(z_k)|^2)^{\frac{2+\alpha}{p}+1}} |g(\varphi(z_k))|, \end{aligned}$$

we have that

$$\lim_{k\to\infty} \frac{(1 - |z_k|^2)^\beta |\varphi'(z_k)|}{(1 - |\varphi(z_k)|^2)^{\frac{2+\alpha}{p}+1}} |g(\varphi(z_k))| = 0.$$

Then (23) follows.

(b) First we assume that (24) holds. It follows from Lemma 3 that $I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta_0$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{A^p_\alpha} \le 1} (1 - |z|^2)^\beta |(I_{g,\varphi} f)'(z)| = 0.$$

Since

$$(1 - |z|^2)^\beta |(I_{g,\varphi} f)'(z)| \le \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}} |g(\varphi(z))| \|f\|_{A^p_\alpha},$$

from the assumption, we get the desired result.

Conversely, we assume that $I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta_0$ is compact. Then $I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta_0$ is bounded and $I_{g,\varphi} : A^p_\alpha \to \mathcal{B}^\beta$ is compact. Hence by (a) and Theorem 3 we have

$$\lim_{|z|\to 1} (1 - |z|^2)^\beta |\varphi'(z)||g(\varphi(z))| = 0 \tag{27}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1 - |z|^2)^\beta |\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}} |g(\varphi(z))| = 0. \tag{28}$$

From (27) and (28), using the similar methods of the proof of Theorem 2, we obtain the desired result. □

From Theorems 3 and 4, we can easily arrive at the following corollaries (see [9, 17]):

Corollary 3. *Let φ be an analytic self-map of the unit disk. Assume that $0 < p, \beta < \infty$ and $\alpha > -1$. Then,*

(a) $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ *is bounded if and only if*

$$\sup_{z\in D} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}} < \infty;$$

(b) $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$ *is bounded if and only if* $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ *is bounded and* $\varphi \in \mathcal{B}^\beta_0$.

Corollary 4. *Let φ be an analytic self-map of the unit disk. Assume that $0 < p, \beta < \infty$ and $\alpha > -1$. Then,*

(a) $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta$ *is compact if and only if* $\varphi \in \mathcal{B}^\beta$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}} = 0;$$

(b) $C_\varphi : A^p_\alpha \to \mathcal{B}^\beta_0$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}+1}} = 0.$$

Corollary 5. *Let $g \in H(D)$. Assume that $0 < p, \beta < \infty$ and $\alpha > -1$. Then,*

(a) $I_g : A^p_\alpha \to \mathcal{B}^\beta$ *is bounded if and only if*

$$\sup_{z\in D} (1-|z|^2)^{\beta-1-\frac{2+\alpha}{p}} |g(z)| < \infty;$$

(b) $I_g : A^p_\alpha \to \mathcal{B}^\beta_0$ *is bounded if and only if* $I_g : A^p_\alpha \to \mathcal{B}^\beta$ *is bounded and* $\lim_{|z|\to 1}(1-|z|^2)^\beta |g(z)| = 0$.

Corollary 6. *Let $g \in H(D)$. Assume that $0 < p, \beta < \infty$ and $\alpha > -1$. Then the following statements are equivalent.*

(a) $I_g : A^p_\alpha \to \mathcal{B}^\beta$ *is compact;*

(b) $I_g : A^p_\alpha \to \mathcal{B}^\beta_0$ *is compact;*

(c)

$$\lim_{|z|\to 1} (1-|z|^2)^{\beta-1-\frac{2+\alpha}{p}} |g(z)| = 0.$$

Remark 2. If $\beta < 1 + \frac{2+\alpha}{p}$, then $I_g : A^p_\alpha \to \mathcal{B}^\beta$ is bounded if and only if $g \equiv 0$ by the maximal module principle. Similarly, if $\beta \le 1 + \frac{2+\alpha}{p}$, then $J_g : A^p_\alpha \to \mathcal{B}^\beta$ is compact if and only if $g \equiv 0$.

4. The boundedness and compactness of $J_{g,\varphi}, I_{g,\varphi} : \mathcal{B}^\beta \to A^p_\alpha$

In this section, we consider the boundedness and compactness of $J_{g,\varphi}, I_{g,\varphi} : \mathcal{B}^\beta \to A^p_\alpha$. Some auxiliary lemmas should be given. The following lemma is well known (for example, see [11]).

Lemma 5. *Let $f \in \mathcal{B}^\beta, 0 < \beta < \infty$. Then*

$$|f(z)| \leq \begin{cases} |f(0)| + ||f||_{\mathcal{B}^\beta} \frac{1-(1-|z|)^{1-\beta}}{1-\beta} & , \quad \beta \neq 1 \\ |f(0)| + ||f||_{\mathcal{B}^\beta} \ln \frac{2}{1-|z|} & , \quad \beta = 1. \end{cases}$$

Let $0 < p < \infty$, μ be a positive Borel measure on D. Define

$$D_p(\mu) = \{f \in H(D), \ \ ||f||^p_{D_p(\mu)} = \int_D |f'(z)|^p d\mu(z) < \infty\}.$$

Lemma 6. *Let μ be a positive measure on D and $0 < p, \beta < \infty$. Then the following statements are equivalent:*

(a) $i : \mathcal{B}^\beta \longmapsto D_p(\mu)$ *is bounded;*
(b) $i : \mathcal{B}^\beta \longmapsto D_p(\mu)$ *is compact;*
(c) $i : \mathcal{B}^\beta_0 \longmapsto D_p(\mu)$ *is bounded;*
(d) $i : \mathcal{B}^\beta_0 \longmapsto D_p(\mu)$ *is compact;*
(e)

$$\int_D \frac{d\mu(z)}{(1-|z|^2)^{\beta p}} < \infty.$$

Remark. The above lemma was obtained by Zhao when $0 < \beta \leq 1$ (see [20]). In fact his proof implies that the results also hold for $\beta > 1$. Partial results can also be found in [4] when $\beta = 1$.

Theorem 5. *Let φ be an analytic self-map of the unit disk and $g \in H(D)$. Assume that $0 < p, \beta < \infty$ and $\alpha > -1$. Then the following statements are equivalent:*

(a) $I_{g,\varphi} : \mathcal{B}^\beta \to A^p_\alpha$ *is bounded;*
(b) $I_{g,\varphi} : \mathcal{B}^\beta \to A^p_\alpha$ *is compact;*
(c) $I_{g,\varphi} : \mathcal{B}^\beta_0 \to A^p_\alpha$ *is bounded;*
(d) $I_{g,\varphi} : \mathcal{B}^\beta_0 \to A^p_\alpha$ *is compact;*
(e)

$$\int_D \frac{|\varphi'(z)|^p (1-|z|^2)^{p+\alpha}}{(1-|\varphi(z)|^2)^{p\beta}} |g(\varphi(z))|^p dA(z) < \infty.$$

Proof. Let $f \in A^p_\alpha$. Since (see [6])

$$\begin{aligned}
\|I_{g,\varphi}f\|^p_{A^p_\alpha} &\asymp \int_D |(I_{g,\varphi}f)'(z)|^p(1-|z|^2)^{p+\alpha}dA(z)\\
&= \int_D |f'(\varphi(z))|^p|\varphi'(z)|^p|g(\varphi(z))|^p(1-|z|^2)^{p+\alpha}dA(z)\\
&= \int_D |f'(\varphi(z))|^p d\mu(z)\\
&= \int_D |f'(z)|^p d\mu\circ\varphi^{-1},
\end{aligned}$$

where

$$d\mu(z) = |\varphi'(z)|^p|g(\varphi(z))|^p(1-|z|^2)^{p+\alpha}dA(z),$$

by Lemma 6, we know that $I_{g,\varphi} : \mathcal{B}^\beta(\mathcal{B}^\beta_0) \to A^p_\alpha$ is bounded(or compact) if and only if

$$\infty > \int_D \frac{d\mu\circ\varphi^{-1}}{(1-|z|^2)^{\beta p}} = \int_D \frac{|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}}{(1-|\varphi(z)|^2)^{p\beta}}|g(\varphi(z))|^p dA(z).$$

□

From Theorem 5, we get the following corollary (see [17]).

Corollary 7. *Let φ be an analytic self-map of the unit disk. Assume that $0 < p, \beta < \infty$ and $\alpha > -1$. Then the following statements are equivalent:*

(a) $C_\varphi : \mathcal{B}^\beta \to A^p_\alpha$ *is bounded;*

(b) $C_\varphi : \mathcal{B}^\beta \to A^p_\alpha$ *is compact;*

(c) $C_\varphi : \mathcal{B}^\beta_0 \to A^p_\alpha$ *is bounded;*

(d) $C_\varphi : \mathcal{B}^\beta_0 \to A^p_\alpha$ *is compact;*

(e)

$$\int_D \frac{|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}}{(1-|\varphi(z)|^2)^{p\beta}}dA(z) < \infty.$$

Theorem 6. *Let φ be an analytic self-map of the unit disk and $g \in H(D)$. Assume that $0 < p < \infty, \beta > 1$ and $\alpha > -1$. Then the following statements are equivalent.*

(a) $J_{g,\varphi} : \mathcal{B}^\beta \to A^p_\alpha$ *is bounded;*

(b) $J_{g,\varphi} : \mathcal{B}^\beta \to A^p_\alpha$ *is compact;*

(c)

$$\int_D \frac{|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}}{(1-|\varphi(z)|^2)^{(\beta-1)p}}|g'(\varphi(z))|^p dA(z) < \infty.$$

Proof. (a) $\Leftrightarrow$ (c). Since(see [6])

$$\begin{aligned}\|J_{g,\varphi}f\|_{A^p_\alpha}^p &\asymp \int_D |(J_{g,\varphi}f)'(z)|^p(1-|z|^2)^{p+\alpha}dA(z)\\ &= \int_D |g'(\varphi(z))|^p|f(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}dA(z)\\ &= \int_D |f(\varphi(z))|^p(1-|\varphi(z)|^2)^{-p}d\mu(z)\\ &= \int_D |f(z)|^p(1-|z|^2)^{-p}d\mu\circ\varphi^{-1},\end{aligned}$$

where

$$d\mu = |g'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}(1-|\varphi(z)|^2)^p dA(z),$$

by Theorem 3.2 of [12], we see that $J_{g,\varphi}:\mathcal{B}^\beta\to A^p_\alpha$ is bounded if and only if

$$\int_D \frac{|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}}{(1-|\varphi(z)|^2)^{(\beta-1)p}}|g'(\varphi(z))|^p dA(z) = \int_D \frac{d\mu\circ\varphi^{-1}}{(1-|z|^2)^{p\beta}} < \infty.$$

(b) $\Rightarrow$ (a). It is clear.

(c) $\Rightarrow$ (b). Assume (c) holds, we obtain that $J_{g,\varphi}$ is bounded. Taking $f(z)=1$, we get

$$\int_D |\varphi'(z)|^p|g'(\varphi(z))|^p(1-|z|^2)^{p+\alpha}dA(z) < \infty. \tag{29}$$

In addition, we find that for any $\varepsilon>0$, there is an $r\in(0,1)$ such that

$$\int_{|\varphi(z)|>r} \frac{|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}}{(1-|\varphi(z)|^2)^{(\beta-1)p}}|g'(\varphi(z))|^p dA(z) < \varepsilon. \tag{30}$$

Let $\{f_k\}$ be any bounded sequence of $\mathcal{B}^\beta$ and converges to 0 uniformly on compact subsets of D. For the above ε, there exists a $k_0>0$ such that $\sup_{|w|\le r}|f_k(w)|<\varepsilon$ as $k>k_0$. Hence by (29) and (30), we have

$$\begin{aligned}&\|J_{g,\varphi}f_k\|_{A^p_\alpha}^p\\ \asymp\ & \Big(\int_{|\varphi(z)|\le r}+\int_{|\varphi(z)|>r}\Big)|(J_{g,\varphi}f_k)'(z)|^p(1-|z|^2)^{p+\alpha}dA(z)\\ =\ & \Big(\int_{|\varphi(z)|\le r}+\int_{|\varphi(z)|>r}\Big)|f_k(\varphi(z))|^p|\varphi'(z)|^p|g'(\varphi(z))|^p(1-|z|^2)^{p+\alpha}dA(z)\\ \le\ & \varepsilon^p\int_{|\varphi(z)|\le r}|\varphi'(z)|^p|g'(\varphi(z))|^p(1-|z|^2)^{p+\alpha}dA(z)\\ & +\|f_k\|_{\mathcal{B}^\beta}^p\int_{|\varphi(z)|>r}\frac{|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}}{(1-|\varphi(z)|^2)^{(\beta-1)p}}|g'(\varphi(z))|^p dA(z)\\ \le\ & C\varepsilon^p+\varepsilon\|f_k\|_{\mathcal{B}^\beta}^p.\end{aligned}$$

In other words, we obtain $\lim_{k\to\infty}\|J_{g,\varphi}f_k\|_{A^p_\alpha}^p=0$. Therefore $J_{g,\varphi}:\mathcal{B}^\beta\to A^p_\alpha$ is compact by Lemma 4. □

Theorem 7. *Let φ be an analytic self-map of the unit disk and $g \in H(D)$. Assume that $0 < p < \infty$, $0 < \beta < 1$ and $\alpha > -1$. Then the following statements are equivalent.*

(a) *$J_{g,\varphi} : \mathcal{B}^\beta \to A^p_\alpha$ is bounded;*

(b) *$J_{g,\varphi} : \mathcal{B}^\beta \to A^p_\alpha$ is compact;*

(c)

$$\int_D |g'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^{p+\alpha} dA(z) < \infty.$$

Proof. (a) $\Leftrightarrow$ (c). Assume that $J_{g,\varphi} : \mathcal{B}^\beta \to A^p_\alpha$ is bounded. Since

$$(31) \qquad \|J_{g,\varphi} f\|^p_{A^p_\alpha} \asymp \int_D |g'(\varphi(z))|^p |f(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^{p+\alpha} dA(z),$$

taking $f = 1$, we get (c).

Conversely we assume that (c) holds. Let $f \in \mathcal{B}^\beta$. Then by (31) and Lemma 5 we see that $J_{g,\varphi} : \mathcal{B}^\beta \to A^p_\alpha$ is bounded.

(b) $\Rightarrow$ (a). It is clear.

(c) $\Rightarrow$ (b). Similarly to the proof of (c) $\Rightarrow$ (b) of Theorem 6 we can get the desired result. □

Finally, we consider the case of $\beta = 1$.

Theorem 8. *Let φ be an analytic self-map of the unit disk and $g \in H(D)$. Assume that $0 < p < \infty$ and $\alpha > -1$. Then,*

(a) *If the operator $J_{g,\varphi} : \mathcal{B} \to A^p_\alpha$ is bounded, then*

$$(32) \qquad \sup_{z \in D} |g'(\varphi(z))| \ln \frac{2}{1-|\varphi(z)|^2} |\varphi'(z)| (1-|z|^2)^{1+\frac{\alpha+2}{p}} < \infty;$$

(b) *If*

$$(33) \qquad \sup_{z \in D} |g'(\varphi(z))| \ln \frac{2}{1-|\varphi(z)|^2} |\varphi'(z)| (1-|z|^2)^{1+\frac{\alpha}{p}} < \infty,$$

then $J_{g,\varphi} : \mathcal{B} \to A^p_\alpha$ is bounded.

Proof. (a) Assume that $J_{g,\varphi} : \mathcal{B} \to A^p_\alpha$ is bounded. Let $f \in \mathcal{B}$. Then $J_{g,\varphi} f \in A^p_\alpha$. By Lemma 2 we have

$$(34) \qquad |(J_{g,\varphi} f)'(z)| \leq C \frac{\|J_{g,\varphi} f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}+1}} \leq C \frac{\|J_{g,\varphi}\|_{\mathcal{B} \to A^p_\alpha} \|f\|_{\mathcal{B}}}{(1-|z|^2)^{\frac{2+\alpha}{p}+1}}.$$

For any $w \in D$, let $f_w(z) = \ln \frac{2}{1-\overline{w}z}$. Since

$$(1-|z|^2)|f_w'(z)| \leq (1-|z|^2) \frac{|w|}{|1-\overline{w}z|} \leq \frac{1-|z|^2}{|1-\overline{w}z|} \leq 2,$$

we have $\|f_w\|_{\mathcal{B}} \leq \ln 2 + 2$. Replacing f with $f_{\varphi(z)}$ in (34), we obtain

$$|g'(\varphi(z))| \ln \frac{2}{1-|\varphi(z)|^2} |\varphi'(z)| (1-|z|^2)^{1+\frac{\alpha+2}{p}} \leq C \|J_{g,\varphi}\|_{\mathcal{B} \to A^p_\alpha} \|f_{\varphi(z)}\|_{\mathcal{B}}.$$

From which we get the desired result.

(b) Assume that (33) holds. For any $f \in \mathcal{B}$, by Lemma 5, we have

$$\begin{aligned}
&\|J_{g,\varphi}f\|^p_{A^p_\alpha} \\
\asymp\ & \int_D |(J_{g,\varphi}f)'(z)|^p(1-|z|^2)^{p+\alpha}dA(z) \\
=\ & \int_D |g'(\varphi(z))|^p|f(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}dA(z) \\
\leq\ & C\|f\|^p_{\mathcal{B}}\int_D |g'(\varphi(z))|^p\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^p|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}dA(z) \\
\leq\ & C\|f\|^p_{\mathcal{B}}\sup_{z\in D}|g'(\varphi(z))|^p\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^p|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}\int_D dA(z) \\
\leq\ & C\|f\|^p_{\mathcal{B}}\sup_{z\in D}|g'(\varphi(z))|^p\Big(\ln\frac{2}{1-|\varphi(z)|^2}\Big)^p|\varphi'(z)|^p(1-|z|^2)^{p+\alpha} \\
<\ & \infty.
\end{aligned}$$

Therefore $J_{g,\varphi}: \mathcal{B} \to A^p_\alpha$ is bounded. □

Theorem 9. *Let φ be an analytic self-map of the unit disk and $g \in H(D)$. Assume that $0 < p < \infty$ and $\alpha > -1$. Then,*

(a) *If the operator $J_{g,\varphi}: \mathcal{B} \to A^p_\alpha$ is compact, then*

$$\lim_{|\varphi(z)|\to 1}|g'(\varphi(z))|\ln\frac{2}{1-|\varphi(z)|^2}|\varphi'(z)|(1-|z|^2)^{1+\frac{\alpha+2}{p}} = 0; \tag{35}$$

(b) *If $J_{g,\varphi}: \mathcal{B} \to A^p_\alpha$ is bounded and*

$$\lim_{|\varphi(z)|\to 1}|g'(\varphi(z))|\ln\frac{2}{1-|\varphi(z)|^2}|\varphi'(z)|(1-|z|^2)^{1+\frac{\alpha}{p}} = 0, \tag{36}$$

then $J_{g,\varphi}: \mathcal{B} \to A^p_\alpha$ is compact.

Proof. (a) Suppose that the operator $J_{g,\varphi}: \mathcal{B} \to A^p_\alpha$ is compact. Let z_n be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Take

$$f_n(z) = \Big(\ln\frac{2}{1-|\varphi(z_n)|^2}\Big)^{-1}\Big(\ln\frac{2}{1-\overline{\varphi(z_n)}z}\Big)^2.$$

Then

$$|f_n(0)| \leq \left(\ln\frac{2}{1-|\varphi(z_n)|^2}\right)^{-1}(\ln 2)^2 \leq \ln 2$$

and

$$f_n'(z) = 2\Big(\ln\frac{2}{1-|\varphi(z_n)|^2}\Big)^{-1}\left(\ln\frac{2}{1-z\overline{\varphi(z_n)}}\right)\frac{\overline{\varphi(z_n)}}{1-z\overline{\varphi(z_n)}}.$$

Thus

$$\begin{aligned}\|f_n\|_{\mathcal{B}} &= |f_n(0)| + \sup_{z\in D}(1-|z|^2)|f_n'(z)| \\ &\leq \ln 2 + 2\sup_{z\in D}(1-|z|^2)\left|\frac{\ln\frac{2}{1-z\overline{\varphi(z_n)}}}{\ln\frac{2}{1-|\varphi(z_n)|^2}}\right|\frac{1}{1-|z|} \leq C.\end{aligned}$$

For $|z| = \rho < 1$, we have

$$|f_n(z)| = \frac{\left|\ln\frac{2}{1-z\overline{\varphi(z_n)}}\right|^2}{\ln\frac{2}{1-|\varphi(z_n)|^2}} \leq \frac{\left(\ln\frac{2}{1-\rho} + C\right)^2}{\ln\frac{2}{1-|\varphi(z_n)|^2}} \to 0 \quad (n\to\infty),$$

that is, $f_n \to 0$ uniformly on compact subsets of D as $n\to\infty$. Similarly to the proof of Theorem 8, we obtain

$$|g'(\varphi(z_n))|\ln\frac{2}{1-|\varphi(z_n)|^2}|\varphi'(z_n)|(1-|z_n|^2)^{1+\frac{\alpha+2}{p}} \leq C\|J_{g,\varphi}f_n\|_{A^p_\alpha}.$$

From which we obtain (35) by Lemma 4.

(b) Assume that $J_{g,\varphi} : \mathcal{B} \to A^p_\alpha$ is bounded and (36) holds. Taking $f = 1$, we obtain that

$$\int_D |g'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}dA(z) < \infty. \tag{37}$$

From (36), for any $\varepsilon > 0$, there exist a r, $0 < r < 1$, such that

$$|g'(\varphi(z))|\ln\frac{2}{1-|\varphi(z)|^2}|\varphi'(z)|(1-|z|^2)^{1+\frac{\alpha}{p}} < \varepsilon, \tag{38}$$

when $|\varphi(z)| > r$. Let $\{f_k\}$ be any bounded sequence of $\mathcal{B}$ and converges to 0 uniformly on compact subsets of D. For the above ε, there exists a $k_0 > 0$ such that $\sup_{|w|\leq r}|f_k(w)| < \varepsilon$ as $k > k_0$. Hence by (37) and (38) we have

$$\begin{aligned}&\|J_{g,\varphi}f_k\|^p_{A^p_\alpha} \\ &\asymp \left(\int_{|\varphi(z)|\leq r} + \int_{|\varphi(z)|>r}\right)|(J_{g,\varphi}f_k)'(z)|^p(1-|z|^2)^{p+\alpha}dA(z) \\ &= \left(\int_{|\varphi(z)|\leq r} + \int_{|\varphi(z)|>r}\right)|f_k(\varphi(z))|^p|g'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}dA(z) \\ &\leq \sup_{|\varphi(z)|\leq r}|f_k(\varphi(z))|^p\int_{|\varphi(z)|\leq r}|g'(\varphi(z))|^p|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}dA(z) \\ &\quad + \|f_k\|^p_{\mathcal{B}}\int_{|\varphi(z)|>r}|g'(\varphi(z))|^p\ln^p\frac{2}{1-|\varphi(z)|^2}|\varphi'(z)|^p(1-|z|^2)^{p+\alpha}dA(z) \\ &\leq C\varepsilon^p + \varepsilon\|f_k\|^p_{\mathcal{B}},\end{aligned}$$

as $k > k_0$. From which we get the desired result by Lemma 4. □

Acknowledgments. The author thank the referee for several helpful comments and suggestions for improvements. The author is supported partly by the Natural Science Foundation of Guangdong Province(No.73006147).

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